International Journal of Mathematical Archive-8(10), 2017, 94-102
IJMA Available online through www.ijma.info ISSN 2229 – 5046

PSEUDO - COMPLEMENTED ALMOST SEMILATTICES

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(Received On: 18-08-17; Revised & Accepted On: 27-09-17)

ABSTRACT

The concept of pseudo-complementation $*$ on an almost semilattice(ASL) with 0 is introduced and proved some elementery properties of the pseudo-complementation $*$. Also, proved that pseudo-complementation $*$ on an ASL is equationally definable. A one-to-one correspondence between the pseudo-complementations on an ASL L with 0 and maximal elements of L is obtained. It is also proved that $L^{**} = \{a^{**} : a \in L\}$ is a Boolean algebra which is independent(up to isomorphism) of the pseudo-complementation $*$ on L.

Key Words: *Almost Semilattice, Pseudo-complementation, Unimaximal element, Maximal element, Equationally definable class, Boolean algebra.*

AMS Subject classification (1991): *06D99, 06D15..*

1. INTRODUCTION

It is well known that for any pseudo-complementation $*$ on a semilattice L, $L^{**} = \{a^{**} : a \in L\}$ becomes a Boolean algebra. In [1], Frink, O. proved that any pseudo- complemetation on a semilattice is equationally definable. In [4], Swamy, U.M., Rao, G.C. and Nanaji Rao, G. introduced the concept of pseudo-complemetation $*$ on an Almost Distributive Lattice(ADL) and proved that this pseudo - complemetation is equationally definable. Also, proved that a one-to-one correspondece between the pseudo-complementations on an ADL L with 0 and maximal elements of L. They proved that if L is an ADL with 0 and $*$ is a pseudo-complementation on L then $L^* = \{a^* : a \in L\}$ is a Boolean algebra which is independent(upto isomorphism) of the pseudo-complementation $*$ on L. In this paper, we introduce the concept of pseudo-complementation $*$ on an ASL with 0 and prove some basic properties of this pseudo-complementation. We prove that the pseudo-complementation on an ASL is equationally definable. It is observed that an ASL with 0 can have more than one pseudo-complemetation. In fact, if there is a pseudo-complementation $*$ on an ASL with 0 and $*$ elements commutes then we prove that each maximal element of L gives rise to a pseudo-complementation and that this correspondence is one-to-one. For any pseudo -complementation $*$ on an ASL with 0 and $*$ elements commutes, we prove that the set $L^{**} = \{a^{**} : a \in L\}$ is a Boolean algebra, which is independent(upto isomorphism) of the pseudo-complementation $*$.

2. PRELIMINARIES

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1 [2]: *Let $(P, \leq)$ be a poset. If P has least element 0 and greatest element 1, then P is said to be a bounded poset.*

If $(P, \leq)$ is a bounded poset with bounds $0,1$, then for any $x \in P$, we have $0 \leq x \leq 1$.

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Definition 2.2 [2]: *Let* $(P, \leq)$ *be a poset. Then* P *is said to be lattice ordered set if for any* $x, y \in P$ *,* $l.u.b\{x, y\}$ *and* $g.l.b\{x, y\}$ *exists in* P *.*

Definition 2.3 [2]: *Let* L *be a non-empty set and* $\vee, \wedge$ *be two binary operations on* L *. Then the triplet* $(L, \vee, \wedge)$ *is called lattice if it satisfies the following conditions:*

(1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)

(2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)

(3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$, for all $x, y \in L$. (Absorption Laws)

Lemma 2.4 [2]: *Let* $(L, \vee, \wedge)$ *be a lattice. Then for any* $x \in L$ *,* $x \wedge x = x$ *and* $x \vee x = x$ *.*

Theorem 2.5 [2]: $(L, \leq)$ *be a lattice ordered set. For any* $x, y \in L$ *, if we define* $x \wedge y$ *is the* $g.l.b\{x, y\}$ *and* $x \vee y$ *is the* $l.u.b\{x, y\}$ *, then* $(L, \vee, \wedge)$ *is a lattice.*

Theorem 2.6 [2]: *Let* $(L, \vee, \wedge)$ *be a lattice. If we define a relation* $\leq$ *on* L *, by* $x \leq y$ *if and only if* $x = x \wedge y$ *, (or equivalently* $x \vee y = y$ *), then* $(L, \leq)$ *is a lattice ordered set.*

Note that, by theorems 2.5 and 2.6 together imply that the concepts of lattice and lattice ordered set are same. We refer to it as a lattice in future.

Theorem 2.7 [2]: *In any lattice* $(L, \vee, \wedge)$ *, the following are equivalent:*

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(4) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.8 [2]: *A lattice* $(L, \vee, \wedge)$ *is called a distributive lattice if it satisfies any one of the four conditions, in theorem 2.7*

Theorem 2.9 [2]: *Let* $(L, \vee, \wedge)$ *be a lattice. Then for any* $x, y, z \in L$ *, the following conditions are equivalent:*

(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(3) $(x \vee y) \wedge z \leq x \vee (y \wedge z)$.

