ON UNIQUE COMMON FIXED POINT THEOREMS FOR THREE AND FOUR SELF MAPPINGS IN SYMMETRIC G-METRIC SPACE

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ABSTRACT

In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric G – metric spaces.

Key words: Symmetric G-metric space, owc maps, common fixed point theorem.

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1. INTRODUCTION:

In 1992, Dhage[1] introduced the concept of D – metric space. Recently, Mustafa and Sims [5] shown that most of the results concerning Dhage’s D – metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure and called it as G – metric space. For more details on G – metric spaces, one can refer to the papers [5]-[9]. In this paper, we prove two unique common fixed point theorems for three and four self mappings in symmetric G – metric spaces.

Now we give basic definitions and some basic results ([5]-[9]) which are helpful for proving our main result.

In 2006, Mustafa and Sims[6] introduced the concept of G-metric spaces as follows:

Definition: 1.1[6] Let X be a nonempty set, and let G: X × X × X → R+ be a function satisfying the following axioms:
(G1) G(x, y, z) = 0 if x = y = z,
(G2) 0 < G(x, x, y), for all x, y ∈ X with x ≠ y,
(G3) G(x, x, y) ≤ G(x, y, z), for all x, y, z ∈ X with z ≠ y,
(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = … (symmetry in all three variables) and
(G5) G(x, y, z) ≤ G(x, a, a) + G (a, y, z) for all x, y, z, a ∈ X, (rectangle inequality)

then the function G is called a generalized metric, or, more specifically a G – metric on X and the pair (X, G) is called a G – metric space.

Definition: 1.2[6] A G-metric space (X, G) is symmetric if
(G6) G(x, y, y) = G(x, x, y) for all x, y ∈ X.

Definition: 1.3[6] Let (X,G) be a G-metric space then for x₀ ∈ X, r > 0, the G-ball with centre x₀ and radius r is

B_G(x₀, r) = {y ∈ X : G(x₀, y, y) < r }.

Proposition: 1.1[6] Let (X,G) be a G-metric space then for any x₀ ∈ X, r > 0, we have,

(1) if G(x₀, y, y) < r then x , y ∈ B_G(x₀, r),
(2) if y ∈ B_G(x₀, r) then there exists a δ > 0 such that B_G(y, δ) ⊆ B_G(x₀, r).

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It follows from (2) of the above proposition that the family of all G-balls, $B = \{B_G(x, r) : x \in X, r > 0\}$ is the base of a topology $\tau(G)$ on $X$, the G-metric topology.

**Proposition: 1.2[6]** Let $(X, G)$ be a G-metric space then for all $x_0 \in X$ and $r > 0$, we have,

$$B_G(x_0, \frac{1}{3}r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r)$$

where $d_G(x, y) = G(x, y, y) + G(x, x, y)$, for all $x, y$ in $X$.

Consequently, the G-metric topology $\tau(G)$ coincides with the metric topology arising from $d_G$. Thus, while ‘isometrically’ distinct, every G-metric space is topologically equivalent to a metric space. This allows us to readily transport many results from metric spaces into G-metric spaces settings.

**Definition: 1.4[6]** Let $(X, G)$ be a G–metric space, and let $\{x_n\}$ a sequence of points in $X$, a point ‘$x$’ in $X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G–convergent to $x$.

Thus, that if $x_n \to x$ or $\lim_{n \to \infty} x_n = x$ in a G-metric space $(X, G)$ then for each $\varepsilon > 0$, there exists a positive integer $N$ such that $G(x, x_n, x_m) \leq \varepsilon$ for all $m, n \geq N$.

**Proposition: 1.3[6]** Let $(X, G)$ be a G – metric space. Then the following are equivalent:

1. $\{x_n\}$ is G-convergent to $x$,
2. $G(x_n, x, x) \to 0$ as $n \to \infty$,
3. $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
4. $G(x_n, x_n, x) \to 0$ as $m, n \to \infty$.

