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RESIDUATED ALMOST DISTRIBUTIVE LATTICES - II

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ABSTRACT

In this paper, we prove some important properties of residuation ' : ' and multiplication ' . 'in a Residuated Almost Distributive Lattice (RADL) L. We prove important results in a residuated ADL L.

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1. INTRODUCTION

Swamy, U.M. and Rao, G.C. [4] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, p rings, biregular rings, associate rings, P_1 rings etc.) on one hand and distributive lattices on the other. In [1], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [5], [6] Ward, M. and Dilworth, R.P., have studied residuated lattices. We introduced the concepts of a residuation ' : ' and a multiplication ' . ' in an ADL in our earlier paper [3]. In this paper, we derive some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. We also prove some equivalent conditions and important results in a residuated ADL L.

In section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL. These are taken from Swamy, U.M. and Rao, G.C. [4] and Rao, G.C. [2]. Also we recall the concepts of residuation and multiplication in an ADL L and the definition of a residuated almost distributive lattice from our earlier paper [3].

In section 3, we derive some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. We prove important results in a residuated ADL L.

2. PRELIMINARIES

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL.

Definition 2.1 ([2]): An **Almost Distributive Lattice (ADL)** is an algebra $(L, \vee, \wedge)$ of type (2, 2) satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$, for all $a, b, c \in L$

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It can be seen directly that every distributive lattice is an ADL.

If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Example 2.1 ([2]): Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in L$,

define $x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases}$ $\quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a **discrete ADL.**

For any $a, b \in L$, we say that a is less than or equals to b and write $a \leq b$, if $a \wedge b = a$. Then "$\leq$ " is a partial ordering on L.

Theorem 2.1 ([2]): Let $(L, \vee, \wedge, 0)$ be an ADL with ' 0 '. Then, for any $a, b \in L$, we have

(1) $a \wedge 0 = 0$ and $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$; $a \vee (b \wedge a) = a$ and $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ whenever $a \leq b$
(6) $a \wedge b \leq b$ and $a \leq a \vee b$
(7) $\wedge$ is associative in L
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possibly the right distributivity of $\vee$ over $\wedge$, the com-mutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

Theorem 2.2 ([2]): Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:

(1) $(L, \vee, \wedge, 0)$ is a distributive lattice
(2) $a \vee b = b \vee a$ for all $a, b \in L$
(3) $a \wedge b = b \wedge a$ for all $a, b \in L$
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c \in L$.

Proposition 2.1 ([2]): Let $(L, \vee, \wedge)$ be an ADL. Then for any $a, b, c \in L$ with $a \leq b$, we have

(1) $a \wedge c \leq b \wedge c$
(2) $c \wedge a \leq c \wedge b$
(3) $c \vee a \leq c \vee b$.

Definition 2.2 ([2]): An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a$ implies $m = a$.

Theorem 2.3 ([2]): Let L be an ADL and $m \in L$. Then the following are equivalent:

(1) m is maximal with respect to $\leq$
(2) $m \vee a = m$, for all $a \in L$
(3) $m \wedge a = a$, for all $a \in L$.

Lemma 2.1 ([2]): Let L be an ADL with a maximal element m and $x, y \in L$. If $x \wedge y = y$ and $y \wedge x = x$ then x is maximal if and only if y is maximal. Also the following conditions are equivalent:

(i) $x \wedge y = y$ and $y \wedge x = x$
(ii) $x \wedge m = y \wedge m$.

Definition 2.3 ([2]): If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element m, then the set I(L) of all ideals of L is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of I, J are given by $I \vee J = \{(x \vee y) \wedge m \,/\, x \in I, y \in J\}$ and $I \wedge J = I \cap J$. The set $PI(L) = \{(a] \,/\, a \in L\}$ of all principal ideals of L forms a sub lattice of I(L). (Since $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$)

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [3].

Definition 2.4 ([3]): Let L be an ADL with a maximal element m. A binary operation : on an ADL L is called a **residuation** over L if, for a, b, c $\in$ L the following conditions are satisfied.

(R1) $a \wedge b = b$ if and only if $a : b$ is maximal

(R2) $a \wedge b = b \Rightarrow$ (i)$(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$ (R3) $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$

(R4) $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$ (R5) $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$

Definition 2.5 ([3]): Let L be an ADL with a maximal element m. A binary operation : on an ADL L is called a **multiplication** over L if, for a, b, c $\in$ L the following conditions are satisfied.

