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SOME NEW FAMILIES OF ODD GRACEFUL GRAPHS

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ABSTRACT

In this paper we give odd graceful labeling to Bistar, $\langle K_{1,n}:2\rangle$, $\langle K_{1,n,n}\rangle$, Coconut tree $CT\langle n,3\rangle$, $\langle K_{1,n}+K_{n,1}\rangle$, $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ for $n\geq 3$, $\langle nC_4{}^2+K_{1,n}\rangle$, $\langle K_{1,n}+nP_2+K_{n,1}\rangle$, $\langle K_{1,n}+nP_3+K_{n,1}\rangle$, C_{2r} with every alternate vertex attached to a pendant vertex .

Key Words: *Odd graceful labeling, Vertex labeling, edge labeling.*

1. INTRODUCTION

A graph *G* with *q* edges is said to be *odd-graceful*, if there is an injection *f* from *V(G)* to $\{0,1,2,\cdots,2q-1\}$ such that, when each edge *{x, y}* of *G* is assigned the label $|f(x)-f(y)|$, the resulting edge labels are $\{1,3,5,\cdots,2q-1\}$. The concept of odd graceful graph was introduced in 1991 by Gnanajothi [1].

Gnanajothi [1] proved that the class of odd-graceful graphs lies between the class of interlaced graphs and the class of bipartite graphs by showing that every interlaced graph has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved the following graphs are odd-graceful: P_n; C_n if and only if *n* is even; $K_{m,n}$; combs $P_n\odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of P_n); books ; crowns $C_n\odot K_1$ (graphs obtained by joining a single pendent edge to each vertex of C_n) if and only if *n* is even ; the disjoint union of copies of C_4; the one-point union of copies of C_4; $C_n\times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from P_n (*n > 3*) by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs obtained from $P_n\times P_2$ by deleting an edge that joins to end points of the P_n paths ; the graphs obtained from a star by adjoining to each end vertex the path P_3 or by adjoining to each end vertex the path P_4 . She conjectures that all trees are odd-graceful and proves the conjecture for all trees with order up to 10. Barrientos [2] has extended this to trees of order up to 12. For details in the progress made so far in this area one can refer to the latest Survey on graph labeling problems due to Gallian [3]. Here we are inspired by the works of Solairaju [4] and Moussa and Badr [5] and give odd graceful labeling to the graphs : Bistar, $\langle K_{1,n}:2\rangle$, $\langle K_{1,n,n}\rangle$, Coconut tree $CT\langle n,3\rangle$, $\langle K_{1,n}+K_{n,1}\rangle$, $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ for $n\geq 3$, $\langle nC_4{}^2+K_{1,n}\rangle$, $\langle K_{1,n}+nP_2+K_{n,1}\rangle$, $\langle K_{1,n}+nP_3+K_{n,1}\rangle$, C_{2r} with every alternate vertex attached to a pendant vertex .

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RESULTS

Theorem 2.1: A Bistar $B_{m,n}$ is odd graceful.

Proof: Let $B_{m,n}$ be a Bistar containing $m+n+2$ vertices consisting of the set $\{u_1, u_2, v_i, w_j\}$, $i=1,\cdots,m$, $j=1,\cdots,n$ where u_1, u_2 are the internal vertices and v_i, w_j are the vertices adjacent to u_1, u_2 respectively. Let q be the total number of edges of $B_{m,n}$ such that *q=m+n+1*. Now the vertex labelling of the Bistar $B_{m,n}$ are given as

$f(u_1)=0$, $f(u_2)=1$, $f(v_i)=2q-2i+1$, $i=1,2,\cdots,m$, $f(w_j)=2q-2m-2j+2$, $j=1,2,\cdots,n$.

The edge labelling of the Bistar is given as $f(u_1v_i)=2q-2i+1$, $i=1,2,\cdots,m$, $f(u_2w_j)=2q-2m-2j+1$, *j=1,2 ,…,n*. So the edge labelling of the Bistar consists of the labelled set $\{1,3,5,\cdots,2q-1\}$. Hence the Bistar $B_{m,n}$ is odd graceful.