**Definition: 1.5[6]** Let $(X, G)$ be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\varepsilon > 0$, there exists a positive integer $N$ such that $G(x_n, x_m, x_l) \leq \varepsilon$ for all $n, m, l \geq N$; i.e. if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

**Proposition: 1.4[6]** If $(X, G)$ is a G – metric space then the following are equivalent:

1. The sequence $\{x_n\}$ is G – Cauchy,
2. for each $\varepsilon > 0$, there exist a positive integer $N$ such that $G(x_n, x_m, x_l) \leq \varepsilon$ for all $n, m, l \geq N$.

**Proposition: 1.5**[6] Let $(X, G)$ be a G – metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

**Definition: 1.6**[6] A G – metric space $(X, G)$ is said to be G–complete if every G-Cauchy sequence in $(X, G)$ is G-convergent in $X$.

**Proposition: 1.7**[6] A G – metric space $(X, G)$ is G – complete if and only if $(X, d_G)$ is a complete metric space.

**Proposition: 1.8**[6] Let $(X, G)$ be a G – metric space. Then, for any $x, y, z, a$ in $X$ it follows that:

(i) If $G(x, y, z) = 0$, then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, z) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{3}{4}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

**Definition: 1.7** Let $(X, G)$ be a G-metric space. $f$ and $g$ be self maps on $X$. A point $x$ in $X$ is called a coincidence point of $f$ and $g$ iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of $f$ and $g$.

**Definition: 1.8** A pair of self mappings $(f, g)$ of a G-metric space $(X, G)$ is said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

**Definition: 1.9** Two self mappings $f$ and $g$ of a G-metric space $(X, G)$ are said to be occasionally weakly compatible (owc) iff there is a point $x$ in $X$ which is coincidence point of $f$ and $g$ at which $f$ and $g$ commute.
2. MAIN RESULTS:

2.1 A unique common fixed point theorem for three mappings

**Theorem 2.1:** Let \((X, G)\) be symmetric G-metric space. Suppose \(f, g,\) and \(h\) are three self mappings of \((X, G)\) satisfying the conditions:

1. For all \(x, y \in X\)
   \[
   \int_0^1 G(fx, gy) \phi(t) dt \leq \int_0^1 \alpha G(hx, hy) + \beta G(fx, hx) + G(gy, hy) + \gamma G(hy, fy) \phi(t) dt
   \]
   where \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}\) is a Lebesgue-integrable mapping which is summable, nonnegative and such that \(\int_0^\epsilon \phi(t) dt > 0\) for each \(\epsilon > 0\), and \(\alpha, \beta, \gamma\) are non-negative reals such that \(\alpha + 2\beta + 2\gamma < 1\).

2. Pair of mappings \((f, h)\) or \((g, h)\) is owc.

Then \(f, g\) and \(h\) have a unique common fixed point.

**Proof:** Suppose that \(f\) and \(h\) are owc then there is an element \(u\) in \(X\) such that \(fu = hu\) and \(fhu = hfu\).

First, we prove that \(fu = gu\). Indeed, by inequality (1), we get

\[
\int_0^1 G(fu, gu) \phi(t) dt \leq \int_0^1 \alpha G(hu, hu) + \beta G(fu, hu) + G(gu, hu) + \gamma G(hu, fu) \phi(t) dt
\]

\[
= \int_0^1 \beta G(gu, fu) + \gamma G(fu, gu) \phi(t) dt
\]

\[
= \int_0^1 (\beta + \gamma) G(fu, gu) \phi(t) dt
\]

\[
< \int_0^1 G(fu, gu) \phi(t) dt
\]

which is a contradiction, hence, \(gu = fu = hu\).

Again, suppose that \(ffu \neq fu\). The use of condition (1), we have

\[
\int_0^1 G(ffu, gu) \phi(t) dt \leq \int_0^1 \alpha G(hfu, hu) + \beta G(ffu, hfu) + G(gu, hu) + \gamma G(hfu, gu) + G(hu, ffu) \phi(t) dt
\]

\[
= \int_0^1 \alpha G(ffu, gu) + 2\gamma G(ffu, gu) \phi(t) dt
\]

\[
= \int_0^1 (\alpha + 2\gamma) G(ffu, gu) \phi(t) dt
\]

\[
< \int_0^1 G(ffu, gu) \phi(t) dt
\]

this contradiction implies that \(ffu = fu = hfu\).