(M1) $(a.b) \wedge m = (b.a) \wedge m$

(M2) $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$ (M3) $(a.m) \wedge m = a \wedge m$

(M4) $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$

Definition 2.6 ([3]): An ADL L with a maximal element m is said to be a **residuated almost distributive lattice (residuated ADL)**, if there exists two binary operations ' : ' and ' . ' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).

(A) $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any x, a, b $\in$ L.

We use the following properties frequently later in the results.

Lemma 2.2 ([3]): Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions M1 - M4. Then for any a, b, c, d $\in$ L;

(i) $a \wedge (a.b) = a.b$ and $b \wedge (a.b) = a.b$
(ii) $a \wedge b = b \Rightarrow (c.a) \wedge (c.b) = c.b$ and $(a.c) \wedge (b.c) = b.c$
(iii) $d \wedge [(a.b).c] = (a.b).c$ if and only if $d \wedge [a.(b.c)] = a(b.c)$
(iv) $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$
(v) $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c) \Rightarrow d \wedge [(a \wedge b).c] = (a \wedge b).c$
(vi) $d \wedge [(a.c) \vee (b.c)] = (a.c) \vee (b.c)$, $d \wedge [(a \vee b).c] = (a \vee b).c$

The following result is a direct consequence of M1 of definition 2.6.

Lemma 2.3 ([3]): Let L be an ADL with a maximal element m and . a binary operation on L satisfying the condition M1. For a, b, x $\in$ L, $a \wedge (x.b) = x.b$ if and only if $a \wedge (b.x) = b.x$

3. PROPERTIES OF RESIDUATED ADL's

In this section, we prove some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. We prove some important results in a residuated ADL L.

First we give the following Lemma, whose proof can be obtained from the definition of Residuated ADL.

Lemma 3.1: Let L be a residuated ADL. Then

(1) $(a : a) : b$ is maximal, for all a, b $\in$ L:
(2) If an element m of L is maximal then $m : a$ is maximal, for all a $\in$ L.

In the following, we prove some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L.

Lemma 3.2: Let L be a residuated ADL with a maximal element m. For a, b, c, d $\in$ L, the following hold in L.

(1) $(a : b) \wedge a = a$
(2) $[a : (a : b)] \wedge (a \vee b) = a \vee b$
(3) $[(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$
(4) $[a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$
(5) $[(a \wedge b) : b] \wedge (a : b) = a : b$
(6) $(a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$
(7) $[a : (a \vee b)] \wedge m = (a : b) \wedge m$
(8) $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$
(9) If $a : b = a$ then $a \wedge (b.d) = b.d \Rightarrow a \wedge d = d$

(10) $\{a : [a : (a : b)]\} \wedge (a : b) = a : b$
(11) $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$
(12) $a \wedge m \geq b \wedge m \Rightarrow (a : c) \wedge m \geq (b : c) \wedge m$
(13) $(a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$
(14) $a \wedge b = b \Rightarrow (a.c) \wedge (b.c) = b.c$
(15) $a \wedge b \wedge (a.b) = a.b$
(16) $[(a.b) : a] \wedge b = b$
(17) $(a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$
(18) $a \vee b$ is maximal $\Rightarrow (a.b) \wedge a \wedge b = a \wedge b$

Proof: Let a, b, c, d $\in$ L. Then

(1) By R1, we have $a : a$ is a maximal element. Then $(a : a) \wedge b = b$, for all $b \in L \Rightarrow (a : a) : b$ is maximal
$\Rightarrow (a : b) : a$ is maximal
$\Rightarrow (a :b) \wedge a = a$