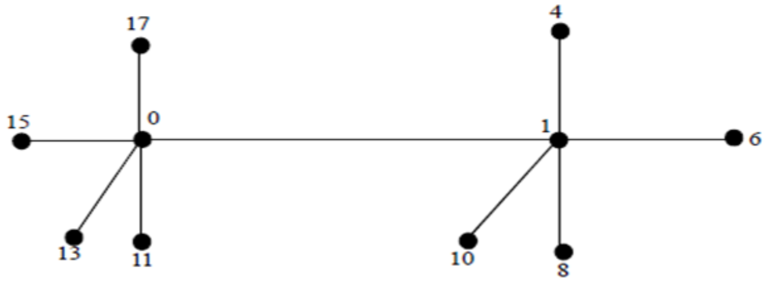

Figure-1: A Bistar $B_{4,4}$ with an odd graceful labeling

Theorem 2.2: $\langle K_{1,n}:2\rangle$ is odd graceful for every natural number *n*.

Proof: Let the set of vertices of $\langle K_{1,n}:2\rangle$ be $\{u, v, w, u_i, v_i\}$, $i=1,\cdots,n$. Let P_3 be the path with vertices *{u , v, w}*, where *u, v, w* are the internal vertices and u_i, v_j the vertices adjacent to *u, v* and *w* respectively. Let $(\langle K_{1,n}:2\rangle)=\{uw, vw, uu_i, vv_i, i=1, 2,\cdots,n\}$ be then edge set of $\langle K_{1,n}:2\rangle$ and q be the total number of edges of $\langle K_{1,n}:2\rangle$ such that $q=2n+2$. The vertex labeling $\langle K_{1,n}:2\rangle$ is given as follows $f(u_i)=2i-1$, $i=1,\cdots,n$, $f(v_i)=2(n+i)+1$, $i=1,2,\cdots,n$, $f(u)=0$, $f(v)=2$, $f(w)=4n+3$, Now the induced edge labelling of $\langle K_{1,n}:2\rangle$ is as follows $f(uu_i)=2i-1$, $i=1,\cdots,n$, $f(vv_j)=2(n+j)-1$, $j=1,\cdots,n$. $f(uw)=4n+3=2q-1$ $f(wv)=4n+1=2q-3$. So the edge labelling of $\langle K_{1,n}:2\rangle$ consists of the set $\{1,3,5,\cdots,2q-1\}$. Hence $\langle K_{1,n}:2\rangle$ is odd graceful.

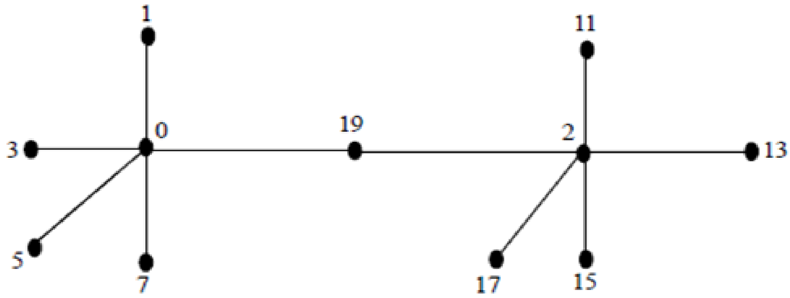

Figure-2: The tree $\langle K_{1,4}:2\rangle$ with an odd graceful labelling

Theorem-2.3: $K_{1,n,n}$ is odd graceful for every natural number *n*.

Proof: Let the set of vertices of $K_{1,n,n}$ be $\{u, u_i, v_i\}$ such that the root vertex u is adjacent to u_i and u_i is adjacent to v_i for $i = 1, \cdots, n$. Let $E(K_{1,n,n}) = \{uu_i, u_i v_i\}$ be the edge set of $K_{1,n,n}$. The vertex labelling of $K_{1,n,n}$ are given as $f(u) = 0$, $f(u_i) = 4n - 2i + 1$, $f(v_i) = 2n + 2i - 2$, $i = 1, \cdots, n$. Now the edge labelling of $K_{1,n,n}$ are given as $f(uu_i) = 4n - 2i + 1 = 2q - 2i + 1$, $f(u_i v_j) = |2n - 4i + 3|$, $i = 1, \cdots, n$. So the edge labelling of $K_{1,n,n}$ consists of the set $\{1, 3, 5, \cdots, 2q - 1\}$. Hence $K_{1,n,n}$ is odd graceful.

