U-Γ-SEMIGROUPS AND V-Γ-SEMIGROUPS

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ABSTRACT

In this paper, the terms, Maximal Γ-ideal, primary Γ-ideal, prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup are introduced. It is proved that Γ-semigroup S is a U- Γ-semigroup if either S has a left (right ) identity or S is generated by a Γ-idempotent. Also it is proved that a Γ-semigroup S is a U- Γ-semigroup if and only if every proper Γ-ideal is contained in a proper prime Γ-ideal. Also it is proved that if A is a proper Γ-ideal in the finite dimensional U- Γ-semigroup S, then A is contained in maximal Γ-ideal and also it is proved that if S is a globally idempotent Γ-semigroup with maximal Γ-ideals, then either S is aV- Γ-semigroup or S has a unique maximal Γ-ideal which is prime.


Keywords: Γ-semigroup, Maximal Γ-ideal, primary Γ-semigroup, commutative Γ-semigroup, left (right) identity, identity, Zero element, Prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup.

1. INTRODUCTION


2. PRELIMINARIES

Definition 2.1: Let S and Γ be any two non-empty sets. Then S is said to be a Γ-semigroup if there exist a mapping from S × Γ × S to S which maps (a, γ, b) → aγb satisfying the condition: (aαb)γc = aα(bγc) for all a, b, c ∈ S and α, β, γ ∈ Γ.

Note 2.2: Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set {aγb : a ∈ A, b ∈ B and γ ∈ Γ} by AΓB.

Definition 2.3: A Γ-semigroup S is said to be commutative Γ-semigroup provided aγb = bγa for all a, b ∈ S and γ ∈ Γ.

Note 2.4: If S is a commutative Γ-semigroup then aΓb = bΓa for all a, b ∈ S.

Definition 2.5: Let S be a Γ-semigroup and a, b ∈ S and a ∈ Γ. Then aaaaab is denoted by (aa)3b and consequently aaaaab is denoted by (aa)3b.

Definition 2.6: A Γ-semigroup S is said to be quasi commutative provided for each a, b ∈ S, there exists a natural number n such that aγb = (bγ)ⁿa ∀ a, b ∈ S.
Note 2.7: If a $\Gamma$-semigroup $S$ is quasi commutative then for each $a, b \in S$, there exists a natural number $n$ such that, $a \Gamma b = (b \Gamma)^n a$.

Definition 2.8: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a left identity of $S$ provided $a \alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.

Definition 2.9: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a right identity of $S$ provided $s \alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

Definition 2.10: An element $a$ of a $\Gamma$-semigroup $S$ is said to be a two sided identity or an identity provided it is both a left identity and a right identity of $S$.

Notation 2.11: Let $S$ be a $\Gamma$-semigroup. If $S$ has an identity , let $S^1 = S$ and if $S$ does not have an identity, let $S^1$ be the $\Gamma$-semigroup $S$ with identity adjoined, usually denoted by the symbol 1.

Definition 2.12: A non empty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a left $\Gamma$-ideal of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $s \alpha a \in A$.

Note 2.13: A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is a left $\Gamma$-ideal of $S$ if $S \Gamma A \subseteq A$.

Definition 2.14: A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a right $\Gamma$-ideal of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ implies $a \alpha s \in A$.

Note 2.15: A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is a right $\Gamma$-ideal of $S$ if $S \Gamma A \subseteq A$.

Definition 2.16: A nonempty subset $A$ of a $\Gamma$-semigroup $S$ is said to be a two sided $\Gamma$-ideal or simply a $\Gamma$-ideal of $S$ if $s \in S, a \in A, \alpha \in \Gamma$ imply $s \alpha a \in A$, $a \alpha s \in A$.

Definition 2.17: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a maximal $\Gamma$-ideal provided $A$ is a proper $\Gamma$-ideal of $S$ and is not properly contained in any proper $\Gamma$-ideal of $S$.

Definition 2.18: A $\Gamma$-ideal $P$ of a $\Gamma$-semigroup $S$ is said to be a prime $\Gamma$-ideal provided $A, B$ are two $\Gamma$-ideals of $S$ and $A \Gamma B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

Definition 2.19: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a semi prime $\Gamma$-ideal provided $x \in S$, $x \Gamma S^1 \Gamma x \subseteq A$ implies $x \in A$.

Definition 2.20: If $A$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$, then the intersection of all prime $\Gamma$-ideals of $S$ containing $A$ is called prime $\Gamma$-radical or simply $\Gamma$-radical of $A$ and it is denoted by $\sqrt{A}$ or $\text{rad}(A)$.

Theorem 2.21[5]: If $A$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$ then $\sqrt{A}$ is a semi prime $\Gamma$-ideal of $S$.