Definition 2.10 [2]: *Let* $(L, \vee, \wedge)$ *be a lattice. Then* L *is said to be bounded lattice if* L *is bounded as a poset.*

It can be easily seen that if $(L, \vee, \wedge)$ is a bounded lattice with bounds $0, 1$, then for any $x \in L$, $0 \wedge x = x \wedge 0 = 0$, $0 \vee x = x \vee 0 = x$, $x \wedge 1 = 1 \wedge x = x$ and $x \vee 1 = 1 \vee x = 1$.

Definition 2.11 [2]: *A bounded lattice* $(L, \vee, \wedge)$ *with bounds* 0 *and* 1 *is said to be complemented if to each* $x \in L$ *, there exists* $y \in L$ *such that* $x \wedge y = 0$ *and* $x \vee y = 1$ *.*

Definition 2.12 [2]: *A complemented distributive lattice is called a Boolean algebra.*

Definition 2.13 [2]: *A ring* R *is called a regular ring if, to each* $a \in R$ *, there exists* $x \in R$ *such that* $axa = a$ *.*

Definition 2.14 [1]: *A semilattice is an algebra* $(S, *)$ *where* S *is non-empty set and* $*$ *is a binary operation on* S *, satisfies the following conditions:*

1. $x * (y * z) = (x * y) * z$ (Associative Law)
2. $x * y = y * x$ (Commutative Law)
3. $x * x = x$, for all $x, y, z \in S$. (Idempotent)

Definition 2.15 [1]: *Let* S *be a meet semilattice with* 0 *in which each element* a *has a pseudo-complement* a^* *such that* $a \wedge x = 0$ *if and only if* $x \leq a^*$.

Definition 2.16 [3]: *An almost semilattice(ASL) is an algebra* $(L, \circ)$ *where* L *is a non-empty set and* $\circ$ *is a binary operation on* L*, satisfies the following conditions:*

1. $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
2. $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
3. $x \circ x = x$, for all $x, y, z \in L$. (Idempotent)

Definition 2.17 [3]: *An ASL with* 0 *is an algebra* $(L, \circ, 0)$ *of type* $(2,0)$ *satisfies the following conditions:*

1. $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
2. $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
3. $x \circ x = x$ (Idempotent)
4. $0 \circ x = 0$, for all $x, y, z \in L$.

Definition 2.18 [3]: *Let* L *be a non-empty set. Define a binary operation* $\circ$ *on* L *by* $x \circ y = y$*, for all* $x, y \in L$. *Then* $(L, \circ)$ *is an ASL and is called discrete ASL.*

Theorem 2.19 [3]: *Let* $(L, \circ)$ *be an ASL. Define a relation* $\leq$ *on* L *by* $a \leq b$ *if and only if* $a \circ b = a$. *Then* $\leq$ *is a partial ordering on* L.

Theorem 2.20 [3]: *Let* $(L, \circ)$ *be an ASL. Then for any* $a, b \in L$ *with* $a \leq b$ *we have* $a \circ c \leq b \circ c$ *and* $c \circ a \leq c \circ b$*, for all* $c \in L$.

Theorem 2.21 [3]: *Let* $(L, \circ)$ *be an ASL. Then for any* $a, b \in L$*, we have the following:*

1. $a \circ b \leq b$.
2. $a \circ b = b \circ a$ whenever $a \leq b$.

Theorem 2.22 [3]: *Let* $(L, \circ)$ *be an ASL with* 0. *Then for any* $a, b \in L$*, we have the following:*

1. $a \circ 0 = 0$.
2. $a \circ b = 0$ if and only if $b \circ a = 0$.
3. $a \circ b = b \circ a$ whenever $a \circ b = 0$.

Definition 2.23 [3]: *Let* $(L, \circ)$ *be an ASL. Then an element* $m \in L$ *is said to be unimaximal if* $m \circ x = x$*, for all* $x \in L$.

Definition 2.24 [2]: *Let* B_1 *and* B_2 *be two Boolean algebras. A mapping* $f : B_1 \to B_2$ *is said to be Boolean homomorphism if it is a lattice homomorphism and preserves complementation. That is, for any* $a, b \in B_1$. $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ *and* $f(a') = (f(a))'$.

It can be observed that if f is a lattice homomorphism from B_1 to B_2 such that $f(0) = 0$ and $f(1) = 1$, then f becomes a Boolean homomorphism. A Boolean isomorphism is a Boolean homomorphism which is a bijection.

3. DEFINITION AND INDEPENDENCY OF THE AXIOMS

In this section, we introduce the concept of the pseudo-complementation on an almost semilattice and we establish the independency of the conditions in the definition. Further, we give few examples of pseudo-complemented almost semilattice.

