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U-Γ-SEMIGROUPS AND V- Γ-SEMIGROUPS

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ABSTRACT

In this paper, the terms, Maximal Γ-ideal, primary Γ-semigroup, prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup are introduced. It is proved that Γ-semigroup S is a U- Γ-semigroup if either S has a left (right) identity or S is generated by a Γ-idempotent. Also it is proved that a Γ-semigroup S is a U- Γ-semigroup if and only if every proper Γ-ideal is contained in a proper prime Γ-ideal. Also it is proved that if A is a proper Γ-ideal in the finite dimensional U- Γ-semigroup S, then A is contained in maximal Γ-ideal and also it is proved that if S is a globally idempotent Γ-semigroup with maximal Γ-ideals, then either S is aV- Γ-semigroup or S has a unique maximal Γ-ideal which is prime.

Mathematical subject classification (2010): *20M07; 20M11; 20M12.*

Keywords: *Γ-semigroup, Maximal Γ-ideal, primary Γ-semigroup, commutative Γ-semigroup, left (right) identity, identity, Zero element, Prime Γ-ideal, simple Γ-semigroup, U- Γ-semigroup and V- Γ-semigroup.*

1. INTRODUCTION

Γ- semigroup was introduced by Sen and Saha [8] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals and radicals in semigroups. Giri and Wazalwar [4] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [5], [6] initiated the study of prime radicals and primary and semiprimary Γ-ideals in Γ-semigroups. In this paper we introduce the notions of U- Γ-semigroups and V- Γ-semigroups in the class of arbitrary Γ-semigroups. We study prime Γ-ideals and maximal Γ-ideals in a U- Γ-semigroup and we characterize V- Γ-semigroups.

2. PRELIMINARIES

Definition 2.1: Let S and Γ be any two non-empty sets. Then S is said to be a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \gamma, b) \to a\gamma b$ satisfying the condition: $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Note 2.2: Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma\}$ by AΓB.

Definition 2.3: A Γ-semigroup S is said to be ***commutative* Γ-semigroup** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

Note 2.4: If S is a commutative Γ-semigroup then $a\Gamma b = b\Gamma a$ for all $a, b \in S$.

Note 2.5: Let S be a Γ-semigroup and $a, b \in S$ and $\alpha \in \Gamma$. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2 b$ and consequently $a\,\alpha\, a\,\alpha\, a\,\alpha \ldots (n \text{ terms}) b$ is denoted by $(a\alpha)^n b$.

Definition 2.6: A Γ-semigroup S is said to be ***quasi commutative*** provided for each $a, b \in S$, there exists a natural number n such that $a\gamma b = (b\gamma)^n a \;\forall \gamma \in \Gamma$.

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Note 2.7: If a Γ-semigroup S is ***quasi commutative*** then for each a, $b \in$ S, there exists a natural number n such that, $a\Gamma b = (b\,\Gamma)^n a$.

Definition 2.8: An element ***a*** of a Γ- semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in$ S and $\alpha \in \Gamma$.

Definition 2.9: **A**n element ***a*** of a Γ- semigroup S is said to be a ***righ tidentity*** of S provided $s\alpha a = s$ for all $s \in$ S and $\alpha \in \Gamma$.

Definition 2.10: An element ***a*** of a Γ- semigroup S is said to be a ***two sided identity*** or an identity provided it is both a left identity and a right identity of S.

Notation 2.11: Let S be a Γ- semigroup. If S has an identity , let S^1 = S and if S does not have an identity, let S^1 be the Γ- semigroup S with identity adjoined, usually denoted by the symbol 1.

Definition 2.12: A non empty subset A of a Γ-semigroup S is said to be a ***left*** **Γ**-***ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

Note 2.13: A nonempty subset A of a Γ-semigroup S is a ***left Γ- ideal*** of S iff S Γ A ⊆ A.

Definition 2.14: A nonempty subset A of a Γ-semigroup S is said to be a ***right*** **Γ**-***ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

Note 2.15: A nonempty subset A of a Γ-semigroup S is a ***right Γ- ideal*** of S iff AΓS ⊆ A.

Definition 2.16: A nonempty subset A of a Γ-semigroup S is said to be a ***two sided*** **Γ**- ***ideal*** or simply a **Γ**- ***ideal*** of S if $s \in$ S, $a \in$ A, $\alpha \in \Gamma$ imply $s\alpha a \in$ A, $a\alpha s \in$ A.

Definition 2.17: A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal*** **Γ**-***ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

Definition 2.18: A Γ- ideal P of a Γ-semigroup S is said to be a ***prime*** **Γ**- ***ideal*** provided A, B are two Γ-ideals of S and AΓB ⊆ P ⇒ either A ⊆ P or B ⊆ P.