Theorem 2.22[5]: A $\Gamma$-ideal $Q$ of $\Gamma$-semigroup $S$ is a semi prime $\Gamma$-ideal of $S$ if $\sqrt{(Q)} = (Q)$ implies $x \Gamma S^1 \Gamma y \subseteq A$.

Definition 2.23: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a left primary $\Gamma$-ideal provided

1) If $X, Y$ are two $\Gamma$-ideals of $S$ such that $X \Gamma Y \subseteq A$ and $Y \not\subseteq A$ then $X \not\subseteq \sqrt{A}$.

2) $\sqrt{A}$ is a prime $\Gamma$-ideal of $S$.

Definition 2.24: A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a right primary $\Gamma$-ideal provided

1) If $X, Y$ are two $\Gamma$-ideals of $S$ such that $X \Gamma Y \subseteq A$ and $X \not\subseteq A$ then $Y \not\subseteq \sqrt{A}$.

2) $\sqrt{A}$ is a prime $\Gamma$-ideal of $S$.

Example 2.25: Let $S = \{a, b, c\}$ and $\Gamma = \{x, y, z\}$. Define a binary operation in $S$ as shown in the following table.

<table>
<thead>
<tr>
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<th>a</th>
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<td>a</td>
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<td>c</td>
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Define a mapping $S \times \Gamma : S \rightarrow S$ by $a \alpha a = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that $S$ is a $\Gamma$-semigroup.
Now consider the Γ-ideal \( <a> = S_1 \Gamma a \Gamma S_1 = \{a\} \). Let \( p \Gamma q \subseteq <a> \), \( p \not\in <a> \Rightarrow q \in \sqrt{<a>} \Rightarrow (q \Gamma)^{-1} q \subseteq <a> \) for some \( n \in N \). Since \( b \Gamma e \subseteq <a> \), \( c \not\in <a> \Rightarrow b \not\in <a> \). Therefore \( <a> \) is left primary. If \( b \not\in <a> \) then \( (c \Gamma)^{n-1} c \subseteq <a> \) for any \( n \in N \Rightarrow c \not\in \sqrt{<a>} \). Therefore \( <a> \) is not right primary.

**Definition 2.26:** A Γ-ideal \( A \) of a Γ- semigroup \( S \) is said to be a **primary Γ-ideal** provided \( A \) is both left primary Γ-ideal and right primary Γ-ideal.

**Definition 2.27:** A Γ-ideal \( A \) of a Γ- semigroup \( S \) is said to be a **principal Γ-ideal** provided \( A \) is a Γ-ideal generated by a single element \( a \). It is denoted by \( J[a] = <a> \).

**Definition 2.28:** An element \( a \) of a Γ-semigroup \( S \) with 1 is said to be **left invertible or left unit** provided there is an element \( b \in S \) such that \( b \Gamma a = 1 \).

**Definition 2.29:** An element \( a \) of a Γ-semigroup \( S \) with 1 is said to be **right invertible or right unit** provided there is an element \( b \in S \) such that \( a\Gamma b = 1 \).

**Definition 2.30:** An element \( a \) of a Γ-semigroup \( S \) is said to be **invertible** or a **Unit** in \( S \) provided it is both left and right invertible element in \( S \).

**Definition 2.31:** A Γ- semigroup \( S \) is said to be a **simple Γ- semigroup** provided \( S \) has no proper Γ- ideals.

**Definition 2.32:** An element \( a \) of a Γ- semigroup \( S \) is said to be a **Γ-idempotent** provided \( a \propto a = a \) for all \( \propto \in \Gamma \).

**Note 2.33:** If an element \( a \) of a Γ- semigroup \( S \) is a Γ-idempotent, then \( a \Gamma a = a \).

**Definition 2.34:** A Γ- semigroup \( S \) is said to be an **idempotent Γ- semigroup** or a **band** provided every element in \( S \) is a Γ-idempotent.

**Definition 2.35:** A Γ- semigroup \( S \) is said to be a **globally idempotent Γ- semigroup** provided \( S \Gamma S = S \).

3). **U-Γ-SEMIGROUPS AND V-Γ-SEMIGROUPS**

**Definition 3.1:** A Γ- semigroup \( S \) is said to be U-Γ-semigroup, provided for any Γ-ideal \( A \) in \( S \), \( \sqrt{A} = S \) implies \( A = S \).

**Example 3.2:** Let \( S \) is a Γ-semigroup with \( S = \Gamma \) under the multiplication given in the following table. \((S \times \Gamma \times S \rightarrow S \) as \( aab = ab)\)

\[
\begin{array}{cccc}
. & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & b \\
c & a & a & a & a \\
d & a & a & c & d \\
\end{array}
\]

Since \( S = \{a, b, c, d\} \) and \( S = \Gamma \). Now \( <a> = \{a, b\} \), \( \{a, c\} \), \( \{a, b, c\} \) and \( \{a, b, c, d\} \) are the Γ-ideals of \( S \).