Definition 3.1: *Let* $(L, \circ, 0)$ *be an almost semilattice with zero. Then a unary operation* $a \mapsto a^*$ *on* L *is said to be pseudo-complementation on* L *if, for any* $a, b \in L$*, it satisfies the following conditions:*

1. $a \circ b = 0 \Rightarrow a^* \circ b = b$
2. $a \circ a^* = 0.$

For brevity, in future, we will refer an Almost Semilattice as ASL and to this Pseudo - Complemented Almost Semilattice as PCASL. Now, we give examples to exhibit independency of the conditions in the above definition.

Example 3.2: *Let $(L,\circ)$ be an ASL with zero with atleast two elements and define a unary operation $*$ on L by $a^* = 0$, for all $a \in L$.*

Here the algebra $(L,\circ)$ satisfies (2) but, it fails to satisfies (1). Because, for any $b \neq 0$, we have $0 \circ b = 0$. But, $0^* \circ b = 0 \circ b = 0 \neq b$.

Example 3.3: *Let L be a meet semilattice with least element 0 and greatest element 1. Now, define a unary operation $*$ on L by $a^* = 1$, for all $a \in L$.*

Here the algebra $(L,\circ)$ satisfies (1) but, it fails to satisfies (2). Because for any $a \neq 0 \in L$, $a \wedge a^* = a \wedge 1 = a \neq 0$

Now, we give some examples of PCASL.

Example 3.4: *Every pseudo - complemented semilattice is a pseudo-complemented almost semilattice.*

In the case of semilattices, if pseudo-complementation exists then it is unique. But, in the case of ASL, there are several pseudo-complementation. For, consider the following examples.

Example 3.5: *Let $(L,\circ)$ be a discrete ASL with zero and fix $x_0 \in L$. Now, define a unary operation $*$ on L by*

$$a^* = \begin{cases} 0 & \text{if a} \neq 0 \\ x_0 & \text{if a} = 0. \end{cases}$$

Then $*$ is a pseudo-complementation on L, and to each $x_0 \in L$, we get a pseudo - complementation on L.

Example 3.6: *Let L = { a, b, c, 0 }. Now, define binary operation $\circ$ on L as follows:*

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	c

Then cleary, $(L,\circ)$ is an ASL. Now, define $0^* = b$, $x^* = 0$ for all $x \neq 0$. Then clearly $*$ is a pseudo-coplementation on L, and hence L is a PCASL.

Note that, we define $0^* = c$ and $x^* = 0$ for all $x \neq 0$, then it can be eaily seen that L is a PCASL.

Example 3.7: *Let L = { a, b, c, 0 }. Now, define binary operation $\circ$ on L as follows:*

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then cleary, $(L,\circ)$ is an ASL. Now, define $0^* = a$, $x^* = 0$ for all $x \neq 0$. Then clearly $*$ is a pseudo-coplementation on L and hence L is a PCASL.

Note that, we define $0^* = b$ and $x^* = 0$ for all $x \neq 0$, then it can be eaily seen that L is a PCASL .

Example 3.8: *Let* $(R,+,.)$ *be a commutative regular ring with unity* 1. *Let* a^0 *be the unique idempotent element in* R *, such that* $aR = a^0R$ *. Now, for any* $a,b \in R$ *, define operations on* R *as follows:* $a \circ b = a^0 b$ *and* $a^* = 1 - a^0$. *Then clearly* $(R,\circ)$ *is an ASL and* $*$ *is a pseudo - complementation on* R.

Example 3.9: *Let* A *be a non-empty set with atleast two elements, and let* B *any set and* $p_0 \in A^B$. *Now, for any* $a,b \in A^B$, *define*

$$(a \circ b)(t) = \begin{cases} b(t) & \text{if } a(t) \neq p_0(t) \\ p_0(t) & \text{if } a(t) = p_0(t). \end{cases}$$

Then $(A^B, \circ, p_0)$ is an ASL with p_0 as zero element. Now, let $p \in A^B$ such that $p(t) \neq p_0(t)$ for all $t \in B$. For any $a \in A^B$, define

$$a^p(t) = \begin{cases} p_0(t) & \text{if } a(t) \neq p_0(t) \\ p(t) & \text{if } a(t) = p_0(t). \end{cases}$$

Then $a \mapsto a^p$ is a pseudo-complementation on A^B and conversely, if $a \mapsto a^*$ is a pseudo-complementation on A^B, then there exists $p \in A^B$ such that $p(t) \neq p_0(t)$ for all $t \in B$ and $a^* = a^P$ for all $a \in A^B$ (take $p = p_0^*$).

In the following we prove some basic properties of PCASL.