Now, suppose that \(gfu \neq fu\). By inequality (1), we have

\[ \int_{0}^{G(fu,gu,gfu)} \phi(t)dt \leq \int_{0}^{\alpha G(hu,hu,hu)+\beta[G(fu,hu,hu)+G(gfu,hfu,hfu)]+\gamma[G(hu,gfu,gu)+G(hfu,hu,hu)]} \phi(t)dt \\
= \int_{0}^{\beta G(gfu,hu,hu)+\gamma G(fu,gfu,gfu)} \phi(t)dt \\
= \int_{0}^{(\beta+\gamma)G(fu,gfu,gfu)} \phi(t)dt \\
< \int_{0}^{G(fu,gfu,gfu)} \phi(t)dt \]

This above contradiction implies that \( gfu = fu \). Put \( fu = gu = hu = t \), so, \( t \) is a common fixed point of mappings \( f \), \( g \) and \( h \).

Now, let \( t \) and \( z \) be two distinct common fixed points of \( f \), \( g \) and \( h \). That is \( ft = gt = ht = t \) and \( fz = gz = hz = z \). As \( t \neq z \), then from condition (1), we have

\[ \int_{0}^{G(t,z,z)} \phi(t)dt = \int_{0}^{G(f,t,g,z)} \phi(t)dt \leq \int_{0}^{\alpha G(h,t,h,t)+\beta[G(f,t,h,t)+G(g,t,g,t)]+\gamma[G(h,t,g,t)+G(t,g,f,t)]} \phi(t)dt \\
= \int_{0}^{\alpha G(t,z,z)+2\gamma G(t,z,z)} \phi(t)dt \\
= \int_{0}^{(\alpha+2\gamma)G(t,z,z)} \phi(t)dt \\
< \int_{0}^{G(t,z,z)} \phi(t)dt \]

Contradiction, hence \( z = t \). Thus the common fixed point is unique.

If we put \( \phi(t) = 1 \) in the above theorem, we get the following result:

**Corollary 2.1:** Let \((X,G)\) be symmetric G-metric space. Suppose \( f, g, \) and \( h \) are three self mappings of \((X,G)\) satisfying the conditions:

1. for all \( x, y \in X \)
   \[ G(fx,gy,hy) \leq \alpha G(hx,hy,hx)+\beta[G(fx,hx,hx)+G(gy,hy,hy)]+\gamma[G(hy,gx,gx)+G(hy,fx,fx)] \]
   \[ \alpha, \beta, \gamma \] are non-negative reals such that \( \alpha + 2\beta + 2\gamma < 1 \)

2. pair of mappings \((f, h)\) or \((g, h)\) is owc.

Then \( f, g \) and \( h \) have a unique common fixed point.

**2.2 A unique common fixed point theorem for four mappings**

Now, we give our second main result:

**Theorem 2.2:** Let \((X, G)\) be symmetric G-metric space. Suppose \( f, g, h \) and \( k \) are four self mappings of \((X,G)\) satisfying the following conditions: (1)

\[ \int_{0}^{G(fx,gy,gy)} \phi(t)dt \leq \int_{0}^{\alpha G(hx,ky,ky)+\beta[G(fx,hx,hx)+G(gy,ky,ky)]+\gamma[G(hx,gy,gy)+G(ky,fx,fx)]} \phi(t)dt \]

for all \( x, y \in X \), where \( \phi : R^+ \rightarrow R \) is a Lebesgue-integrable mapping which is summable, nonnegative and such that \( \int_{0}^{\phi(t)}dt > 0 \) for each \( \epsilon > 0 \), and \( \alpha, \beta, \gamma \) are non-negative reals such that \( \alpha + 2\beta + 2\gamma < 1 \)

(2) pair of mappings \((f, h)\) and \((g, k)\) are owc.
Then f, g, h and k have a unique common fixed point.

**Proof:** Since pairs of mappings (f, h) and (g, k) are owc, then, there exists two points u and v in X such that fu = hu and fhv = hvu, gv = kv and gkv = kv.