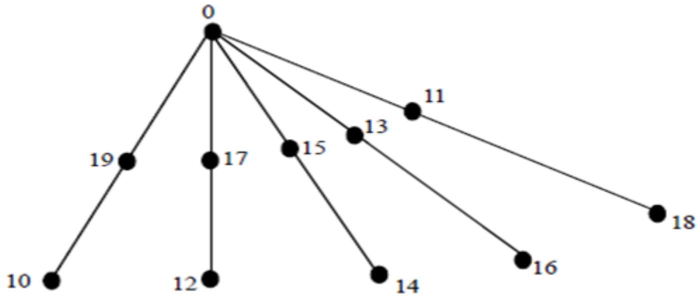

Figure-3: The tree $K_{1,5,5}$ with an odd graceful labeling

Theorem-2.4: The Coconut tree $CT\langle n, 3\rangle$ is odd graceful.

Proof: Let the set of vertices of $CT\langle n, 3\rangle$ be $\{u, v, w, u_i, i = 1, \cdots, n\}$ where *u, v, w* lie on the central path of $CT\langle n, 3\rangle$. Let $E\langle CT\langle n, 3\rangle\rangle = \{uu_i, uv, vw\}$, $i = 1, \cdots, n$ be the edge set of $CT\langle n, 3\rangle$. The vertex labelling of $CT\langle n, 3\rangle$ are given as $f(u) = 0$, $f(v) = 2n + 3$, $f(w) = 2$, $f(u_i) = 2i - 1$, $i = 1, \cdots, n$. Now the edge labelling of $CT\langle n, 3\rangle$ are given as $f(uv) = 2n + 3 = 2q - 1$, $f(vw) = 2n + 1 = 2q - 3$, $f(uu_i) = 2i - 1$, $i = 1, \cdots, n$. So the edge labelling of $CT\langle n, 3\rangle$ consists of the set {1, 3, 5, - - - , 2q-1}. Hence the Coconut tree $CT\langle n, 3\rangle$ is odd graceful.

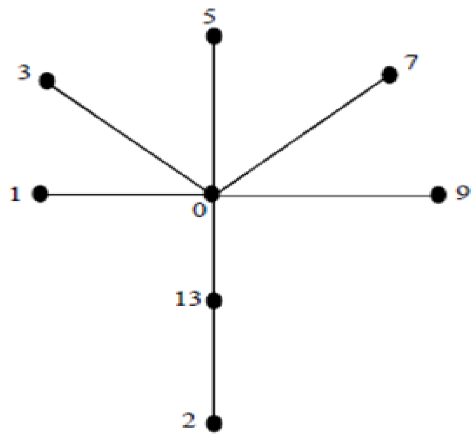

Figure-4: The coconut tree $CT\langle 5, 3\rangle$ with an odd graceful labeling

Theorem-2.5: $\langle K_{1,n} + K_{n,1}\rangle$ is odd graceful for every natural number *n.*

Proof: Let the set of vertices of $\langle K_{1,n} + K_{n,1}\rangle$ be $\{u, v, u_i, i = 1, \cdots, n\}$ where u_i is adjacent to both *u* and *v*. The edge set of $\langle K_{1,n} + K_{n,1}\rangle$ is given by $\{u u_i, u_i v, \ i = 1, \cdots, n\}$.

Now the vertex labelling of $\langle K_{1,n}+K_{n,1}\rangle$ is given by $f(u)=0, f(v)=2n, f(u_i)=4n-2i+1, i=1,\cdots,n$. And the edge labelling of $\langle K_{1,n}+K_{n,1}\rangle$ is given by $f(uu_i)=4n-2i+1,=2q-2i+1$ $f(u_iv)=2n-2i+1,$ $i=1,\cdots,n$. So the edge labelling of $\langle K_{1,n}+K_{n,1}\rangle$ consists of the set $\{1,3,5,\cdots,2q-1\}$. Hence $\langle K_{1,n}+K_{n,1}\rangle$ is odd graceful.