Lemma 3.10: *Let* L *be a PCASL. Then for any* $a,b \in L$, *we have the following:*

1. $0^* \circ a = a$
2. 0^* is unimaximal
3. 0^* is maximal
4. $a^{**} \circ a = a$
5. $a \circ a^{***} = 0$
6. $a^* \circ a^{***} = a^{***}$
7. $a^{****} \circ a = a$
8. $a \leq b \Rightarrow a^* \circ b^* = b^*$
9. a is unimaximal $\Rightarrow$ $a^* = 0$
10. $0^{**} = 0$
11. a^{**} is unimaximal $\Leftrightarrow a^* = 0$
12. $a = 0 \Leftrightarrow a^{**} = 0$
13. $(a \circ b)^* \circ a^* = a^*$
14. $(a \circ b)^* \circ b^* = b^*$

Proof:

1. Since $0 \circ a = 0$ for all $a \in L$, we have $0^* \circ a = a,$ for all $a \in L$.
2. Proof follows by condition (1).
3. Let $x \in L$ such that $0^* \leq x$. Then $0^* = 0^* \circ x = x$ since 0^* is unimaximal. Thus 0^* is maximal.
4. Since $a^* \circ a = 0$, we have $a^{**} \circ a = a$.
5. By (4), we have $a^{**} \circ a = a$. Now, consider $a \circ a^{***} = (a^{**} \circ a) \circ a^{***} = (a \circ a^{**}) \circ a^{***}$ $= a \circ (a^{**} \circ a^{***}) = a \circ 0 = 0.$
6. By (5), $a \circ a^{***} = 0$, it follows that $a^* \circ a^{***} = a^{***}$.
7. By (5), $a \circ a^{***} = 0$. Hence $a^{***} \circ a = 0$. It follows that $a^{****} \circ a = a$.
8. Supose $a \leq b$. Then $a \circ b^* \leq b \circ b^*$. Hence $a \circ b^* = 0$. It follows that $a^* \circ b^* = b^*$.
9. Suppose a is unimaximal. Then $a \circ t = t$ for all $t \in L$. Now, $0 = a \circ a^* = a^*$. Thus $a^* = 0.$
10. We have $0 = 0^* \circ 0^{**} = 0^{**}$ since 0^* is unimaximal. Thus $0^{**} = 0$.

11. Suppose a^{**} is unimaximal. Then $a^{***} = 0$ since by (9). Now, consider $a^{*} = a^{***} \circ a^{*} = 0 \circ a^{*} = 0$. Therefore $a^{*} = 0$. Conversely, suppose $a^{*} = 0$. Then $a^{**} = 0^{*}$ which is unimaximal.
12. Suppose $a = 0$. Then $a^{**} = 0^{**} = 0$. Conversely, suppose $a^{**} = 0$. Consider $a = a^{**} \circ a = 0 \circ a = 0$. Thus $a = 0$.
13. We have $(a \circ b) \circ a^{*} = 0$. Therefore $(a \circ b)^{*} \circ a^{*} = a^{*}$. Similarly, we can prove (14).

Next, we prove some equivalent conditions in PCASL.

Theorem 3.11: *Let L be a PCASL.Then for any $a,b \in L$, the following are equivalent:*

1. $a \circ b = 0$
2. $a^{**} \circ b = 0$
3. $a \circ b^{**} = 0$
4. $a^{**} \circ b^{**} = 0$

Proof:
$(1) \Rightarrow (2)$: Suppose $a \circ b = 0$. Then $a^{*} \circ b = b$. Now consider $a^{**} \circ b = a^{**} \circ (a^{*} \circ b) = (a^{**} \circ a^{*}) \circ b = 0 \circ b = 0.$
$(2) \Rightarrow (1)$: Suppose $a^{**} \circ b = 0$. Now, consider $a \circ b = (a^{**} \circ a) \circ b = (a \circ a^{**}) \circ b = a \circ (a^{**} \circ b) = a \circ 0 = 0.$
Therefore $a \circ b = 0$. $(1) \Rightarrow (3)$: Suppose $a \circ b = 0$.
Then $b \circ a = 0$. Therefore $b^{*} \circ a = a$. Now, consider $a \circ b^{**} = (b^{*} \circ a) \circ b^{**} = (a \circ b^{*}) \circ b^{**} = a \circ (b^{*} \circ b^{**}) = a \circ 0 = 0.$ Thus $a \circ b^{**} = 0.$
$(3) \Rightarrow (4)$: Suppose $a \circ b^{**} = 0.$ Then $a^{*} \circ b^{**} = b^{**}$. Now , consider

$$a^{**} \circ b^{**} = a^{**} \circ (a^{*} \circ b^{**}) = (a^{**} \circ a^{*}) \circ b^{**} = 0 \circ b^{**} = 0.$$ Thus $a^{**} \circ b^{**} = 0.$

$(4) \Rightarrow (1)$: Suppose $a^{**} \circ b^{**} = 0.$ Now, consider

$$a \circ b = (a^{**} \circ a) \circ (b^{**} \circ b) = a^{**} \circ (a \circ (b^{**} \circ b)) = a^{**} \circ ((a \circ b^{**}) \circ b) = a^{**} \circ ((b^{**} \circ a) \circ b)$$
$$= a^{**} \circ (b^{**} \circ (a \circ b)) = (a^{**} \circ b^{**}) \circ (a \circ b) = 0 \circ (a \circ b) = 0.$$ Thus $a \circ b = 0$.