If \( A = <a> \) then \( \sqrt{<a>} = \) intersection of all prime Γ- ideals containing \( <a> = \{a, b, c\} \cap \{a, b, c, d\} = \{a, b, c\} \neq S \). Similarly \( \sqrt{\{a, b\}} = \{a, b, c\} \neq S \). \( \sqrt{\{a, b, c\}} = \{a, b, c\} \neq S \) and if \( A = \{a, b, c, d\} \) then \( \sqrt{A} = \sqrt{\{a, b, c, d\}} = S \) implies \( A = S \). Therefore \( \sqrt{A} = S \) is true for only \( A = S \). Therefore \( S \) is U-Γ-semigroup.

**Theorem 3.3:** A Γ-semigroup \( S \) is a U-Γ-semigroup if either \( S \) has a left (right) identity or \( S \) is generated by a Γ- idempotent.

**Proof:** Suppose \( S \) has a left identity \( e \). Let \( A \) be any proper Γ-ideal such that \( \sqrt{A} = S \). Since \( \sqrt{A} \subseteq \{x \in S; (x \Gamma)^n x \subseteq A \text{ for some natural number } n \} = S \). So there is a natural number \( n \) such that \( (e \Gamma)^n e \subseteq A \) and hence \( e \in A \). Thus \( S = e \Gamma S \subseteq A \), it is a contradiction. Therefore \( S \) is a U-Γ-semigroup. Suppose \( S \) is generated by a Γ-idempotent \( e \). As above we can prove that for any Γ- ideal \( A \) in \( S \), if \( \sqrt{A} = S \), then \( e \in A \) and hence \( A = S \). So \( S \) is a U-Γ-semigroup.
Theorem 3.4: A $\Gamma$-semigroup $S$ is a U- $\Gamma$-semigroup if and only if every proper $\Gamma$- ideal is contained in a proper prime $\Gamma$-ideal.

Proof: Suppose $S$ is a U-$\Gamma$-semigroup. Let $A$ be any proper $\Gamma$- ideal in $S$. If $A$ is not contained in any proper prime $\Gamma$- ideal, then $\sqrt{A} = S$. Since $S$ is a U-$\Gamma$-semigroup, we have $A = S$, this is a contradiction. So every proper $\Gamma$-ideal is contained in a proper prime $\Gamma$-ideal. Conversely if every proper $\Gamma$- ideal is contained in a proper prime $\Gamma$-ideal, Then $\sqrt{A} \neq S$ implies $A \neq S$ then clearly $S$ is a U-$\Gamma$-semigroup.

Theorem 3.5: Let $S$ be a U-$\Gamma$-semigroup. Then $S = S \setminus \cup \{P_{\alpha}\}$ and hence every maximal $\Gamma$- ideal is prime.

Conversely if $\{P_{\alpha}\}$ is the collection of all prime $\Gamma$- ideals in $S$ and if $P$ is a maximal element in this collection, then $P$ is a maximal $\Gamma$- ideal in $S$.

Proof: Clearly $\sqrt{S} \cap \Gamma S = S$. Since $S$ is a U-$\Gamma$-semigroup, we have $S \cap \Gamma S = S$ and hence every maximal $\Gamma$- ideal is prime. If $P$ is not a maximal $\Gamma$- ideal in $S$, then there exists a proper $\Gamma$- ideal $A$ in $S$, containing $P$ properly. Since $P$ is a maximal element in the collection of all proper prime $\Gamma$- ideals in $S$, we have $A$ is not contained in any proper prime $\Gamma$- ideal. So $\sqrt{A} = S$. Since $S$ is a U-$\Gamma$-semigroup, $A = S$. This is a contradiction. Therefore $P$ is a maximal $\Gamma$- ideal in $S$.

Definition 3.6: A $\Gamma$-semigroup $S$ is said to have dimension $n$ or $n$ – dimensional if there exist a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of prime (proper) $\Gamma$- ideals in $S$, but no such a chain of $n+2$ proper prime $\Gamma$- ideals exists in $S$.

Theorem 3.7: If $A$ is a proper $\Gamma$- ideal in the finite dimensional U-$\Gamma$-semigroup $S$, then $A$ is contained in a maximal $\Gamma$-ideal.