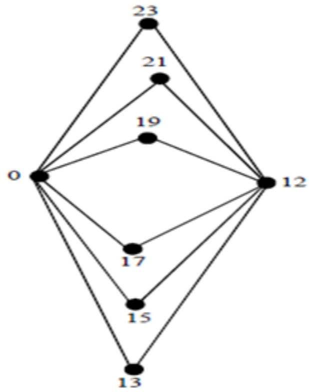

Figure-5: The graph $\langle K_{1,6}+K_{6,1}\rangle$ with an odd graceful

Theorem-2.6: $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ is odd graceful for every natural number $n \geq 3$.

Proof: Let the set of vertices of $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ be $\{u, v, u_i, v_i, w_i, i=1,\cdots,n\}$ where the vertices are in the order $u_i-u-v_i-v-w_i$. Then edge set of $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ is $\{u_iu, uv_i, v_iv, vw_i\}$, $i=1,\cdots,n$. We define the vertex labelling f of G by $f(u)=0$, $f(v)=8n-2$,

$$f(u_i)=\begin{cases}1 & , i=1\\ 8n-2i+3, & i=2,\cdots,n\end{cases}$$
$$f(v_i)=12n-2i+1, i=1,\cdots, 2n$$
$$f(w_i)=2n+2i-3, i=0,---,n-1.$$

The label of the edges of $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ are given by

$$f(u_iu)=\begin{cases}1 & , \quad i=1\\ 8n-2i+3, & i=2,3,---,n\end{cases},$$
$$f(vw_i)=6n-2i+1, i=0,---,n-1$$
$$f(v_iv)=4n-2i+3, i=1,\cdots,2n$$
$$f(uv_i)=12n-2i+1, i=1,---,2n.$$

We observe that the label of the edges of $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ constitute the set $\{1,3,5,\cdots,2q-1\}$. Hence $\langle K_{n,1}+nC_4+K_{1,n}\rangle$ is odd graceful.

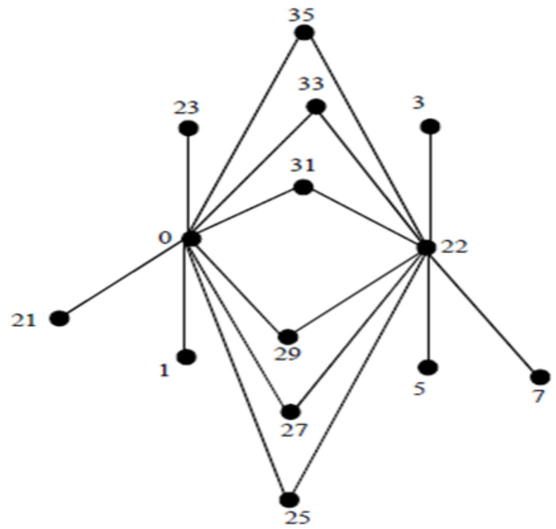

Figure-6: The graph $\langle K_{n,1} + nC_4 + K_{1,n} \rangle$ with an odd graceful

Theorem 2.7: $\langle nC_4{}^2 + K_{1,n} \rangle$ is odd graceful.

Proof: Let the vertices of $\langle nC_4{}^2 + K_{1,n} \rangle$ consists of the set $\{u, v, u_i, v_i, w, w_i, i = 1, \cdots, n\}$, where the vertices are adjacent in the order $u - u_i - v - v_i - w - w_i$. The edge set of $\langle nC_4{}^2 + K_{1,n} \rangle$ is $\{uu_i, u_i v, vv_i, v_i w, ww_i\}$, $i = 1, \cdots, n$. The vertex labelling f of $\langle nC_4{}^2 + K_{1,n} \rangle$ are assigned as $f(u) = 4n$, $f(u_i) = 8n + 2i - 1$, $f(v) = 0$, $f(v_i) = 20n - 2i + 1, i = 1, \cdots, 2n$, $f(w) = 16n$, $f(w_i) = 2i - 1$, $i = 1, \cdots, 2n$ .The label of the edges of $\langle nC_4{}^2 + K_{1,n} \rangle$ are given by
$f(uu_i) = 4n + 2i - 1$, $f(u_i v) = 8n + 2i - 1$, $f(vv_i) = 20n - 2i + 1$, $f(v_i w) = 4n - 2i + 1, i = 1, \cdots, n$.
$f(ww_i) = 16n - 2i + 1$, $i = 1, \ldots 2n$. We observe that the label of the edges of $\langle nC_4{}^2 + K_{1,n} \rangle$ constitute the set $\{1, 3, 5, \cdots, 2q - 1\}$. Hence $\langle nC_4{}^2 + K_{1,n} \rangle$ is odd graceful.