Corollary 3.12: *Let L be a PCASL. Then for any $a,b \in L$, we have the following:* $(a \circ b)^{**} \circ a^{**} \circ b^{**} = a^{**} \circ b^{**}$.

Proof: We have $a \circ b \circ (a \circ b)^{*} = 0$. Therefore by theorem 3.11, we get $a^{**} \circ b \circ (a \circ b)^{*} = 0$. This implies $b \circ a^{**} \circ (a \circ b)^{*} = 0.$ Again, by theorem 3.11, we get $b^{**} \circ a^{**} \circ (a \circ b)^{*} = 0$. It follows that $(a \circ b)^{*} \circ a^{**} \circ b^{**} = 0.$ Therefore $(a \circ b)^{**} \circ a^{**} \circ b^{**} = a^{**} \circ b^{**}$.

In the following, we prove that pseudo-complementation $*$ on an ASL L is equationally definable.

Theorem 3.13: *Let L be an ASL with 0. Then a unary operation $* : L \to L$ is a pseudo - complementation on L if and only if it satisfies the following conditions:*

(1) $a^{*} \circ b = (a \circ b)^{*} \circ b$

(2) $0^{*} \circ a = a$

(3) $0^{**} = 0$

Proof: Suppose $*$ is a pseudo-complementation on L. Then we have $a \circ b \circ (a \circ b)^{*} = 0$.
Therefore $a^{*} \circ b \circ (a \circ b)^{*} = b \circ (a \circ b)^{*}$. This implies $a^{*} \circ b \circ (a \circ b)^{*} \circ b = b \circ (a \circ b)^{*} \circ b$. Hence $a^{*} \circ (a \circ b)^{*} \circ b = (a \circ b)^{*} \circ b$. Therefore $(a \circ b)^{*} \circ a^{*} \circ b = (a \circ b)^{*} \circ b$. Hence $a^{*} \circ b = (a \circ b)^{*} \circ b$ since $(a \circ b) \circ (a^{*} \circ b) = 0$. Proofs of conditions (2) and (3) follows by lemma 3.10. Conversely, suppose $*$ satisfies the given conditions. Let $a,b \in L$ such that $a \circ b = 0$. Now, from (1) we get $a^{*} \circ b = (a \circ b)^{*} \circ b = 0^{*} \circ b = b.$ Therefore $a^{*} \circ b = b$. Again, consider $a^{*} \circ a = (0^{*} \circ a)^{*} \circ a = 0^{**} \circ a = 0 \circ a = 0.$ It follows that $a \circ a^{*} = 0$. Thus $*$ is a pseudo-complementation on L.

Remark: Whether $*$ elements commutes are not, is not known so far in pseudo-complementated ASL with pseudo-complementation $*$. Investigations are still going on.

Definition 3.14: *Let* $(L,\circ,0)$ *be a pseudo-complemented almost semilattice, with pseudo - complementation* $*$. *Then* L *is said to be* $*$ *- commutative if* $a^* \circ b^* = b^* \circ a^*$, *for all* $a,b \in L$.

Next, we prove that, for any $*$ - commutative PCASL L the set $L^{**} = \{a^{**} : a \in L\}$ becomes a Boolean algebra. It is remarked that an ASL with 0 can have more than one pseudo - complementation and examples were given to this effect. In fact, we prove that if L is an ASL with a pseudo-complementation $*$, then to each maximal element m in L, we obtain a pseudo-coplementation $*_m$ and this correspondence between maximal elements of L and pseudo-complementation on L is one-to-one. Also prove that the Boolean algebra L^{**} is independent (upto isomorphism) of the pseudo-complementation $*$. For, this, first we need the following.

Theorem 3.15: *Let* L *be a* $*$ *- commutative PCASL. Then for any* $a,b \in L$, *we have the following:*

1. $a \le b \Rightarrow b^* \le a^*$
2. $a^* \le 0^*$
3. $a^{***} = a^*$
4. $a^* \le b^* \Leftrightarrow b^{**} \le a^{**}$
5. $a^* \le (b \circ a)^*$ and $b^* \le (a \circ b)^*$

Proof:

1. Suppose $a \le b$. Then $a \circ b^* \le b \circ b^*$. Therefore $a \circ b^* = 0$. It follows that $a^* \circ b^* = b^*$. Hence $b^* \circ a^* = b^*$. We get $b^* \le a^*$.
2. Since $0 \circ a^* = 0$. It follows that $0^* \circ a^* = a^*$. Hence $a^* \circ 0^* = a^*$. Therefore $a^* \le 0^*$.
3. We have $a^{**} \circ a^* = 0$ and hence $a^{***} \circ a^* = a^*$. On the other hand, we have $a \circ a^{***} = 0$ since by lemma 3.10(5). Therefore $a^* \circ a^{***} = a^{***}$. Hence by $*$-commutative we get $a^{***} = a^*$.
4. Suppose $a^* \le b^*$. Then $b^{**} \le a^{**}$ since by (1). Conversely, suppose $b^{**} \le a^{**}$. Then again by (1), we get $a^{***} \le b^{***}$. This implies $a^* \le b^*$ since by (3).
5. We have $a \circ b \le b$. Hence by (1), $b^* \le (a \circ b)^*$. Also, we have $b \circ a \le a$. Therefore by (1), $a^* \le (b \circ a)^*$.