(2) $[(a : b).(a \vee b)] \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow \{[(a : b).a] \vee [(a : b).b]\} \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow ([(a : b).a] \wedge m) \vee ([(a : b).b] \wedge m) = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow ([a.(a : b)] \wedge m) \vee ([b.(a : b)] \wedge m) = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow \{[a.(a : b)] \vee [b.(a : b)]\} \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow [\{a \wedge [a.(a : b)]\} \vee \{a \wedge [b.(a : b)]\}] \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow \{a \wedge [a.(a : b)] \wedge m\} \vee \{a \wedge [b.(a : b)] \wedge m\} = [(a : b).(a \vee b)]$ ^ m
$\Rightarrow \{a \wedge [(a : b).a] \wedge m\} \vee \{a \wedge [(a : b).b] \wedge m\} = [(a : b)(a \vee b)] \wedge m$
$\Rightarrow [\{a \wedge [(a : b).a]\} \vee \{a \wedge [(a : b).b]\}] \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow [a \wedge \{[(a : b).a] \vee [(a : b).b]\}] \wedge m = [(a : b).(a \vee b)] \wedge m$
$\Rightarrow a \wedge [(a : b).(a \vee b)] \wedge m = [(a : b).(a \vee b)] \wedge m \Rightarrow a \wedge [(a : b).(a \vee b)] = [(a : b).(a \vee b)]$
$\Rightarrow [a : (a : b)] \wedge (a \vee b) = a \vee b$

(3) we have $[a : (b.c)] \wedge [a : (b.c)] = a : (b.c)$
$\Rightarrow a \wedge \{(b.c).[a : (b.c)]\} = (b.c).[a : (b.c)]$
$\Rightarrow a \wedge (b.\{c.[a : (b.c)]\}) = b.\{c.[a : (b.c)]\}$
$\Rightarrow (a : b) \wedge \{c.[a : (b.c)]\} = c.[a : (b.c)] \Rightarrow [(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$

(4) We have $[(a : b) : c] \wedge [(a : b) : c] = (a : b):c$
$\Rightarrow (a : b) \wedge \{c.[(a : b) : c]\} = c.[(a : b) : c]$
$\Rightarrow a \wedge (b.\{c.[(a : b) : c]\}) = b.\{c.[(a : b) : c]\}$
$\Rightarrow a \wedge \{(b.c).[(a : b) : c]\} = (b.c).[(a : b) : c]$
$\Rightarrow [a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$

(5) $[(a \wedge b) : b] \wedge (a : b) = [(a \wedge b) : b] \wedge m \wedge (a:b)$
$= [(a : b) \wedge (b : b)] \wedge m \wedge (a : b)$
$= [(a : b) \wedge (b : b)] \wedge (a :b)$
$= [(b : b) \wedge (a : b)] \wedge (a: b)$
$= (a : b) \wedge (a : b)$
$= (a : b)$

(6) We have $(a : b) \wedge (a : b) \wedge (b : b) \wedge m = (a : b) \wedge (b :b) \wedge m$
$\Rightarrow (a : b) : [(a : b) \wedge (b : b) \wedge m]$ is maximal
$\Rightarrow (a : b) : \{[(a \wedge b) : b] \wedge m\}$ is maximal
$\Rightarrow (a:b) \wedge [(a \wedge b) : b] \wedge m = [(a \wedge b) : b] \wedge m \Rightarrow (a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$

(7) $[a : (a \vee b)] \wedge m = [(a : a) \wedge (a : b)] \wedge m = (a : b) \wedge m$

(8) We have $a \wedge a \wedge b = a \wedge b$
$\Rightarrow [c : (a \wedge b)] \wedge (c : a) = c : a$ and $b \wedge a \wedge b = a \wedge b$
$\Rightarrow [c : (a \wedge b)] \wedge (c : b) = c : b$
Therefore $\{[c : (a \wedge b)] \wedge (c : a)\} \vee \{[c : (a \wedge b)] \wedge (c : b)\} = (c : a)$ v $(c : b)$
Hence $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$

(9) Assume that $a : b = a$

Suppose $a \wedge (b.d)=b.d \Rightarrow (a : b) \wedge d = d$
$\Rightarrow$ $a \wedge d = d$ (Since $a : b = a$)

(10) $[a : (a : b)] \wedge [a : (a : b)] = a : (a : b)$
$\Rightarrow$ $a \wedge \{(a : b).[a : (a : b)]\} = (a : b).[a : (a : b)]$
$\Rightarrow$ $a \wedge \{(a : b).[a : (a : b)]\} \wedge m = \{(a : b).[a : (a : b)]\} \wedge m$
$\Rightarrow$ $a \wedge \{[a : (a : b)].(a : b)\} \wedge m = \{[a : (a : b)].(a : b)\} \wedge m$
$\Rightarrow$ $a \wedge \{[a : (a : b)].(a : b)\} = \{[a : (a : b)].(a : b)\}$
$\Rightarrow \{a : [a : (a : b)]\} \wedge (a : b) = a : b$

