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FIXED POINT THEOREM IN MENGER PROBABILISTIC METRIC SPACE

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ABSTRACT

The Banach fixed point theorem guarantees the existence of unique fixed point under a contraction mapping on a complete metric space. A similar theorem does not hold in a complete Menger Probabilistic metric space. The problem is that the triangular function in such spaces is not enough to guarantee that the sequence of iterates of a point under a certain map is Cauchy sequence. Two different approaches have been pursued. One is to identify those triangle functions which guarantee that the sequence of iterates is a Cauchy sequence. The other is to modify the original definition of contraction map. First this was done by Hicks. In this paper I prove some fixed point in Menger space.

2. INTRODUCTION

Menger [2] generalized the metric axioms by associating a distribution function with each pair of points of an abstract set *X*. (A distribution functions is a mapping $f : R \to R^+$ which is non-decreasing, left continuous, with inf $f = 0$ and sup $f = 1$). Thus for any ordered pair of points *p*, *q* of *X*, we associate a distribution function denoted by $F_{p,q}$ and, for any positive number *x*, we interpret $F_{p,q}(x)$ as the probability that the distance between *p* and *q* is less than *x*. This gives rise to a new theory of 'probabilistic metric spaces' which started developing rapidly after the publication of the paper of Schweizer and Sklar [5].

PROBABILISTIC METRIC SPACES [2]

Definition 2.1: A mapping $f : R \to R^+$ is called a distribution function if it is non decreasing, left continuous and inf $f(x) = 0$, sup $f(x) = 1$.

We shall denote by *L* the set of all distribution functions. The specific distribution function $H \in L$ is defined by

$$\left.\begin{aligned} H(x) &= 0, \quad x \le 0 \\ &= 1, \quad x > 0 \end{aligned}\right\}$$

Definition 2.2: A probabilistic metric space (PM space) is an ordered pair, *X* is a nonempty set and $F : X \times X \to L$ is mapping such that, by denoting $F(p,q)$ by $F_{p,q}$ for all *p*, *q* in *X*, we have

(I) $F_{p,q}(x) = 1 \;\; \forall\, x > 0$ iff $p = q$

(II) $F_{p,q}(0) = 0$

(III) $F_{p,q} = F_{q,p}$

(IV) $F_{p,q}(x) = 1$, $F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1$

We note that $F_{p,q}(x)$ is value of the distribution function $F_{p,q} = F(p,q) \in L$ at $x \in R$.

Definition 2.3: A mapping $t : [01] \times [01] \to [01]$ is called t-norm if it is non- decreasing (by non-decreasing, we mean $a \le c, b \le d \Rightarrow t(a,b) \le t(c,d)$), commutative, associative and $t(a,1) = a$ for all *a* in [0, 1], $t(0,0) = 0$.

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Definition 2.4: A Menger PM space is a triple (*X*, *F*; *t*) where (*X*, *F*) is a PM space and *t* is t-norm such that,

$$F_{p,r}(x+y) \geq t\left(F_{p,q}(x), F_{q,r}(y)\right) \quad \forall \quad x, y \geq 0 .$$

If (*X*, *F*; *t*) is Menger Probabilistic metric space with $\sup t\left(x,x\right)=1,\ 0<x<1$, then (*X*, *F; t*) is a Hausdorff topological space in the topology *T* induced by the family of (ε,λ) neighborhoods $\{U_p(\varepsilon,\lambda): p \in X, \in > 0, \lambda > 0\}$ where $U_p(\varepsilon,\lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1-\lambda\}$ ([8]) .

Definition 2.5: A sequence $\{p_n\}$ in *X* is said to converge to $p \in X$ iff $\forall\ \varepsilon > 0$ and $\forall \lambda > 0,$ there exists an integer *M* such that $F_{p_n,p}(\varepsilon) > 1-\lambda\ \forall n \geq M$. Again $\{p_n\}$ is a Cauchy sequence if $\forall \in > 0$ and $\forall \lambda > 0,$ there exists an integer *M* such that,

$$F_{p_n,p_m}(\varepsilon) > 1-\lambda \text{ for all } m, n \geq M.$$

Some common fixed point theorems using sequence which are not necessarily obtained as a sequence of iterates of certain mappings are motivated by a result of Jungck [1]. He proved that a continuous self mapping *f* of a complete metric space (*X*,*d*) has a fixed point provided there exists $q \in (0,1)$ and a mapping $g : X \to X$ which commute with *f* and satisfies

(a) $g(X) \subseteq f(X)$

(b) $d(gx, gy) \leq qd(fx, fy)$, for all $x, y \in X$. Then g and f have unique common fixed point.

In 1960. B. Schweizer and A. Sklar have been studied these spaces in depth. These spaces have also been considered by several other authors. The first result for a contractive self mapping on a Menger PM space was obtained by Sehgal and Bharucha Reid [3]. Let (*X, F*) be PM space and $f : X \to X$ be a mapping. Then *f* is said to contraction if $\exists\ k \in (0\ 1)$ s.t. $\forall\ p, q \in X$, $F_{f(p)f(q)}(kx) \geq F_{pq}(x)$, $x > 0$.

Recently Piyush Kumar Tripathi [6], [7] defined dual contraction and using to it he proved some fixed point theorems.

2.1 Definition: Let (*X, F, t*) be a Menger space. A mapping $f : X \to X$ is called dual contraction if $\exists\ k > 1$ such that $F_{fpfq}(kx) \leq F_{pq}(x)$, $x > 0$

2.3 Theorem: Let (*X*,*F*,*t*) be complete Menger space. Suppose $f : X \to X$ is onto and continuous mapping satisfying the condition of dual contraction. Then f has a unique fixed point.