Theorem 3.16: *Let* L *be a* $*$ *- commutative PCASL. Then for any* $a,b \in L$, *we have the following:*

1. $(a \circ b)^{**} = a^{**} \circ b^{**}$
2. $(a \circ b)^* = (b \circ a)^*$
3. $a^*, b^* \le (a \circ b)^*$.

Proof:

1. Let $a,b \in L$. Then we have $(a \circ b)^* \circ a \circ b = 0$. This implies $b \circ (a \circ b)^* \circ a = 0$. Therefore $b^* \circ (a \circ b)^* \circ a = (a \circ b)^* \circ a$. Now, consider $(a \circ b)^* \circ a \circ b^{**} = b^* \circ (a \circ b)^* \circ a \circ b^{**} =$ $(a \circ b)^* \circ a \circ b^* \circ b^{**} = (a \circ b)^* \circ a \circ 0 = 0.$ Therefore $a \circ (a \circ b)^* \circ b^{**} = 0.$ Hence $a^* \circ (a \circ b)^* \circ b^{**}$ $= (a \circ b)^* \circ b^{**}.$ Now, $(a \circ b)^* \circ b^{**} \circ a^{**} = a^* \circ (a \circ b)^* \circ b^{**} \circ a^{**} = (a \circ b)^* \circ a^* \circ b^{**} \circ a^{**} = (a \circ b)^*$ $\circ b^{**} \circ a^* \circ a^{**} = (a \circ b)^* \circ b^{**} \circ 0 = 0.$ Therefore $(a \circ b)^* \circ b^{**} \circ a^{**} = 0$ and hence $(a \circ b)^* \circ a^{**} \circ b^{**} = 0.$ It follows that $(a \circ b)^{**} \circ a^{**} \circ b^{**} = a^{**} \circ b^{**}$. On the other hand, we have $(a \circ b)^* \circ a^* = a^*$.Therefore $(a \circ b)^{**} \circ (a \circ b)^* \circ a^* = (a \circ b)^{**} \circ a^*$. Hence $(a \circ b)^{**} \circ a^* = 0$. This implies $a^* \circ (a \circ b)^{**} = 0$. Hence $a^{**} \circ (a \circ b)^{**} = (a \circ b)^{**}$. Similarly, we can prove that $b^{**} \circ (a \circ b)^{**} = (a \circ b)^{**}$. Hence we get $a^{**} \circ b^{**} \circ (a \circ b)^{**} = (a \circ b)^{**}$. Therefore $(a \circ b)^{**} \circ a^{**} \circ b^{**} = (a \circ b)^{**}.$ It follows by $*$-

commutativity, $(a\circ b)^{**}=a^{**}\circ b^{**}$.

2. Consider, $(a\circ b)^{*}=(a\circ b)^{***}=((a\circ b)^{**})^{*}=(a^{**}\circ b^{**})^{*}=(b^{**}\circ a^{**})^{*}=((b\circ a)^{**})^{*}=(b\circ a)^{***}$
$=(b\circ a)^{*}$. Therefore $(a\circ b)^{*}=(b\circ a)^{*}$.

3. Proof of (3) follows by condition (5) in theorem 3.15 and condition (2) in theorem 3.16.

In a $*$- commutative PCASL L, it can be easily observed that, if $x=a^{*}$ then $x^{**}=x$ and $a^{*}\circ b^{*}=(a^{*}\circ b^{*})^{**}$. Also, it can be easily seen that if x,y are $*$ - elements in L then $x\circ y=0$ if and only if $x\le y^{*}$ if and only if $y\le x^{*}$. Now, we prove that if L is $*$ - commutative PCASL then the set $L^{**}=\{a^{**}:a\in L\}$ is a Boolean algebra.

Theorem 3.17: *Let $(L,\circ)$ be a $*$- commutative PCASL. Then the set L^{**} is a Boolean algebra with the original determination of the meet operation $a\circ b$ and of the order relation $a\le b$, the Boolean complement of an element being its pseudo-complement for these element, the Boolean join operation is given by the formula $a\underline{\vee} b=(a^{*}\circ b^{*})^{*}$.*