Definition 2.19: A Γ- ideal A of a Γ-semigroup S is said to be a ***semi prime*** **Γ**- ***ideal*** provided $x \in$ S, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in$ A .

Definition 2.20: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime*** **Γ**-***radical*** or simply **Γ**-***radical*** of A and it is denoted by **√A** or ***rad A***.

Theorem 2.21[5]: If A is a **Γ**-ideal of a **Γ**-semigroup S then √A is a semi prime **Γ**-ideal of S.

Theorem 2.22[5]: A Γ- ideal Q of Γ-semigroup S is a semi prime Γ-ideal of S iff √(Q) = (Q) implies $x\,\Gamma S^1\Gamma y \subseteq$ A.

Definition 2.23: A Γ-ideal A of a Γ- semigroup S is said to be a ***left primary*** **Γ**-***ideal*** provided

1) If X, Y are two Γ-ideals of S such that X ΓY ⊆ A and Y ⊈ A then X ⊆ √A.
2) √A is a prime Γ-ideal of S.

Definition 2.24: A Γ-ideal A of a Γ- semigroup S is said to be a ***right primary*** **Γ**-***ideal*** provided

1) If X, Y are two Γ-ideals of S such that X ΓY ⊆ A and X ⊈A then Y ⊆ √A.
2) √A is a prime Γ-ideal of S.

Example 2.25: Let S = {a, b, c} and Γ = {x, y, z}. Define a binary operation. in S as shown in the following table.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a maping S XΓ X S → S by $a\ \alpha\ b = ab$, for all a, $b \in$ S and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup.

Now consider the Γ-ideal $<a> = S^1 \Gamma a \Gamma S^1 = \{a\}$. Let $p \Gamma q \subseteq <a>$, $p \notin <a> \Rightarrow q \in \sqrt{<a>} \Rightarrow (q\Gamma)^{n-1}q \subseteq <a>$ for some $n \in N$. Since $b \Gamma c \subseteq <a>$, $c \notin <a> \Rightarrow b \in <a>$. Therefore $<a>$ is left primary. If $b \notin <a>$ then $(c\Gamma)^{n-1}c \notin <a>$ for any $n \in N \Rightarrow c \notin \sqrt{<a>}$. Therefore $<a>$ is not right primary.

Definition 2.26: A Γ-ideal A of a Γ- semigroup S is said to be a **primary Γ-ideal** provided A is both left primary Γ-ideal and right primary Γ-ideal.

Definition 2.27: A Γ-ideal A of a Γ- semigroup S is said to be a **principal Γ-ideal** provided A is a Γ-ideal generated by a single element ***a***. It is denoted by $J[a] = <a>$.

Definition 2.28: An element ***a*** of a Γ-semigroup S with 1 is said to be ***left invertible*** or ***left unit*** provided there is an element $b \in S$ such that $b \Gamma a = 1$.

Definition 2.29: An element ***a*** of a Γ-semigroup S with 1 is said to be ***right invertible*** or ***right unit*** provided there is an element $b \in S$ such that $a\Gamma b = 1$.

Definition 2.30: An element ***a*** of a Γ-semigroup S is said to be ***invertible*** or a ***Unit*** in S provided it is both left and right invertible element in S.

Definitoin 2.31: A Γ- semigroup S is said to be a ***simple* Γ- *semigroup*** provided S has no proper Γ- ideals.

Definition 2.32: An element ***a*** of a Γ- semigroup S is said to be a **Γ-*idempotent*** provided $a \propto a = a$ for all $\propto \in \Gamma$.

Note 2.33: If an element***a*** of a Γ- semigroup S is a Γ-**idempotent**, then $a \Gamma a = a$.

Definition 2.34: A Γ- semigroup S is said to be an **idempotent Γ- semigroup** or a **band** provided every element in S is a Γ-idempotent.

Definition 2.35: A Γ- semigroup S is said to be a **globally idempotent Γ- semigroup** provided $S \Gamma S = S$**.**

3). U-Γ-SEMIGROUPS AND V-Γ-SEMIGROUPS

Definition 3.1: A Γ- semigroup S is said to be U-Γ-semigroup, provided for any Γ-ideal A in S, $\sqrt{A} = S$ implies $A = S$.

Example 3.2: Let S is a Γ-semigroup with S = Γ under the multiplication given in the following table. ($S \times \Gamma \times S \longrightarrow S$ as $a\alpha b = ab$)

.	*a*	*b*	*c*	*d*
a	*a*	*a*	*a*	*a*
b	*a*	*a*	*a*	*b*
c	*a*	*a*	*a*	*a*
d	*a*	*a*	*c*	*d*

Since $S = \{a, b, c, d\}$ and S = Γ. Now $<a>$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$ and $\{a, b, c, d\}$ are the Γ-ideals of S.

