AFFINE TRANSFORMATIONS AND ISOMETRIES
IN A COMPLETE RIEMANNIAN MANIFOLD

K. C. Petwal* and Shikha Uniyal

Department of Mathematics, H. N. B. Garhwal University campus, Badshahi Thaul,
Tehri Garhwal-249199, Uttarakhand (India)

E-mail: kcpatwal@gmail.com

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ABSTRACT

The groups of affine transformations of an affinely connected manifold were studied by Nomizu and also Han-Morimato. Further, Myers and Steenrod gave the theory of group of isometries of Riemannian manifold. In the present study we describe certain aspects of a complete Riemannian manifold. We have investigated that in a complete irreducible Riemannian manifolds the group of all affine transformations and the group of all isometries are equal. Further, if \( X \) be an infinitesimal affine transformation on a complete Riemannian manifold \( M \), then \( X \) is an infinitesimal isometry. Also, we have related the affine transformations on a complete Riemannian manifold to the conditions of isometry.

Key words: Complete, Riemannian, affine transformation, isometry.


1. INTRODUCTION

If \( M \) is a differentiable Riemannian manifold with a fundamental metric tensor field \( G \) which is positive definite, for any vector field \( X \) we denote by \( \nabla (X) \) the covariant differentiation in the direction of \( X \) with respect to the Riemannian connection.

Now let \( M_1 \) and \( M_2 \) be two Riemannian manifolds with \( G_1 \) and \( G_2 \) as their fundamental metric tensor fields and \( \nabla_1(X_1) \) and \( \nabla_2(X_2) \) are the corresponding covariant differentiations respectively. Let \( \varphi \) be a differentiate homeomorphism of \( M_1 \) onto \( M_2 \). If \( \varphi \) commutes with the covariant differentiations i. e. for any vector field \( X \) on \( M_1 \) such that:

\[
\varphi (\nabla_1 X) = \nabla_2 (\varphi X) \varphi,
\]

then \( \varphi \) is called an affine transformation. If we have \( \varphi G_1 = \rho G_2 \), then \( \varphi \) is said to be an isometric transformation or an isometry. If for some real constant \( \rho > 0 \) we have \( \varphi G_1 = \rho G_2 \), \( \varphi \) is called a homothetic transformation [1].

A manifold with an affine connection (or a Riemannian manifold) \( M \) is complete if every geodesic curve can be extended for any large value of the canonical parameter. When the completeness is satisfied on \( M \), any infinitesimal affine transformation (or a Killing vector field) generates a one-parameter group of affine transformations from \( M \) onto itself [2].

Let \( M \) be a manifold with an affine connection. The group \( A(M) \) of all affine transformations of \( M \) onto itself is a Lie group with respect to the compact-open topology ([3], [4], [7]). When \( M \) has a Riemannian metric, the group \( I(M) \) of all isometries of \( M \) onto itself is a closed subgroup of \( A(M) \). \( I(M) \) is also a Lie group [5]. The mapping from \( A(M) \times M \) onto \( M \) gives the transformation law which is differentiable, as is known from a theorem of S. Bochner and D. Montgomery [3]. Any one-parameter subgroup in \( A(M) \) (i.e. \( I(M) \)) induces an infinitesimal affine transformation (i.e. a Killing vector field) on \( M \).

As \( \varphi \) is an affine transformation, the two Riemannian metrics \( g \) and \( g^\ast \) determine the same Riemannian connection, say \( L \). Let \( \varphi(x) \) be the linear holonomy group of \( L \) with reference point \( x \). Since it is irreducible and leaves both \( g \) and \( g^\ast \) invariant, there exists a positive constant \( c_x \) such that \( g^\ast(X,Y) = c_x^2 g(X,Y) \) for all \( X, Y \in T_x(M) \), i.e., \( g^\ast_x = c_x^2 g_x \). Since both \( g \) and \( g^\ast \) are parallel tensor fields with respect to \( t, c_x \) is constant.

*Corresponding author: K. C. Petwal*, *E-mail: kcpatwal@gmail.com*
Let us assume that $\varphi$ is non-isometric homothetic transformation of $M$. Considering the inverse transformation if necessary, we may assume that the constant $c$ associated with $\varphi$ is less than $1$. Take an arbitrary point $x$ of $M$. If the distance between $x$ and $\varphi(x)$ is less than $\delta$, then the distance between $\varphi^m(x)$ and $\varphi^{m+1}(x)$ is less than $c^m\delta$. It follows that $\{\varphi^m(x); m=1, 2, \ldots\}$ is a Cauchy sequence and hence converges to some point $x^*$, since $M$ is complete. We may further conclude at the following two definitions:

**Definition (1.1):** If $M$ is an irreducible Riemannian manifold, then every affine transformation $\varphi$ of $M$ homothetic [2].