Proof: Suppose L is a $*$- commutative PCASL. Then clearly $L^{**}=\{a^{**}:a\in L\}$ is a poset with respect to $\le$ defined as in L. Suppose $a^{**},b^{**}\in L^{**}$. Then $a^{**}\circ b^{**}=(a\circ b)^{**}\in L^{**}$ and clearly $(a\circ b)^{**}$ is the greatest lower bound of $\{a^{**},b^{**}\}$. Now, $a^{**}\underline{\vee} b^{**}=(a^{***}\circ b^{***})^{*}=(a^{*}\circ b^{*})^{*}$. Since $a^{*}\circ b^{*}\le a^{*},b^{*}$ it follows that $a^{**},b^{**}\le(a^{*}\circ b^{*})^{*}$. Therefore $(a^{*}\circ b^{*})^{*}$ is an upper bound of $\{a^{**},b^{**}\}$. Let $t\in L^{**}$ such that t is an upper bound of $\{a^{**},b^{**}\}$. Then $a^{**}\le t$ and $b^{**}\le t$. Since $t\in L^{**}$, $t=c^{**}$ for some $c\in L$. Therefore $a^{**}\le c^{**}$ and $b^{**}\le c^{**}$. It follows that $c^{*}\le a^{*}$ and $c^{*}\le b^{*}$. Hence $c^{*}\le a^{*}\circ b^{*}$. Thus $(a^{*}\circ b^{*})^{*}\le c^{**}=t$. Therefore $(a^{*}\circ b^{*})^{*}$ is the least upper bound of $\{a^{**},b^{**}\}$. Hence L^{**} is a lattice. Now, we have $0=0^{**}$ and hence $0\in L^{**}$. Clearly 0 and 0^{*} are the least and greatest elements in L^{**} respectively. Also, for any $a\in L^{**}$ we have $a^{*}\in L^{**}$ since $a^{*}=a^{***}$ and $a\circ a^{*}=0$. Now, consider, $a\underline{\vee} a^{*}=(a^{*}\circ a^{**})^{*}=0^{*}$. Thus a^{*} is a complement of a in L^{**}. Finally, for $a,b,c\in L^{**}$, we have $b\circ c\circ(a^{*}\circ(b\circ c)^{*})=0$. It follows that $c\circ(a^{*}\circ(b\circ c)^{*})\le b^{*}$. Again, we have $a\circ c\circ(a^{*}\circ(b\circ c)^{*})=0$. Therefore $c\circ(a^{*}\circ(b\circ c)^{*})\le a^{*}$. It follows that $c\circ(a^{*}\circ(b\circ c)^{*})\le a^{*}\circ b^{*}$. Hence $(c\circ(a^{*}\circ(b\circ c)^{*}))\circ(a^{*}\circ b^{*})^{*}=0$. This implies $((a^{*}\circ(b\circ c)^{*})\circ c)\circ(a^{*}\circ b^{*})^{*}=0$ and hence $(a^{*}\circ(b\circ c)^{*})\circ(c\circ(a^{*}\circ b^{*})^{*})=0$. Therefore $c\circ(a^{*}\circ b^{*})^{*}\le(a^{*}\circ(b\circ c)^{*})^{*}$ and hence $(a^{*}\circ b^{*})^{*}\circ c\le(a^{*}\circ(b\circ c)^{*})^{*}$. It follows that $(a\underline{\vee} b)\circ c\le a\underline{\vee}(b\circ c)$. Therefore by theorem 2.9, $(L^{**},\underline{\vee},\circ,0,0^{*})$ is a distributive lattice and hence is a Boolean algebra.

Finally, we prove that if L is an ASL with a pseudo-complementation $*$, then to each maximal element m in L, we obtain a pseudo-copmlementation $*_m$ and this correspondence between maximal elements of L and pseudo-complementation on L is one-to-one. Also, prove that if an ASL L with two pseudo-complements say $*$ and $\perp$ then the corresponding Boolean algebras L^{**} and $L^{\perp\perp}$ are isomorphic. For this first we need the following.

Lemma 3.18: *Let L be a PCASL and let $*$ and $\perp$ be two pseudo-complementations on L. Then for any $a,b\in L$, we have the following:*

1. $a^{*}\circ a^{\perp}=a^{\perp}$
2. $a^{*\perp}=a^{\perp\perp}$
3. $a^{*}=b^{*}\Leftrightarrow a^{\perp}=b^{\perp}$
4. $a^{*}=0\Leftrightarrow a^{\perp}=0\Leftrightarrow(a\circ b=0\Rightarrow b=0)$
5. $a^{*}\circ 0^{\perp}=a^{\perp}$

Proof:

1. Since $a\circ a^{\perp}=0$. It follows that $a^{*}\circ a^{\perp}=a^{\perp}$.
2. Consider $a^{*\perp}=(0^{*}\circ a^{*})^{\perp}=(a^{*}\circ 0^{*})^{\perp}=(a^{\perp}\circ a^{*}\circ 0^{*})^{\perp}=(a^{*}\circ a^{\perp}\circ 0^{*})^{\perp}=(a^{\perp}\circ 0^{*})^{\perp}=$

$(0^* \circ a^\perp)^\perp$ $(since\ by\ theorem\ 3.16,\ condition(2)) = (a^\perp)^\perp = a^{\perp\perp}$. Therefore $a^{*\perp} = a^{\perp\perp}$.