If $A = <a>$ then $\sqrt{<a>}$ = intersection of all prime Γ- ideals containing $<a> = \{a, b, c\} \cap \{a, b, c, d\} = \{a, b, c\} \neq S$. Similarly $\sqrt{\{a, b\}} = \{a, b, c\} \neq S$. $\sqrt{\{a, b, c\}} = \{a, b, c\} \neq S$, $\sqrt{\{a, c\}}=\{a, b, c\} \neq S$ and if $A = \{a, b, c, d\}$ then $\sqrt{A}=\sqrt{\{a, b, c, d\}}=\{a, b, c, d\}=S$ implies A=S. Therefore $\sqrt{A}= S$ is true for only A=S. Therefore S is U-Γ-semigroup.

Theorem 3.3: A Γ-semigroup S is a U-Γ-semigroup if either S has a left (right) identity or S is generated by a Γ- idempotent.

Proof: Suppose S has a left identity e. Let A be any proper Γ-ideal such that $\sqrt{A}=S$. Since $\sqrt{A} \subseteq \{x \in S: (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\} = S$. So there is a natural number n such that $(e\Gamma)^{n-1} e \subseteq A$ and hence $e \in A$. Thus $S = e \Gamma S \subseteq A$, it is a contradiction. Therefore S is a U- Γ-semigroup. Suppose S is generated by a Γ–idempotent e. As above we can prove that for any Γ- ideal A in S, if $\sqrt{A} = S$, then $e \in A$ and hence $A = S$. So S is a U- Γ-semigroup.

Theorem 3.4: A Γ-semigroup S is a U- Γ-semigroup if and only if every proper Γ- ideal is contained in a proper prime Γ-ideal.

Proof: Suppose S is a U-Γ-semigroup. Let A be any proper Γ- ideal in S. If A is not contained in any proper prime Γ- ideal, then $\sqrt{A} = S$. Since S is a U-Γ-semigroup. We have A = S, this is a contradiction. So every proper Γ-ideal is contained in a proper prime Γ-ideal. Conversely if every proper Γ- ideal is contained in a proper prime Γ-ideal, Then $\sqrt{A} \neq S$ implies $A \neq S$ then clearly S is a U-Γ-semigroup.

Theorem 3.5: Let S be a U-Γ-semigroup. Then S = S Γ S and hence every maximal Γ- ideal is prime.

Conversely if $\{P_\alpha\}$ is the collection of all prime Γ- ideals in S and if P is a maximal element in this collection, then P is a maximal Γ- ideal in S.

Proof: Clearly $\sqrt{S \Gamma S} = S$. Since S is a U-Γ-semigroup, we have S Γ S = S and hence every maximal Γ- ideal is prime. If P is not a maximal Γ- ideal in S, then there exists a proper Γ- ideal A in S, containing P properly. Since P is a maximal element in the collection of all proper prime Γ- ideals in S, we have A is not contained in any proper prime Γ- ideal. So $\sqrt{A} = S$. Since S is a U-Γ-semigroup, A = S. This is a contradiction. Therefore P is a maximal Γ- ideal in S.

Definition 3.6: A Γ-semigroup S is said to have dimension n or n – dimensional if there exist a strictly ascending chain $P_0 \subset P_1 \subset P_2 \subset \ldots\ldots\subset P_n$ of prime (proper) Γ- ideals in S, but no such a chain of n+2 proper prime Γ- ideals exists in S.

Theorem 3.7: If A is a proper Γ- ideal in the finite dimensional U-Γ-semigroup S, then A is contained in a maximal Γ-ideal.

Proof: By theorem 3.4, A is contained in a proper prime Γ- ideal P_0, If P_0 is not a maximal Γ- ideal, then by theorem 3.5, there exists a proper prime Γ- ideal P such that $P_0 \subset P_1$. If P_1 is maximal we are through. Otherwise P_1 is properly contained in a proper prime Γ- ideal P_2 in S. The process of choosing P_i's must cease in a finite number of steps because of the finite dimensionality of S. Hence A is contained in a maximal Γ- ideal.

Note 3.8: In a commutative ring, it is proved that every finite dimentional v -ring is a union of maximal Γ- ideals. But in Γ- semigroups this is not true, as the Γ- semigroup S in example 3.2 is a finite dimensional U-Γ-semigroup with the unique maximal Γ- ideal {a, b, c}.

Definition 3.9: A Γ- semigroup S is said to be V- Γ- semigroup provided for any element $a \in S$, $\sqrt{<a>} = S$ implies $<a> = S$.