(11) We have $(a \vee b) \wedge a = a$ and $(a \vee b) \wedge b = b$
$\Rightarrow [(a \vee b) : c] \wedge (a : c) = a : c$ and $[(a \vee b) : c] \wedge (b : c) = b : c$
$\Rightarrow \{[(a \vee b) : c] \wedge (a : c)\} \vee \{[(a \vee b) : c] \wedge (b : c)\} = (a : c) \vee (b : c)$
$\Rightarrow [(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$

(12) Suppose $a \wedge m \geq b \wedge m$. Then $b \wedge m \wedge a \wedge m = b \wedge m \Rightarrow a \wedge b = b$

Now, $(b : c) \wedge m \wedge (a : c) \wedge m = (a : c) \wedge (b : c) \wedge m = [(a \wedge b) : c] \wedge m = (b : c) \wedge m$

Therefore, $(a : c) \wedge m \geq (b : c) \wedge m$

(13)
$$\begin{aligned}[(a : b) : \{a : [a : (a : b)]\}] \wedge m &= ([a : \{a : [a : (a : b)]\}] : b) \wedge m \\ &\geq [(a \vee [a : (a : b)]) : b] \wedge m \\ &\geq [(a : b) \vee \{[a : (a : b)] : b\}] \wedge m \\ &= [\{[a : (a : b)] : b\} \vee (a : b)] \wedge m \\ &\geq \{[a : (a : b)] : b\} \wedge m \\ &= [(a : b) : (a : b)] \wedge m = m\end{aligned}$$
But $[(a : b) : \{a : [a : (a : b)]\}] \wedge m \leq m$ Therefore $[(a : b) : \{a : [a : (a : b)]\}] \wedge m = m \Rightarrow (a : b) : \{a : [a : (a : b)]\}$ is maximal.
$\Rightarrow (a :b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$

(14) Suppose $a \wedge b = b$.
Then
$$\begin{aligned}[(a.c) \wedge m] \vee [(b.c) \wedge m] &= [(c.a) \wedge m] \vee [(c.b) \wedge m] \\ &= [(c.a) \vee (c.b)] \wedge m \\ &= [c.(a \vee b)] \wedge m \\ &= (c.a) \wedge m \\ &= (a.c) \wedge m\end{aligned}$$

$\Rightarrow (a.c) \wedge m \wedge (b.c) \wedge m = (b.c) \wedge m$
$\Rightarrow (a.c) \wedge (b.c) = b.c$

(15)
$$\begin{aligned}[(a \wedge b) : a] \wedge b &= [(a \wedge b) : a] \wedge m \wedge b \\ &= (a : a) \wedge (b : a) \wedge m \wedge b \\ &= (b : a) \wedge b = b\end{aligned}$$

$\Rightarrow a \wedge b \wedge (a.b) = a:b$

(16) $(a.b) \wedge (a.b) = a.b$
$\Rightarrow [(a.b) : a] \wedge b = b$

(17) $b \wedge a \wedge b = a \wedge b$
$\Rightarrow (b.a) \wedge [(a \wedge b).a] = (a \wedge b).a$
$\Rightarrow (b.a) \wedge m \wedge [(a \wedge b).a] = (a \wedge b).a$
$\Rightarrow (a.b) \wedge m \wedge [(a \wedge b).a] = (a \wedge b).a$
$\Rightarrow (a.b) \wedge [(a \wedge b).a] = (a \wedge b).a$

Similarly, $(a.b) \wedge [(a \wedge b).b] = (a \wedge b).b$

Then $\{(a.b) \wedge [(a \wedge b).a]\} \vee \{(a.b) \wedge [(a \wedge b).b]\} = [(a \wedge b).a] \vee [(a \wedge b).b]$
$\Rightarrow (a.b) \wedge \{[(a \wedge b).a] \vee [(a \wedge b).b]\} = [(a \wedge b).a] \vee [(a \wedge b).b]$
$\Rightarrow (a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$