Proof: By theorem 3.4, $A$ is contained in a proper prime $\Gamma$- ideal $P_0$. If $P_0$ is not a maximal $\Gamma$- ideal, then by theorem 3.5, there exists a proper prime $\Gamma$- ideal $P$ such that $P_0 \subset P$. If $P_1$ is maximal we are through. Otherwise $P_1$ is properly contained in a proper prime $\Gamma$- ideal $P_2$ in $S$. The process of choosing $P_1$’s must cease in a finite number of steps because of the finite dimensionality of $S$. Hence $A$ is contained in a maximal $\Gamma$-ideal.

Note 3.8: In a commutative ring, it is proved that every finite dimensional $\psi$ -ring is a union of maximal $\Gamma$- ideals. But in $\Gamma$-semigroups this is not true, as the $\Gamma$- semigroup $S$ in example 3.2 is a finite dimensional U-$\Gamma$-semigroup with the unique maximal $\Gamma$- ideal $\{a, b, c\}$.

Definition 3.9: A $\Gamma$- semigroup $S$ is said to be V- $\Gamma$- semigroup provided for any element $a \in S$, $\sqrt{\langle a \rangle} = S$ implies $\langle a \rangle = S$.

Note 3.10: Every U-$\Gamma$-semigroup is a V-$\Gamma$-semigroup. But a V-$\Gamma$-semigroup is not necessarily a U-$\Gamma$-semigroup.

Example 3.11: Let $S$ be the $\Gamma$-semigroup of all natural numbers greater than 1, under usual multiplication. The $\Gamma$-ideal $A = \{3, 4, \ldots\}$ is not contained in any proper prime $\Gamma$-ideal and hence by theorem 3.4, $S$ is not a U-$\Gamma$-semigroup. Clearly every principal $\Gamma$-ideal is contained in a proper prime $\Gamma$-ideal. So $S$ is a V-$\Gamma$-semigroup.

Theorem 3.12: If $S$ is a globally idempotent $\Gamma$-semigroup with maximal $\Gamma$-ideals, then either $S$ is a V-$\Gamma$-semigroup or $S$ has a unique maximal $\Gamma$-ideal which is prime.

Proof: Let $T = \{a \in S: \sqrt{\langle a \rangle} \neq S\}$. If $T = \emptyset$, then for every $a \in S$, $\sqrt{\langle a \rangle} = S$ and so $S$ has no proper prime $\Gamma$-ideals. But maximal $\Gamma$-ideals are prime. Hence this case is inadmissible. Clearly $T$ is a $\Gamma$-ideal in $S$. If $T \neq S$ then $T$ is the unique maximal $\Gamma$-ideal. Since $S = S \setminus T$, $M$ is a prime $\Gamma$-ideal and so $M = M$. Now if $a \in M \setminus T$ then $\sqrt{\langle a \rangle} \subseteq \sqrt{M} = M$. Thus $M \subseteq T$ and so $M = T$. Then only other possibility is $T = S$, in which case $S$ is a V-$\Gamma$-semigroup.

Note 3.13: It is clear that a $\Gamma$-semigroup $S$ is globally idempotent if and only if maximal $\Gamma$-ideals in $S$ is prime. So if a $\Gamma$-semigroup $S$ contains unique maximal $\Gamma$-ideal which is prime, then $S$ is globally idempotent. But from the example 3.11, we remark that there are V-$\Gamma$-semigroups containing maximal $\Gamma$-ideals which are not globally idempotent.

Theorem 3.14: A $\Gamma$-semigroup $S$ is a V-$\Gamma$-semigroup if and only if $S$ has atmost one proper prime $\Gamma$-ideal and if $\{P_{\alpha}\}$ is the family of all proper prime $\Gamma$-ideals then $\langle a \rangle = S$ for $x \in S \setminus \cup P_{\alpha}$ or $S$ is a simple $\Gamma$-semigroup.

Proof: Let $S$ be a V- $\Gamma$-semigroup which is not a simple $\Gamma$-semigroup. If $S$ has no proper prime $\Gamma$-ideals, then $\sqrt{\langle a \rangle} = S$ for $a \in S$. This implies $\langle a \rangle = S$ and hence $S$ is a simple $\Gamma$-semigroup. So assume $S$ has proper prime $\Gamma$-ideals. Then for any $a \in S \setminus \cup P_{\alpha}$, $\sqrt{\langle a \rangle} = S$, since $a$ does not belong to any proper prime $\Gamma$-ideals. Then $\langle a \rangle = S$. Conversely let ‘$a$’ be any element of $S$ such that $\langle a \rangle \neq S$. If $a \in S \setminus \cup P_{\alpha}$ then $\langle a \rangle = S$. So $a \in \cup P_{\alpha}$ and hence $\sqrt{\langle a \rangle} \neq S$. Therefore $S$ is a V- $\Gamma$-semigroup.
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