INTRODUCTION

Dhage [1], [2], [3] has given the foundation of a new structure of D-metric space and proved some basic results concerning topology, completeness and compactness etc, of the D-metric space.

Definition 1.1: A function $D$ on $X \times X \times X$ into $R$ is said to be a D-metric on nonempty set $X$ if it satisfies the following properties

(M1): $D(x, y, z) \geq 0$; for all $x, y, z \in X$ (Non negativity)

(M2): $D(x, y, z) = 0$ if and only if $x = y = z$

(M3): $D(x, y, z) = D(x, z, y) = ...$ (Symmetry)

(M4): $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$, for all $x, y, z, a \in X$ (Rectangle inequality),

A nonempty set $X$ together with a D-metric is called generalized metric space or Dhage metric space or D-metric space and is denoted by $(X, D)$.

We give some examples of D-metrics paces

Example 1.1: Define a function $D_1$ on $X$ by

$D_1(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$ for $x, y, z \in X$, and $d$ is an ordinary metric on $X$. Then $D_1$ is D-metric and $(X, D_1)$ is D-metric space.

Example 1.2: Define a function $D_2: X \times X \times X \rightarrow R$ by

$D_2(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, for $x, y, z \in X$ and $d$ is ordinary metric on $X$. Then $(X, D_2)$ is D-metric space

2. D-CONTRACTION PRINCIPLE

The fundamental and most interesting result in D-metric spaces is D-contraction principle due to Dhage [2] & it is proved by same author that the Banach contraction principle is the particular case of this theorem. Dhage [2] proved the following fixed point theorem for D-contraction mapping in D-metric spaces called D-contraction principle.

Theorem 2.1: Let $f$ be a self mapping of a complete and bounded D-metric space $X$ satisfying

$D(f(x), f(y), f(z)) \leq \alpha \cdot D(x, y, z)$

for all $x, y, z \in X$ and $\alpha < 1$. Then $f$ has a unique fixed point.

The following lemma and theorem of [1] is useful to prove main result.

Lemma 2.1: Let $\{x_n\}$ be a sequence of bounded D-metric space $X$ such that

$D(x_{n+1}, x_{n+2}) \leq \alpha D(x_n, x_{n+1})$

for all $n \in N$, where $0 \leq \alpha < 1$. Then $\{x_n\}$ is D-cauchy.
Theorem 2.2: Let \( f \) be a self-map of a complete and bounded D-metric space \( X \) satisfying
\[
D(fx, fy, fz) \leq \alpha \max \{D(x, fx, fy), D(y, fy, fz)\}
\]
(2.1.3)
For all \( x, y, z \in X \) and \( 0 \leq q < 1 \). Then \( f \) has a unique fixed point.

3. MAIN RESULT

Theorem 3.1: Let \( X \) be a Tri-D-metric space with three D-metrics \( D, D_1, D_2 \). Let \( f: X \to X \) be a mapping and suppose that the following conditions hold in \( X \).

(i) \( X \) is bounded w.r.t. \( D \)

(ii) \( D_2(x, y, z) \leq D_1(x, y, z) \leq D(x, y, z) \) for all \( x, y, z \in X \)

(iii) \( X \) is complete w.r.t. \( D_1 \)

(iv) \( X \) is continuous w.r.t. \( D_2 \)

(v) \( f \) satisfies the condition (2.1.3) w.r.t. \( D \).

Then \( X \) has a unique fixed point.

Proof: Suppose \( x = x_0 \in X \) is an arbitrary point and consider a sequence \( \{x_n\} \) in \( X \) defined by
\[
x_0 = x, \quad x_{n+1} = f x_n, \quad n \in \mathbb{N} \cup \{0\}
\]
(3.1.1)
where \( \mathbb{N} \) denotes the set of natural numbers.

If \( x_r = x_{r+1} \) for some \( r \in \mathbb{N} \) then \( x_r = u \) is a fixed point of \( f \). Therefore we assume that \( x_n \neq x_{n+1} \) for each \( n \in \mathbb{N} \), we show that \( \{x_n\} \) is a D-Cauchy sequence in \( X \).