(18) Suppose $a \vee b$ is maximal.
By (16), we have $(a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$
$\Rightarrow (a.b) \wedge [(a \wedge b).(a \vee b)] \wedge m = [(a \wedge b).(a \vee b)] \wedge m$
$\Rightarrow (a.b) \wedge (a \wedge b) \wedge m = (a \wedge b) \wedge m$
$\Rightarrow (a.b) \wedge a \wedge b = a \wedge b.$

In the definition of a residuated ADL, we have
(R4) $[(a\wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$ and (R5) $[a : (b\vee c)] \wedge m = (a : b) \wedge (a : c) \wedge m$. Regarding $[(a \vee b) : c] \wedge m$ and $[a : (b \wedge c)] \wedge m$ we have the following equivalent conditions.

Theorem 3.1: Let L be a residuated ADL with a maximal element m and a, b, c 2 L. Then the following conditions are equivalent.
(R7) $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$
(R8) $[(a : b) \vee (b : a)] \wedge m = [(a \wedge b) : (b \wedge a)] \wedge m$
(R9) $(a : b) \vee (b : a)$ is maximal
(R10) $[(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m$

Proof: Let a, b, c $\in$ L.

(R7) ⇒ (R8): Suppose $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$.
Then $[(a : b) \vee (b : a)] \wedge m = [(a : b) \wedge m] \vee [(b : a) \wedge m]$
$= [(a : b) \wedge (b : b) \wedge m] \vee [(a : a) \wedge (b : a) \wedge m]$
$= \{[(a \wedge b) : b] \wedge m\} \vee \{[(a \wedge b) : a] \wedge m\}$
$= [(a \wedge b) : (b \wedge a)] \wedge m$ (By (R7))

Hence (R8) holds in L.

(R8) ⇒ (R9) : Suppose $[(a : b) \vee (b : a)] \wedge m = [(a \wedge b) : (b \wedge a)] \wedge m$ $b \wedge a \wedge [(b \wedge a).m] = (b \wedge a).m$
$\Rightarrow a \wedge b \wedge [(b \wedge a).m] = (b \wedge a).m$
$\Rightarrow [(a \wedge b) : (b \wedge a)] \wedge m = m$
$\Rightarrow [(a : b) \vee (b : a)] \wedge m = m$ (By (R8))
Hence $(a : b) \vee (b : a)$ is maximal

Thus (R9) holds in L.

(R9) ⇒ (R10): Suppose $(a : b) \vee (b : a)$ is maximal.

Consider $[(a : c) : (a : b)] \wedge m = \{[a : (a : b)] : c\} \wedge m \geq [(a \vee b) : c] \wedge m \rightarrow (1)$

Now, $[\{(a : c) : [(a \vee b) : c]\} : (a : b)] \wedge m = \{[(a : c) : (a : b)] : [(a \vee b) : c]\} \wedge m$
$\geq \{[(a \vee b) : c] : [(a \vee b) : c]\} \wedge m$, Hence it is maximal.

Therefore, $\{(a : c) : [(a \vee b) : c]\} : (a : b)$ is maximal.

Hence $\{(a : c) : [(a \vee b) : c]\} \wedge (a : b) = (a : b)$

Thus $\{(a : c) : [(a \vee b) : c]\} \wedge m \geq (a : b) \wedge m \rightarrow (2)$

Similarly, we get $\{(b : c) : [(a \vee b) : c]\} \wedge m \geq (b : a) \wedge m \rightarrow (3)$

Now, $\{[(a : c) \vee (b : c)] : [(a \vee b) : c]\} \wedge m \geq [\{(a : c) : [(a \vee b) : c]\} \wedge m] \vee [\{(b : c) : [(a \vee b) : c]\} \wedge m]$
$\geq [(a : b) \wedge m] \vee [(b : a) \wedge m]$ (By above (2) and (3))

$= [(a : b) \vee (b : a)] \wedge m$, which is maximal by (R9)
= Therefore, by (R1), we get
$[(a : c) \vee (b : c)] \wedge [(a \vee b) : c] = (a \vee b) : c$

This combined with property (11) of Lemma 3.2, we get
$[(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m$ Thus (R10) holds in L.