Piyush Kumar Tripathi [4] also proved the following lemma which is used in our results.

2.1 Lemma: Let (*X, F, t*) be a Menger space, where *t* is continuous. If $\exists\ k > 1$ such that $F_{fpf^2p}(kx) \leq F_{pfp}(x)$, $x > 0$. Suppose $f : X \to X$ is onto mapping then $\exists$ a Cauchy sequence in *X*.

3. MAIN RESULTS

In this section, I have also prove some fixed point theorems under different contractive conditions using contraction constant *k* > 1 or *k* < 1.

3.1 Theorem: Let (*X*,*F*;*t*) be a complete Menger probabilistic metric space where $F_{p,q}$ is strictly increasing distribution function and $f : X \to X$ is continuous mapping. If $\exists\ k \in (0,1)$ s. t.

$$F_{f(p),f(q)}(kx) \geq \min\{F_{p,q}(x), F_{p,f(p)}(x), F_{q,f(q)}(x), F_{q,f(p)}(x)\} .$$

Then $\exists$ a unique fixed point.

Proof: Let $p_0 \in X$. Construct a sequence $p_n = f(p_{n-1})$, *n* = 1,2,3 ……………….Then

$$F_{p_n,p_{n+1}}(kx) = F_{f(p_{n-1}),f(p_n)}(kx)$$

$$\geq \min\left\{F_{p_{n-1},p_n}(x), F_{p_{n-1},p_n}(x), F_{p_n,p_{n+1}}(x), F_{p_n,p_n}(x)\right\}$$

$$\text{i.e. } F_{p_n,p_{n+1}}(kx) \geq \min\left\{F_{p_{n-1},p_n}(x), F_{p_n,p_{n+1}}(x)\right\}$$

$$F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x)\ , x > 0$$

Therefore by lemma 2.1{P_n }is a Cauchy sequency. Since (*X*, *F*, *t*) is complete so $p_n \to p \in X$. Then by theorem 2.1, *p* is a unique fixed point of *f*. For uniqueness suppose *f(p) = p*, *f(q) = q*. Then

$$F_{p,q}(kx) = F_{f(p),g(q)}(x) \geq \min\left\{F_{p,q}(x), F_{p,p}(x), F_{q,q}(x), F_{q,p}(x)\right\}$$

i.e. $F_{pq}(kx) \geq F_{p,q}(x)$.

Which is not possible so *p* = *q*. Because $F_{p,q}$ is strictly increasing function and *kx < 0*

3.2 Theorem: Let (*X*, *F; t*) be a complete Menger probabilistic metric space where $F_{p,q}$ strictly increasing distribution function is and $f, g : X \to X$ is continuous mapping. If $\exists\ k \in (0,1)$ such that

$$F_{f(p),g(q)}(kx) \leq \max\left\{F_{p,q}(x), F_{p,f(p)}(x), F_{q,g(q)}(x)\right\}.$$

Then *f* and *g* have a unique common fixed point.

Proof: Let $p_0 \in X$. Construct a sequence {p_n} defined by $f(p_{2n}) = p_{2n+1}, g(p_{2n+1}) = p_{2n+2}$, *n* = 1,2,3. If n = 2r + 1 then

$$F_{p_n, p_{n+1}}(kx) \geq \min\left\{F_{p_{n-1}, p_n}(x), F_{p_n, p_{n+1}}(x)\right\}$$

$$F_{p_n, p_{n+1}}(kx) \geq F_{p_{n-1}, p_n}(x) \text{ because } F_{p,q} \text{ is strictly increasing and } kx < x$$

Again if *n = 2r* then

$$F_{p_n, p_{n+1}}(kx) = F_{p_{2r}, p_{2r+1}}(kx) = F_{g(p_{2r-1}), f(p_{2r})}(kx) \leq \max\{F_{p_{2r}, p_{2r+1}}(x), F_{p_{2r}, p_{2r+1}}(x), F_{p_{2r-1}, p_{2r}}(x)\}$$

$$F_{p_n, p_{n+1}}(kx) \leq \max\{F_{p_{2r}, p_{2r-1}}(x), F_{p_{2r}, p_{2r+1}}(x)\}$$

$$F_{p_n, p_{n+1}}(kx) \geq F_{p_n, p_{n-1}}(x) \text{ , } x > 0 \text{ therefore } \forall \text{ +ve integer } n$$

$$F_{p_n, p_{n+1}}(kx) \geq F_{p_n, p_{n-1}}(x)$$

Therefore by lemma 2.1.1, {p_n}is a Cauchy sequence. Then $p_n \to p \in X$.

Since $\left\{p_{2n+1}\right\}, \left\{p_{2n}\right\}$ is subsequence of $\left\{p_n\right\}$ so $p_{2n+1} \to p, p_{2n} \to p$. Then *f(p) = p* and *g(p) = p* that is *p* is common fixed point of *f* and *g*. For uniqueness suppose *p* and *q* are two common fixed-point *f* and *g*. Then,

$$F_{p,q}(kx) = F_{f(p),g(q)}(kx) \leq \max\{F_{p,q}(x), F_{p,p}(x), F_{q,q}(kx)\} \Rightarrow F_{p,q}(kx) \geq F_{p,q}(x),$$

which is not possible because $F_{p,q}$ is strictly increasing function and *kx < x* . Therefore *f* and *g* have unique common fixed point.

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