3. Suppose $a^* = b^*$. Now, consider $a^\perp = a^{\perp\perp\perp} = a^{*\perp\perp} = b^{*\perp\perp} = b^{\perp\perp\perp} = b^\perp$. Therefore $a^\perp = b^\perp$. Similarly, we can prove that if $a^\perp = b^\perp$ then $a^* = b^*$.
4. Suppose $a^* = 0$. Then we have $a^\perp = a^* \circ a^\perp = 0 \circ a^\perp = 0$. Therefore $a^\perp = 0$. Now, suppose $a^\perp = 0$ and suppose $a \circ b = 0$. Then we have $a^\perp \circ b = b$. It follows that $b = 0$. Suppose $a \circ b = 0$ implies that $b = 0$. Now, we have $a \circ a^* = 0$. Therefore $a^* = 0$.
5. Consider, $a^* \circ 0^\perp = a^\perp \circ a^* \circ 0^\perp = a^* \circ a^\perp \circ 0^\perp = a^* \circ 0^\perp \circ a^\perp = a^* \circ a^\perp = a^\perp$. Therefore $a^* \circ 0^\perp = a^\perp$.

Now, we prove the following theorem.

Theorem 3.19: *Let L be an ASL and $*$ be a psuedo-complementation on L. Let M be the set of all maximal elements in L and let $PC(L)$ be the set of all pseudo-complementations on L. For any $m \in M$, define $*_m : L \to L$ by $a^{*_m} = a^* \circ m$, for all $a \in L$. Then $m \mapsto *_m$ is a bijection of M onto $PC(L)$.*

Proof: Let $m, n \in M$ such that $*_m = *_n$. Then $0^{*_m} = 0^{*_n}$. Therefore $0^* \circ m = 0^* \circ n$. Hence $m = n$. Let $\perp \in PC(L)$. If $m = 0^\perp$, then consider $a^{*_m} = a^* \circ m = a^* \circ 0^\perp = a^\perp$. Therefore $a^{*_m} = a^\perp$. Hence $*_m$ is the same as $\perp$ and m is maximal. Thus $m \mapsto *_m$ is a bijection of M onto $PC(L)$.

In the following we prove that, if L is an ASL with the pseudo-complementation $*$ and $\perp$ then the Boolean algebra L^{**} and $L^{\perp\perp}$ are isomorphic.

Theorem 3.20: *Let L be an ASL and $*, \perp$ be two pseudo-complementations on L. Then the map $f : L^{**} \to L^{\perp\perp}$ defined by $f(a^{**}) = a^{\perp\perp}$ is an isomorphism of Boolean algebras.*

Proof: Suppose $a^{**}, b^{**} \in L^{**}$ such that $f(a^{**}) = f(b^{**})$. Then $a^{\perp\perp} = b^{\perp\perp}$. It follows by lemma 3.18 condition(3), we get $a^{**} = b^{**}$. Therefore f is one-one. Suppose $a^{\perp\perp} \in L^{\perp\perp}$.

Then we have $a^{**} \in L^{**}$ and $f(a^{**}) = a^{\perp\perp}$. Hence f is onto. Let $a^{**}, b^{**} \in L^{**}$. Now, consider $f(a^{**} \circ b^{**}) = f((a \circ b)^{**}) = (a \circ b)^{\perp\perp} = a^{\perp\perp} \circ b^{\perp\perp} = f(a^{**}) \circ f(b^{**}.)$ Again, consider $f(a^{**} \veebar b^{**}) = f(a^{***} \circ b^{***})^* = f((a^* \circ b^*)^*) = f[((a^* \circ b^*)^*)^{**}] = [(a^* \circ b^*)^*]^{\perp\perp} = (a^* \circ b^*)^{*\perp\perp} = (a^* \circ b^*)^{\perp\perp\perp} = (a^{*\perp\perp} \circ b^{*\perp\perp})^\perp = (a^{\perp\perp\perp} \circ b^{\perp\perp\perp})^\perp = a^{\perp\perp} \veebar b^{\perp\perp} = f(a^{\perp\perp}) \veebar f(b^{\perp\perp})$. Hence f is a homomorphism. Now, consider $f(0) = f(0^{**}) = 0^{\perp\perp} = 0$ and $f(0^*) = 0^\perp$. Thus f is a Boolean isomorphism.

REFERENCES

1. Frink, O *Pseudo-complements in semi-lattice*, Duke Math J. 29,505-515(1962).
2. Gratzer, G.: *Lattice Theory*: First Concepts and Distributive Lattices, W.H. Freeman and Company, San Francisco, 1971.
3. Nanaji Rao, G.,Terefe, G.B.: *Almost Semilattice*, International Journal of Mathematical Archive - 7(3), 2016, 52-67.
4. Swamy, U.M., Rao, G.C., Nanaji Rao, G.: *Pseudo - Complementation on Almost Distributive Lattice*, Southeast Asian Bulletin of mathematics- (2000) 24: 95-104.

Source of support: Nil, Conflict of interest: None Declared.