(R10) ⇒ (R7): Suppose $[(a \vee b) : c] \wedge m = [(a : c) \vee (b : c)] \wedge m$

Since $[a \vee (b \wedge c)] \wedge m \geq (b \wedge c) \wedge m$, we get
$\{[a \vee (b \wedge c)] : b\} \wedge m \geq [(b \wedge c) : b] \wedge m$ and $\{[a \vee (b \wedge c)] : c\} \wedge m \geq [(b \wedge c) : c] \wedge m$

Now,
$$\begin{aligned}\{[(a : b) \vee (a : c)] : [a : (b \wedge c)]\} \wedge m &\geq [\{(a : b) : [a : (b \wedge c)]\} \wedge m] \vee [\{(a : c) : [a : (b \wedge c)]\} \wedge m]\\ &= [\{(a : [a : (b \wedge c)]) : b\} \wedge m] \vee [\{(a : [a : (b \wedge c)]) : c\} \wedge m]\\ &\geq \{([a \vee (b \wedge c)] : b) \wedge m\} \vee \{([a \vee (b \wedge c)] : c) \wedge m\}\\ &\geq \{[(b \wedge c) : b] \wedge m\} \vee \{[(b \wedge c) : c] \wedge m\} \text{ (By (3) and (4))}\\ &= [(b : b) \wedge (c : b) \wedge m] \vee [(b : c) \wedge (c : c) \wedge m]\\ &= [(c : b) \wedge m] \vee [(b : c) \wedge m]\\ &= \{[c : (b \vee c)] \wedge m\} \vee \{[b : (b \vee c)] \wedge m\}\\ &= \{[c : (b \vee c)] \vee [b : (b \vee c)]\} \wedge m\\ &= [(c \vee b) : (b \vee c)] \wedge m \text{ (By R10)}\end{aligned}$$

Thus $[(a : b) \vee (a : c)] : [a : (b \wedge c)]$ is maximal.
$\Rightarrow [(a : b) \vee (a : c)] \wedge [a : (b \wedge c)] = a : (b \wedge c)$

Now, by property (8) of Lemma 3.2, we get
$[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$.

Thus (R7) holds in L.

Definition 3.1: Let L be an ADL and $a \in L$. An element $a^1 \in L$ is said to be a complement of a in L if $a \wedge a^1 = 0$ and $a \vee a^1$ is maximal. In this case we say that a is a complemented element of L.

Theorem 3.2: Let L be a residuated ADL with a maximal element m and $a, b \in L$. If b^1 is a complement of b in L then $(a : b) \wedge m = (a \vee b^1) \wedge m$

Proof: Suppose b^1 is a complement of b in L.

Then
$$\begin{aligned}\{(a : b) : (a \vee b^1)\} \wedge m &= f[(a : b) : a] \wedge [(a : b) : b^1]g \wedge m\\ &= [(a : a) : b] \wedge [(a : b) : b^1] \wedge m\\ &= [(a : b) : b^1] \wedge m\\ &\geq [a : (b.b^1)] \wedge m\\ &= (a : 0) \wedge m \text{ (Since } b.b^1 = b \wedge b^1 \wedge (b.b^1) = 0)\\ &= m \text{ (Since } (a : 0) \wedge m = (a : a) \wedge (a : 0) \wedge m = [a : (a \vee 0)] \wedge m = (a : a) \wedge m = m)\end{aligned}$$

Therefore $[(a : b) : a \vee b^1] \wedge m = m$ and

Hence $(a : b) \wedge (a \vee b^1) = a \vee b^1$

Now,
$$\begin{aligned}[(a \vee b^1) : (a : b)] \wedge m &\geq \{[a : (a : b)] \vee [b^1 : (a : b)]\} \wedge m\\ &\geq [(a \vee b) \wedge m] \vee \{[b^1 : (a : b)] \wedge m\} \geq (b \wedge m) \vee (b^1 \wedge m)\\ &= (b \vee b^1) \wedge m\\ &= m\end{aligned}$$

$\Rightarrow (a \vee b^1) : (a : b)$ is maximal
$\Rightarrow (a \vee b^1) \wedge (a : b) = a : b$

Hence $(a : b) \wedge m = (a \vee b^1) \wedge m$

Theorem 3.3: Let L be a residuated ADL with a maximal element m and $a, b \in L$. If b is a complemented element of L, then $(a.b) \wedge m = a \wedge b \wedge m$

Proof: Let L be a residuated ADL with a maximal element m and $a, b \in L$. Suppose b is a complement element of L and b^1 be a complement of b in L. Then $b \wedge b^1 = 0$ and $b \vee b^1$ is maximal.