**Definition (1.2):** If $M$ is a complete Riemannian manifold which is not locally Euclidian of $M$ is an isometry [2].

### 2. AFFINE TRANSFORMATION AND ISOMETRIES IN A COMPLETE RIEMANNIAN MANIFOLD

In the Euclidian space of three dimensions the distance $ds$ between adjacent points whose rectangular cartesian coordinates are $(x, y, z)$ and $(x + dx, y + dy, z + dz)$ is given by $ds^2 = dx^2 + dy^2 + dz^2$. More generally for any system of oblique curvilinear coordinates $(u, v, w)$ we have, $ds^2 = du^2 + dv^2 + dw^2 + 2dwdv + 2gdwdy + 2hdudy$, where $a, b, c, f, g, h$ are the function of coordinates. This idea was generalized and extended to space of $n$-dimensions of Riemannian, who had defined the infinitesimal distance $ds$ between the adjacent points whose coordinates in any system are $x^1$ and $x^1 + dx^1$, $(i = 1, 2, \ldots, n)$, by the relation $ds^2 = g_{ij}dx^1dx^j$, where the coefficient $g_{ij}$ are the function of the coordinates $x^1$. This quadratic differential form is called a Riemannian metric and a space which is characterized by such a metric is called Riemannian manifold. A Riemannian manifold or a Riemannian metric $g$ on $M$ is said to be complete if the Riemannian connection defined by $g$ coincides with $g$. This means that every homothetic transformation of $M$ homothetic [2].

For a connect Riemannian manifold the following conditions are mutually equivalent:

1. $M$ is a complete Riemannian manifold.
2. $M$ is a complete metric space with respect to the distance function $d$.
3. Every bounded subset of $M$ is relatively compact.
4. For an arbitrary point $x$ of $M$ and for an arbitrary curve $C$ in tangent space $T_x(M)$ starting from $x$ which is developed upon the given curve $C$.

**Definition:** If $M$ is a connected complete Riemannian manifold, then any two points $x$ and $y$ of $M$ can be joined by a minimizing geodesic.

**Theorem (2.1):** If $M$ is a complete, irreducible Riemannian manifold of dim $n$ $>1$, then the group of all affine transformations and the group of all isometries are equal.

**Proof:** A transformation $\varphi$ of a Riemannian manifold is said to be homothetic, if there is a positive constant $c$ such that $g(\varphi(X), \varphi(Y)) = c^2g(X, Y)$ for all $X, Y \in T_x(M)$ Where $T_x(M)$ is a tangent space of $M$ at $x$, and $x \in M$, consider the Riemannian metric $g$ define by $g(X, Y) = g(\varphi(X), \varphi(Y))$. By the [2] the Riemannian connection defined by $g$ coincides with $g$. This means that every homothetic transformation of a Riemannian manifold $M$ is an affine transformation of $M$ [def 1.1, def 1.2]. Let $U$ be a neighborhood of $x^*$ such that $\overline{U}$ is compact. Let $K^*$ be a positive number such that $|g(R(Y_1, Y_2) Y_1, Y_2)| < K$ for any unit vector $Y_1$ and $Y_2$ at $Y \in U$, where $R$ denotes the curvature tensor field. Let $z \in M$ and $q$ any plane in $T_z(M)$. Let $X, Y$ be an orthonormal basis for $q$, since $\varphi$ is an affine transformation and let $f: M \rightarrow M'$ be an affine mapping and $X, Y, Z$ the vector fields on $M$ which are $f$-related to vector fields $X', Y', Z'$ on $M'$ respectively, then $R(X, Y) Z$ is $f$-related to $R'(X', Y') Z'$. Here $R$ and $R'$ are the curvature tensor fields of $M$ and $M'$ respectively, which implies

$$R'(\varphi^m X', \varphi^m Y')(\varphi^m Y) = \varphi^m R(X, Y) Y).$$

Hence we have,

$$g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y, \varphi^m X) = g(\varphi^m(R(X, Y) Y), \varphi^m X)$$

$$= c^{2m} g(R(X, Y) Y, X)$$

$$= c^{2m} K(q).$$

Also, the distance between $x^* = \varphi^m(x^*)$ and $\varphi^m(z)$ approaches 0 as $m$ tends to infinity. Thus, there exists an integer $m_0$ such that $\varphi^m(z) \in U$ for every $m \geq m_0$. Since the lengths of the vectors $\varphi^m X$ and $\varphi^m Y$ are equal to $c^m$, we have

$$c^{2m} K \geq |g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y), \varphi^m X)| \text{ for } m \geq m_0.$$
This implies
\[ c^{2m}K \geq |K(q)| \text{ for } m \geq m_0 \]

Provided \( m \) tends to infinity, we have \( K(q) = 0 \). In view of def (1.1) and def (1.2) and the above result, we say that if \( M \) is a complete, irreducible Riemannian manifold of \( \dim n \geq 1 \), then group of all affine transformations is equal to all group of all isometries.