Note 3.10: Every U-Γ-semigroup is a V- Γ-semigroup. But a V- Γ-semigroup is not necessarily a U- Γ-semigroup.

Example 3.11: Let S be the Γ-semigroup of all natural numbers greater than 1, under usual multiplication. The Γ-ideal A = {3, 4…..} is not contained in any proper prime Γ-ideal and hence by theorem 3.4, S is not a U-Γ-semigroup. Clearly every principal Γ-ideal is contained in a proper prime Γ-ideal. So S is a V- Γ-semigroup.

Theorem 3.12: If S is a globally idempotent Γ-semigroup with maximal Γ-ideals, then either S is a V-Γ-semigroup or S has a unique maximal Γ-ideal which is prime.

Proof: Let $T = \{a \in S: \sqrt{<a>} \neq S\}$ If $T = \phi$, then for every $a \in S$, $\sqrt{<a>} = S$ and so S has no proper prime Γ-ideals. But maximal Γ-ideals are prime. Hence this case is inadmissible. Clearly T is a Γ-ideal in S. If $T \neq S$ then T is the unique maximal Γ-ideal. Since S = S ΓS, M is a prime Γ-ideal and so $\sqrt{M} = M$. Now if $a \in M\backslash T$ then $S = \sqrt{<a>} \subseteq \sqrt{M} = M$. Thus $M \subseteq T$ and so M = T. Then only other possibility is T = S, in which case S is a V- Γ-semigroup.

Note 3.13: It is clear that a Γ-semigroup S is globally idempotent if and only if maximal Γ-ideals in S is prime. So if a Γ-semigroup S contains unique maximal Γ-ideal which is prime, then S is globally idempotent. But from the example 3.11, we remark that there are V- Γ-semigroups containing maximal Γ-ideals which are not globally idempotent.

Theorem 3.14: A Γ-semigroup S is a V- Γ-semigroup if and only if S has atmost one proper prime Γ-ideal and if $\{P_\alpha\}$ is the family of all proper prime Γ-ideals then $<x> = S$ for $x \in S\backslash\cup P_\alpha$ or S is a simple Γ-semigroup.

Proof: Let S be a V- Γ-semigroup which is not a simple Γ-semigroup. If S has no proper prime Γ-ideals, then $\sqrt{<a>} = S$ for $a \in S$. This implies $<a> = S$ and hence S is a simple Γ-semigroup. So assume S has proper prime Γ-ideals. Then for any $a \in S\backslash\cup P_{\alpha,}$ $\sqrt{<a>} = S$, since a does not belong to any proper prime Γ-ideals. Then $<a> = S$. Conversly let 'a' be any element of S such that $<a> \neq S$. If $a \in S\backslash\cup P_{\alpha,}$ then $<a> = S$. So $a \in \cup P_\alpha$ and hence $\sqrt{<a>} \neq S$. Therefore S is a V- Γ-semigroup.

REFERENCES

1. Anjaneyulu. A, and Ramakotaiah. D., *On a class of semigroups*, Simon stevin, Vol.54 (1980), 241-249.
2. Anjaneyulu. A., *Structure and ideal theory of Duo semigroups*, Semigroup Forum, Vol.22 (1981), 257-276.
3. Anjaneyulu. A., *Semigroup in which Prime Ideals are maximal*, Semigroup Forum, Vol.22 (1981), 151-158.
4. Giri. R. D. and Wazalwar. A. K., *Prime ideals and prime radicals in non- commutative semigroup*, Kyungpook Mathematical Journal Vol.33 (1993), no.1, 37-48.
5. Madhusudhana Rao. D, Anjaneyulu. A and Gangadhara Rao. A, *Prime Γ-radicals in Γ-semigroups*, International e-Journal of Mathematics and Engineering 138(2011) 1250 - 1259.
6. Madhusudhana Rao. D, Anjaneyulu. A and Gangadhara Rao.A, *Primary and Semiprimary Γ-ideals in Γ-semigroup*, International Journal of Mathematical Sciences, Technology and Humanities 29 (2012) 282-293.
7. Petrich. M., *Introduction to semigroups*, Merril Publishing Company, Columbus, Ohio,(973).
8. Sen. M.K. and Saha. N.K., *On* Γ- *Semigroups-I*, Bull. Calcutta Math. Soc. 78(1986), No.3, 180-186.
9. Sen. M.K. and Saha. N.K., *On* Γ- *Semigroups-II*, Bull. Calcutta Math. Soc. 79(1987), No.6, 331-335
10. Sen. M.K. and Saha. N.K., *On* Γ- *Semigroups-III*, Bull. Calcutta Math. Soc. 8o (1988), No.1, 1-12.

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