FIXED POINT THEOREM IN MENER PROBABILISTIC METRIC SPACE

PIYUSH KUMAR TRIPATHI*

Amity University, Uttar Pradesh, India.
(Amity School of Applied Sciences, Lucknow Campus)

(Received On: 13-09-17; Revised & Accepted On: 09-10-17)

ABSTRACT

The Banach fixed point theorem guarantees the existence of unique fixed point under a contraction mapping on a complete metric space. A similar theorem does not hold in a complete Menger Probabilistic metric space. The problem is that the triangular function in such spaces is not enough to guarantee that the sequence of iterates of a point under a certain map is Cauchy sequence. Two different approaches have been pursued. One is to identify those triangle functions which guarantee that the sequence of iterates is a Cauchy sequence. The other is to modify the original definition of contraction map. First this was done by Hicks. In this paper I prove some fixed point in Menger space.

2. INTRODUCTION

Menger [2] generalized the metric axioms by associating a distribution function with each pair of points of an abstract set $X$. A distribution function is a mapping $f: R \rightarrow R^+$ which is non-decreasing, left continuous, with $\inf f = 0$ and $\sup f = 1$. Thus for any ordered pair of points $p, q$ of $X$, we associate a distribution function denoted by $F_{p,q}$ and, for any positive number $x$, we interpret $F_{p,q}(x)$ as the probability that the distance between $p$ and $q$ is less than $x$. This gives rise to a new theory of ‘probabilistic metric spaces’ which started developing rapidly after the publication of the paper of Schweizer and Sklar [5].

PROBABILISTIC METRIC SPACES [2]

**Definition 2.1:** A mapping $f: R \rightarrow R^+$ is called a distribution function if it is non-decreasing, left continuous and $\inf f(x) = 0$, $\sup f(x) = 1$.

We shall denote by $L$ the set of all distribution functions. The specific distribution function $H \in L$ is defined by

$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$

**Definition 2.2:** A probabilistic metric space (PM space) is an ordered pair, $(X, t)$ is a nonempty set and $F: X \times X \rightarrow L$ is mapping such that, by denoting $F(p,q)$ by $F_{p,q}$ for all $p, q$ in $X$, we have

(I) $F_{p,q}(x) = 1$ $\forall x > 0$ iff $p = q$

(II) $F_{p,q}(0) = 0$

(III) $F_{p,q} = F_{q,p}$

(IV) $F_{p,q}(x) = 1$, $F_{p,q}(y) = 1 \Rightarrow F_{p,q}(x+y) = 1$

We note that $F_{p,q}(x)$ is value of the distribution function $F_{p,q} = F(p,q) \in L$ at $x \in R$.

**Definition 2.3:** A mapping $t:[0,1] \times [0,1] \rightarrow [0,1]$ is called $t$-norm if it is non- decreasing (by non-decreasing, we mean $a \leq c, b \leq d \Rightarrow t(a,b) \leq t(c,d))$, commutative, associative and $t(a,1) = a$ for all $a$ in $[0,1]$, $t(0,0) = 0$.

*Corresponding Author: Piyush Kumar Tripathi*

Amity University, Uttar Pradesh, India. (Amity School of Applied Sciences, Lucknow Campus).
**Definition 2.4:** A Menger PM space is a triple \((X, F, t)\) where \((X, F)\) is a PM space and \(t\) is \(t\)-norm such that,

\[
F_{p,q}(x+y) \geq t\left(F_{p,q}(x), F_{q,p}(y)\right) \quad \forall \ x, y \geq 0.
\]

If \((X, F, t)\) is Menger Probabilistic metric space with \(\sup \{t(x,y)\} = 1, 0 < x < 1\), then \((X, F, t)\) is a Hausdorff topological space in the topology \(T\) induced by the family of \((\epsilon, \lambda)\) neighborhoods \(\{U_p(x, \epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0\}\) where \(U_p(x, \epsilon, \lambda) = \{x \in X : F_{p,x}(\epsilon) > 1 - \lambda\} \) \([8]\).

**Definition 2.5:** A sequence \(\{p_n\}\) in \(X\) is said to converge to \(p \in X\) iff \(\forall \epsilon > 0\) and \(\forall \lambda > 0\), there exists an integer \(M\) such that \(F_{p,x}(\epsilon) > 1 - \lambda, \forall n \geq M\). Again \(\{p_n\}\) is a Cauchy sequence if \(\forall \epsilon > 0\) and \(\forall \lambda > 0\), there exists an integer \(M\) such that, \(F_{p,x}(\epsilon) > 1 - \lambda\) for all \(m, n \geq M\).

Some common fixed point theorems using sequence which are not necessarily obtained as a sequence of iterates of certain mappings are motivated by a result of Jungck \([1]\). He proved that a continuous self mapping \(f\) of a complete metric space \((X, d)\) has a fixed point provided there exists \(q \in (0, 1)\) and a mapping \(g : X \rightarrow X\) which commute with \(f\) and satisfies

\[
\begin{align*}
(a) & \quad g(X) \subseteq f(X) \\
(b) & \quad d(gx, gy) \leq qd(fx, fy), \quad \forall x, y \in X. \quad \text{Then } g \text{ and } f \text{ have unique common fixed point.}
\end{align*}
\]

In 1960. B. Schweizer and A. Sklar have been studied these spaces in depth. These spaces have also been considered by several other authors. The first result for a contractive self mapping on a Menger PM space was obtained by Sehgal and Bharucha Reid \([3]\). Let \((X, F)\) be PM space and \(f : X \rightarrow X\) be a mapping. Then \(f\) is said to contraction if \(\exists k \in (0, 1)\) such that \(F_{p,q}(\epsilon) > 1 - \lambda, \forall \epsilon > 0\) and \(\forall \lambda > 0\), there exists an integer \(M\) such that,

\[
F_{p,x}(\epsilon) > 1 - \lambda \text{ for all } m, n \geq M.
\]

Recently Piyush Kumar Tripathi \([4]\), \([7]\) defined dual contraction and using it he proved some fixed point theorems.