**Theorem (2.2):** If \( X \) be an infinitesimal affine transformation on a complete Riemannian manifold \( M \), then \( X \) is an infinitesimal isometry.

**Proof:** Consider \( X \) an infinitesimal affine transformation on a complete Riemannian manifold \( M \). To prove above theorem, we need the theorem (2.1) and the following theorem:

Let \( M= M_0 \times M_1 \times \ldots \times M_k \) be the de Rham decomposition of a complete, simply connected Riemannian manifold \( M \), then

\[ \mathfrak{g}(M) = \mathfrak{g}(M_0) \times \mathfrak{g}(M_1) \times \ldots \times \mathfrak{g}(M_k), \]

\[ \mathfrak{a}(M) = \mathfrak{a}(M_0) \times \mathfrak{a}(M_1) \times \ldots \times \mathfrak{a}(M_k), \]

where \( \mathfrak{a}(M) \) is a closed subgroup of the group of all affine transformations \( \mathfrak{a}(M) \), while \( \mathfrak{g}(M) \) and \( \mathfrak{a}(M) \) are their respective identity components.

Considering that \( M \) is connected, let \( M^* \) be the universal covering manifold with the naturally induced Riemannian metric \( g^* = p^*(g) \), where \( p: M^* \to M \) is natural projection.

Let \( X^* \) be the vector field on \( M^* \) induced by \( X \) and \( X^* \) is \( p \)-related to \( X \). Then \( X^* \) is an infinitesimal affine transformation of \( M^* \). Clearly \( X^* \) is an infinitesimal affine transformation of \( M^* \) and \( X \) is an infinitesimal isometry of \( M^* \) if and only if \( X \) is an infinitesimal isometry of \( M \).

**Corollary (2.1):** If \( M \) is connected, complete Riemannian manifold whose restricted linear holonomy group leaves no any non-zero vector at fix point, then group of all affine transformation is equal to group of all isometries.

**Proof:** The linear holonomy group of \( M \) is naturally isomorphic with the restricted linear holonomy group of \( M \). This means that \( M_0 \) reduces to a point and hence \( X_0 = 0 \) in the above corollary.

**Corollary (2.2):** If \( X \) is an infinitesimal affine transformation of a complete Riemannian manifold and if the length of \( X \) is bounded, then \( X \) is an infinitesimal isometry.

**Proof:** Let \( M \) to be connected. If the length of \( X \) is bounded on \( M \), the length of \( X_0 \) is also bounded on \( M_0 \). Let \( x^1, x^2, \ldots, x^r \) be the Euclidean coordinate system in \( M_0 \) and set,

\[ X_0 = \sum_{a=1}^r \xi^a \frac{\partial}{\partial x^a}. \]

Applying the formula

\[ (L_{X_0} \nabla_y - \nabla_y L_{X_0})Z = \nabla_{[X_0,y]}Z \]

to

\[ y = \frac{\partial}{\partial x^a} \quad \text{and} \quad Z = \frac{\partial}{\partial x^a}, \]

we see that

\[ \frac{\partial^2 \xi^a}{\partial x^a \partial x^r} = 0. \]

This means that \( X_0 \) is of the form

\[ \sum_{a=1}^r \left( \sum_{\beta=1}^r \omega_{\beta a} x^\beta + b^a \right) \frac{\partial}{\partial x^a}. \]
It is easy to see that length of \( X_0 \) is bounded on \( M_0 \) if and only if \( a_{\alpha\beta}^0 = 0 \) for \( \alpha, \beta = 1, \ldots, r \). Thus if \( X_0 \) is of bounded length, then \( X_0 \) is an infinitesimal isometry of \( M_0 \) ([2], [4]).

**Corollary (2.3):** On a compact Riemannian manifold \( M \), we have \( \Psi^0(M) = \mathcal{Y}^0(M) \).

**Proof:** On a compact manifold \( M \), every vector field is of bounded length. By Corollary (2.3), every infinitesimal affine transformation \( X \) is an infinitesimal isometry.

**REFERENCES**


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