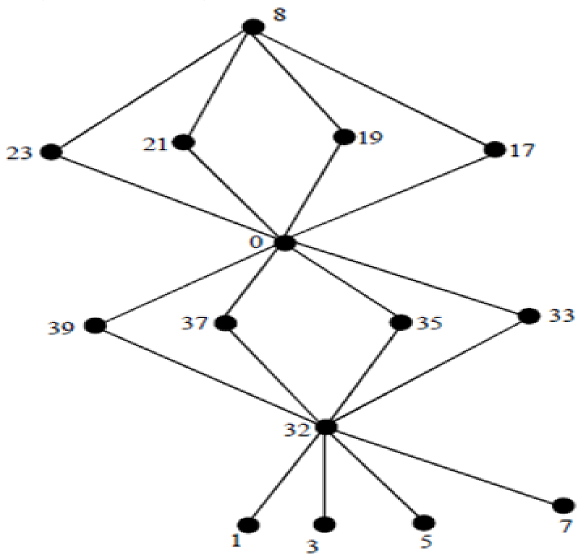

Figure-7: The graph $\langle 4C_4{}^2 + K_{1,4} \rangle$ with an odd graceful labelling.

Theorem 2.8: $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ is odd graceful.

Proof: Let the vertices of $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ consists of the set $\{u, u_i, v, v_i\}$ where the vertices are adjacent in the order $u - u_i - v_i - v$. The edge set of $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ is $\{u\,u_i, u_i v_i, v_i v\,;\, i = 1, \cdots, n\}$. The labeling of the vertices of $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ are given by $f(u) = 0$, $f(u_i) = 6n - 2i + 1$, $f(v) = 2n - 1$, $f(v_i) = 4n - 4i + 2$, $i = 1, \cdots, n$. The labeling of the edges of $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ are given by $f(uu_i) = 6n + 2i + 1$, $f(u_i v_i) = 2n + 2i - 1$, $f(v_i v) = |2n - 4i + 3|$, $i = 1, \cdots, n$. Now the edge labeling of $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ constitute the set $\{1, 3, 5, \cdots, 2q - 1\}$. Hence the graph $\langle K_{1,n} + nP_2 + K_{n,1} \rangle$ is an odd graceful graph.

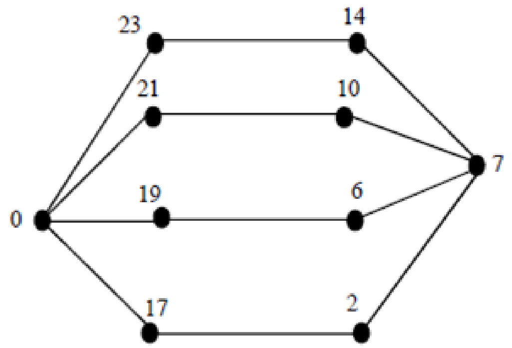

Figure-8: The graph $\langle K_{1,4} + 4P_2 + K_{4,1} \rangle$ with an odd graceful

Theorem 2.9: $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ is odd graceful.

Proof: Let the vertices of $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ consists of the set $\{u, u_i, v_i, w_i, w\}$ where the vertices are adjacent in the order $u - u_i - v_i - w_i - w$. The edge set of $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ is $\{uu_i, u_i v_i, v_i v; . i = 1, \cdots, n\}$ The labeling of the vertices of $\langle K_{1,n} + np_3 + K_{n,1} \rangle$ are given by $f(u) = 0$, $f(u_i) = 8n - 2i + 1,$ $f(v_i) = 4n - 4i + 2,$ $f(w_i) = 6n - 2i + 1),$ $f(w) = 4n$. The labeling of the edges of $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ are given by $f(u\ _i) = 8n + 2i - 1,$ $f(u_i v_i) = 4n + 2i - 1,$ $f(v_i w_i) = 2n + 2i - 1,$ $f(w_i w) = 2n - 2i + 1 i = 1, \cdots, n$. We observe that the labels of the edges of $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ constitute the set $\{1, 3, 5, \cdots, 2q - 1\}$. Hence the graph $\langle K_{1,n} + nP_3 + K_{n,1} \rangle$ is an odd graceful graph.