By property (15) of Lemma 3.2, we get $a \wedge b \wedge (a.b) = a.b$

Now, $[a.(b \vee b^1)] \wedge (b \vee b^1) = a \wedge (b \vee b^1)$

$\Rightarrow [a.(b \vee b^1)] \wedge (b \vee b^1) \wedge m = a \wedge (b \vee b^1) \wedge m$
$\Rightarrow [a.(b \vee b^1)] \wedge m = a \wedge m$
$\Rightarrow a \wedge m = [(a.b) \vee (a.b^1)] \wedge m$
$= [(a.b) \vee (b^1.a)] \wedge m$
$\leq [(a.b) \vee b^1] \wedge m$ (Since $(b^1.a) \wedge m \leq b^1 \wedge m$)

$\Rightarrow a \wedge b \wedge m = b \wedge a \wedge m$
$\leq b \wedge [(a.b) \vee b^1] \wedge m$
$= \{[b \wedge (a.b)] \vee (b \wedge b^1)\} \wedge m$
$= \{[b \wedge (b.a) \wedge m] \vee (0 \wedge m)\}$
$= [(b.a) \vee 0] \wedge m$
$= (b.a) \wedge m$
$= (a.b) \wedge m$

Therefore $a \wedge b \wedge m \leq (a.b) \wedge m$

Hence $(a.b) \wedge m = a \wedge b \wedge m$

In Theorem 3.1, we have proved that (R8), (R9) and (R10) are equivalent to (R7). Now we prove (R7) in a residuated ADL L under certain condition. We conclude this paper with the following Theorem.

Theorem 3.4: Let L be a residuated ADL with a maximal element m and a, b, c $\in$ L. If $b \vee c$ is a maximal element of L then $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m$

Proof: Let L be a residuated ADL with a maximal element m and a, b, c $\in$ L such that $b \vee c$ is maximal.

Now, $b \wedge b \wedge c = b \wedge c$ and $c \wedge b \wedge c = b \wedge c$
$\Rightarrow [a : (b \wedge c)] \wedge (a : b) = a : b$ and $[a : (b \wedge c)] \wedge (a : c) = a : c$
$\Rightarrow [a : (b \wedge c)] \wedge [(a : b) \vee (a : c)] = (a : b) \vee (a : c)$

First, we prove the following property (i).
(i) $[(a : b) \vee (a : c)] \wedge m = \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m$
(i) $([(a : b) \vee (a : c)] : \{[(a : b) \vee (a : c)] : (b \vee c)\}) \wedge m \geq \{[(a : b) \vee (a : c)] \vee (b \vee c)\} \wedge m$
$= (b \vee c) \wedge m$
$= m$
$\Rightarrow$ $[(a : b) \vee (a : c)] : \{[(a : b) \vee (a : c)] : (b \vee c)\}$ is maximal
$\Rightarrow$ $[(a : b) \vee (a : c)] \wedge m \geq \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m$

By property (1) of Lemma 3.2, we get
$\{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m \geq [(a : b) \vee (a : c)] \wedge m$

Therefore, $[(a : b) \vee (a : c)] \wedge m = \{[(a : b) \vee (a : c)] : (b \vee c)\} \wedge m$
$= \{[(a : b) \vee (a : c)] : b\} \wedge \{[(a : b) \vee (a : c)] : c\} \wedge m$
$\geq \{[(a : c) : b] \wedge m\} \wedge \{[(a : b) : c] \wedge m\}$
$= [(a : b) : c] \wedge m \wedge [(a : b) : c] \wedge m$
$= (a : b) : c] \wedge m \geq [a : (b.c)] \wedge m \geq [a : (b \wedge c)] \wedge m.$

Therefore, $[(a : b) \vee (a : c)] \wedge m \geq [a : (b \wedge c)] \wedge m$

Hence $[a : (b \wedge c)] \wedge m = [(a : b) \vee (a : c)] \wedge m.$

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