**2.1 Definition:** Let \((X, F, t)\) be a Menger space. A mapping \(f : X \rightarrow X\) is called dual contraction if \(\exists k > 1\) such that \(F_{p,q}(\epsilon) \leq F_{p,q}(x), x > 0\)

**2.3 Theorem:** Let \((X,F,t)\) be complete Menger probabilistic metric space where \(\sup \{t(x,y)\} = 1, 0 < x < 1\) and \((X, F, t)\) is Hausdorff topological space in the topology \(T\) induced by the family of \((\epsilon, \lambda)\) neighborhoods \(\{U_p(x, \epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0\}\) where \(U_p(x, \epsilon, \lambda) = \{x \in X : F_{p,x}(\epsilon) > 1 - \lambda\} \) \([8]\).

Suppose \(f : X \rightarrow X\) is onto mapping then \(\exists\) a unique fixed point.

**3. MAIN RESULTS**

In this section, I have also prove some fixed point theorems under different contractive conditions using contraction constant \(k > 1\) or \(k < 1\).

**3.1 Theorem:** Let \((X,F,t)\) be a complete Menger probabilistic metric space where \(F_{p,q}\) is strictly increasing distribution function and \(f : X \rightarrow X\) is continuous mapping. If \(\exists k \in (0, 1)\) such that \(F_{f,p}(x, x) \leq F_{p,q}(x), x > 0\),

\[
\begin{align*}
F_{f,p}(x, y) & \geq \min\{F_{p,q}(x, x), F_{p,f}(x, y), F_{q,f}(y, x), F_{q,f}(y, x)\}.
\end{align*}
\]

Then \(\exists\) a unique fixed point.

**Proof:** Let \(p_0 \in X\). Construct a sequence \(p_n = f(p_{n-1})\), \(n = 1, 2, 3 \ldots \). Then

\[
F_{p_0, p_0}(x) = F_{f(p_0-1), f(p_0)}(x) \geq \min\{F_{p_0, p_0}(x), F_{p_0, p_0}(x), F_{p_0, p_0}(x), F_{p_0, p_0}(x)\}
\]

i.e. \(F_{p_0, p_0}(x) \geq \min\{F_{p_0, p_0}(x), F_{p_0, p_0}(x)\}\)

\[
F_{p_0, p_0}(x) \geq F_{p_0, p_0}(x) , x > 0
\]
Therefore by lemma 2.1 \{p_n\} is a Cauchy sequence. Since \((X, F, t)\) is complete so \(p_n \to p \in X\). Then by theorem 2.1, \(p\) is a unique fixed point of \(f\). For uniqueness suppose \(f(p) = p, f(q) = q\). Then
\[
F_{p,q}(x) = F_{f(p),g(q)}(x) \geq \min \left\{ F_{p,q}(x), F_{p,p}(x), F_{q,q}(x), F_{q,p}(x) \right\}
\]
i.e. \(F_{p,q}(x) \geq F_{p,q}(x)\).

Which is not possible so \(p = q\). Because \(F_{p,q}\) is strictly increasing function and \(kx < 0\).

3.2 Theorem: Let \((X; F; t)\) be a complete Menger probabilistic metric space where \(F_{p,q}\) strictly increasing distribution function is and \(f, g : X \to X\) is continuous mapping. If \(\exists k \in (0,1)\) such that
\[
F_{f(p),g(q)}(x) \leq \max \left\{ F_{p,q}(x), F_{p,f(p)}(x), F_{q,g(q)}(x) \right\}.
\]
Then \(f\) and \(g\) have a unique common fixed point.

Proof: Let \(p_0 \in X\). Construct a sequence \(\{p_n\}\) defined by \(f(p_{2n}) = p_{2n+1}, g(p_{2n+1}) = p_{2n+2} \), \(n = 1,2,3\). If \(n = 2r + 1\) then
\[
F_{p_{2r+1},p_{2r+2}}(x) \geq \min \left\{ F_{p_{2r+1},p_{2r+2}}(x), F_{p_{2r+1},p_{2r+2}}(x) \right\}
\]
Again if \(n = 2r\) then
\[
F_{p_{2r},p_{2r+1}}(x) = F_{f(p_{2r+1}),g(p_{2r+1})}(x) \leq \max \left\{ F_{p_{2r+1},p_{2r+1}}(x), F_{p_{2r+1},p_{2r+1}}(x), F_{p_{2r+1},p_{2r+1}}(x) \right\}
\]
Therefore by lemma 2.1, \(\{p_n\}\) is a Cauchy sequence. Then \(p_n \to p \in X\). Since \(\{p_{2n+1}\}, \{p_{2n}\}\) is subsequence of \(\{p_n\}\) so \(p_{2n+1} \to p, p_{2n} \to p\). Then \(f(p) = p\) and \(g(p) = p\) that is \(p\) is common fixed point of \(f\) and \(g\). For uniqueness suppose \(p\) and \(q\) are two common fixed-point \(f\) and \(g\). Then,
\[
F_{p,q}(x) = F_{f(p),g(q)}(x) \leq \max \left\{ F_{p,q}(x), F_{p,p}(x), F_{q,q}(x) \right\} \Rightarrow F_{p,q}(x) \geq F_{p,q}(x),
\]
which is not possible because \(F_{p,q}\) is strictly increasing function and \(kx < 0\). Therefore \(f\) and \(g\) have unique common fixed point.

REFERENCES


Source of support: Nil, Conflict of interest: None Declared.

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