First, we prove that fu = gv. Indeed, by inequality (1), we get

\[ \int_0^{G(fu,gv)} \phi(t)dt \leq \int_0^{\alpha G(hu,kv,kv)+\beta[G(fu,hu)+G(gv,kv,kv)]+\gamma[G(hu,gv,gv)+G(kv,fu,fu)]} \phi(t)dt \]

\[ = \int_0^{\alpha G(hu,kv,kv)+\gamma[G(fu,gv,gv)]} \phi(t)dt \]

\[ = \int_0^{(\alpha+\gamma)G(fu,gv,gv)} \phi(t)dt \]

\[ < \int_0^{G(fu,gv,gv)} \phi(t)dt \]

which is a contradiction, hence, gv = fu = hu = kv.

Again, suppose that ffu = fhu = fhu ≠ fu. The use of condition (1), we have

\[ \int_0^{G(ffu,gv,gv)} \phi(t)dt \leq \int_0^{\alpha G(hfu,kv,kv)+\beta[G(ffu,hfu,hfu)+G(gv,kv,kv)]+\gamma[G(hfu,gv,gv)+G(kv,ffu,ffu)]} \phi(t)dt \]

\[ = \int_0^{\alpha G(hfu,kv,kv)+2\gamma[G(ffu,gv,gv)]} \phi(t)dt \]

\[ = \int_0^{(\alpha+2\gamma)G(ffu,gv,gv)} \phi(t)dt \]

\[ < \int_0^{G(ffu,gv,gv)} \phi(t)dt \]

this contradiction implies that ffu = fu = hfu = fhv.

Similarly gfu = kfu = fu. Put fu = t, therefore t is a common fixed point of mappings f, g, h and k.

Now, let t and z be two distinct common fixed points of f, g, h and k. That is ft = gt = ht = t and fz = gz = hz = kx = z. As t ≠ z, then from condition (1), we have

\[ \int_0^{G(t,z,z)} \phi(t)dt = \int_0^{G(fz,gz)} \phi(t)dt \leq \int_0^{\alpha G(ht,kz,kz)+\beta[G(fz,ht,ht)+G(gz,kz,kz)]+\gamma[G(ht,gz,gz)+G(hz,fz,fz)]} \phi(t)dt \]

\[ = \int_0^{\alpha G(t,z,z)+2\gamma[G(t,z,z)]} \phi(t)dt \]

\[ = \int_0^{(\alpha+2\gamma)G(t,z,z)} \phi(t)dt \]

\[ < \int_0^{G(t,z,z)} \phi(t)dt \]

a contradiction, hence z = t. Thus the common fixed point is unique.

If we put \( \phi(t) = 1 \) in the above theorem, we get the following result:

**Corollary:** 2.2 Let (X,G) be symmetric G-metric space. Suppose f, g, h and k are four self mappings of (X,G) satisfying the following conditions:
(1) \[ G(fx, gy, gy) \leq \alpha G(hx, ky, ky) + \beta [G(fx, hx, hx) + G(gy, ky, ky)] + \gamma [G(hx, gy, gy) + G(ky, fx, fx)] \]

for all \( x, y \in X \), and \( \alpha, \beta, \gamma \) are non-negative reals such that \( \alpha + 2\beta + 2\gamma < 1 \).

(2) pair of mappings \( (f, h) \) and \( (g, k) \) are owc.

Then \( f, g, h \) and \( k \) have a unique common fixed point.

**Example 2.1:** Let \( X = [0, \infty) \) with the symmetric G-metric \( G(x, y, z) = (x - y)^2 + (y - z)^2 + (z - x)^2 \). Define

\[
\begin{align*}
  f(x) &= g(x) = \begin{cases} 
    0 & x \in [0, 1), \\
    1 & x \in [1, \infty),
  \end{cases} \\
  h(x) &= \begin{cases} 
    3 & x \in [0, 1), \\
    1/x & x \in [1, \infty),
  \end{cases} \\
  k(x) &= \begin{cases} 
    1 & x \in [0, 1), \\
    \sqrt{x} & x \in [1, \infty).
  \end{cases}
\end{align*}
\]

Clearly \( (f, h) \) and \( (g, k) \) are occasionally weakly compatible. By taking \( \phi(x) = 3x^2, \alpha = \frac{1}{4}, \beta = \frac{1}{5}, \gamma = \frac{1}{6} \), all the hypothesis of theorem 2.2 are satisfied and \( x = 1 \) is the unique common fixed point of mappings \( f, g, h \) and \( k \).

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3. REFERENCES:


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