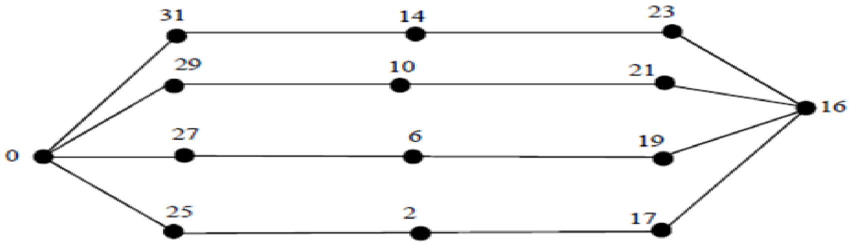

Figure-9: The graph $K_{1,4} + 4P_3 + K_{4,1}$ with an odd graceful labeling

Theorem-2.10: The graph C_{2r} with every alternate vertex attached to a pendant vertex is odd graceful.

Proof: Let the graph have *n* edges. Observe that $n = 3r$. Let us designate the vertices on the cycle as $u_1, u_2, \cdots, u_r$. Let the vertices u_i be adjacent to one pendant vertex v_i for $i = 1, 3, 5, \cdots, 2r - 1$. We assign the labels to the vertices of the graph with the labeling f given by

$$f(u_i)=\begin{cases} 2n-i & if \quad i=1,3,\ldots,2r-1 \\ i & if \quad i=2,4,6,\ldots,2r-2 \\ 4r-2 & if \quad i=2r \end{cases} \text{ and}$$

$$f(u_i)=\begin{cases} 0 & if \quad i=1 \\ 2n-2i+2 & if \quad i=3 \\ 2n-2i & if \quad i=5,7,\ldots,2r-1 \end{cases}$$

$|f(u_{i-1})-f(u_i)|=2n-(i-1)-i=2n-2i+1, i=2,3,\cdots,2r-1; |f(u_{2r-1})-f(u_{2r})|=3;$
$|f(u_{2r})-f(u_1)|=2n-1-(4r-2)=6r-1-(4r-2)=2r+1, |f(u_1)-f(v_1)|=2n-1,$
$|f(u_3)-f(v_3)|=1$, $|f(u_{2i-1})-f(v_{2i-1})|=2i-1$, $i=3,4,\cdots,r$. Observe that the set of all vertex labels of the graph is the set $\{1,3,5,\cdots,2n-1\}$. Hence, the given graph is odd graceful.

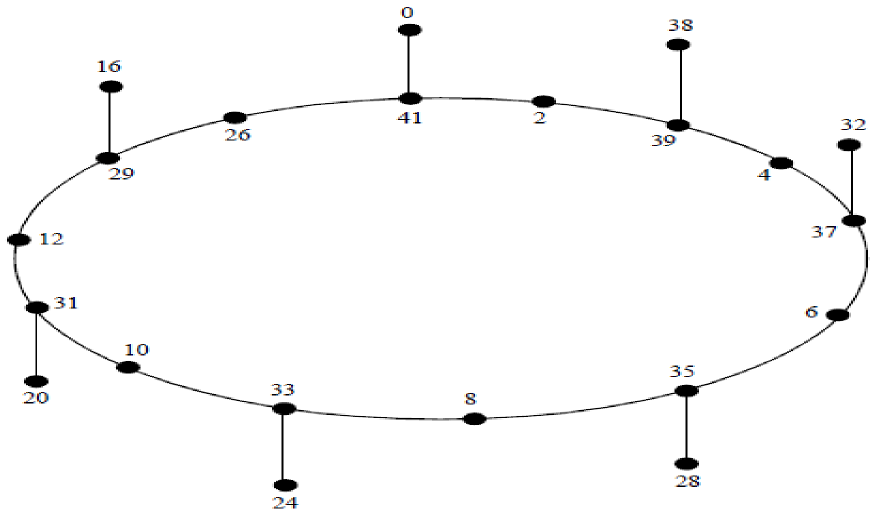

Figure 10: C_{14} with every alternate vertices attached to a pendant vertex is odd graceful.

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