Let \( x = x_0, \ y = x_1, \ z = x_2 \) then by condition (2.1.3) we get
\[
D(fx_0, fx_1, fx_2) \leq \alpha \max\{D(x_0, fx_0, fx_1), D(x_1, fx_1, fx_2)\}
\]
i.e, \( D(x_1, x_2, x_3) \leq \alpha \max\{D(x_0, x_1, x_2), D(x_1, x_2, x_3)\} \)

since, \( D(x_1, x_2, x_3) \leq \alpha \max D(x_1, x_2, x_3) \) is not possible, we have
\[
D(x_1, x_2, x_3) \leq \alpha D(x_0, x_1, x_2) \quad (3.1.2)
\]
Similarly letting \( x = x_1, \ y = x_2, \ z = x_3 \) in condition (2.1.3) we obtain
\[
D(fx_1, fx_2, fx_3) \leq \alpha \max\{D(x_1, fx_1, fx_2), D(x_2, fx_2, fx_3)\}
\]
i.e, \( D(x_2, x_3, x_4) \leq \alpha \max\{D(x_1, x_2, x_3), D(x_2, x_3, x_4)\} \)

since, \( D(x_2, x_3, x_4) \leq \alpha \max D(x_2, x_3, x_4) \) is not possible, we have
\[
D(x_2, x_3, x_4) \leq \alpha D(x_1, x_2, x_3) \quad (3.1.3)
\]
Proceeding in this way by induction we obtain
\[
D(x_n, x_{n+1}, x_{n+2}) \leq \alpha D(x_{n-1}, x_n, x_{n+1}) \quad (3.1.4)
\]
for all \( n, n=1, 2, \ldots \). Then by Lemma (2.1.1) \( \{x_n\} \) is D-Cauchy sequence.

i.e, \( \lim D(x_n, x_m, x_p) = 0 \)

\[ m, n, p \to \infty \]

The hypothesis (ii) implies that
\[
\lim D(x_n, x_m, x_p) \leq \lim D(x_m, x_n, x_p) \to 0 \]
\[ m, n, p \in \infty \]

This shows that \( \{x_n\} \) is a D-cauchy sequence w.r.t. \( D_1 \), there is a point \( u \in X \) such that
\[
\lim D(x_n, x_m, u) = 0 \]
i.e, \( \lim x_n = 0 \)

\[ m, n, \infty \to \infty \]
w.r.t. \( D_1 \). Again \( D_2 \leq D_1 \) on \( X \), we get \( x_n \to u \) w.r.t. \( D_2 \).
\[
u = \lim x_n = \lim f x_n = f \lim x_n = f u \]
\[ n \to \infty \]
\[ n \to \infty \]
\[ n \to \infty \]
showing that \( u \) is a fixed point of \( f \).

To prove uniqueness, let \( v (\neq u) \) be another fixed point of \( f \) then by condition (2.1.3) we obtain
\[
D(u, u, v) = D(fu, fu, fv) \leq \alpha \max\{D(u, fu, fu), D(u, fu, fv)\}
\]
\[ = \alpha \max\{D(u, u, u), D(u, u, v)\} \]
\[ = \alpha \max\{0, D(u, u, v)\} \]
\[ D(u, u, v) \leq \alpha D(u, u, v) \]
Which is contradiction since $\alpha < 1$. Hence $u = v$. Therefore $f$ has a unique fixed point.

**Corollary 3.1:** Let $X$ be a Tri-D-metric space with three D-metrics $D$, $D_1$, $D_2$. Let $f: X \to X$ be a mapping and suppose that following conditions are satisfied.

(i) The conditions (i) - (iv) of Theorem 3.1

(ii) There exists a positive integer $p$ such that $f^p$ satisfies condition

$$D(f^p x, f^p y, f^p z) \leq \alpha \max \{D(x, f^p x, f^p y), D(y, f^p y, f^p z)\} \quad (3.1.5)$$

For all $x, y, z \in X$ and $0 \leq p < 1$.

Then $f$ has a unique fixed point.

**Proof:** Let $T = f^p$, then $T$ is continuous no $X$ w. r. to $D_2$, and since $f$ and consequently $f^p$ is continuous on $X$ w. r. to $D_2$.

Now by an application of Theorem 3.1 implies that $T$ has a unique fixed point, say $u$ in $X$. i. e, it is a point such that

$$Tu = f^p u = u$$

But $fu = f(f^p u) = f^p(fu) = T(fu)$, which shows that $fu$ is again a fixed point of $T$. By uniqueness of $u$, we get $fu = u$.

Again the uniqueness of $u$ follows from the condition (2.1.3).

This completes the proof.

**REFERENCES**


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