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SOME FIXED POINT THEOREMS FOR EXTENDED EXPANSION MAPPINGS ON CONE METRIC SPACES

SHERLY GEORGE*, SHAINI PULICKAKUNNEL

Department of Mathematics,
Sam Higginbottom University of Agriculture,
Technology and Sciences, Allahabad-211007, India.

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ABSTRACT

In this paper, we prove some fixed point theorems for expansion mappings in the framework of cone metric spaces. Our results in this paper extends and improves upon, among others, the corresponding results of Aage and Salunke [1]

MSC: *primary 47H09, 47H10, 54H25.*

Keywords: *Cone metric space, normal constants, fixed point.*

1. INTRODUCTION

One of the most widely used fixed point theorems in all analysis is Banach contraction theorem. It has been generalized in different directions by Mathematicians over the years. In contemporary time, fixed point theory has evolved speedily in cone metric spaces equipped with partial ordering. Huang and Zhang introduced in [5] the concept of cone metric space as generalisation of metric space where the set of real numbers is replaced by an ordered Banach space. Thereafter various authors have generlised the results of Huang and Zhang and studied fixed point theorems for normal and non-normal cones [2, 3, 6, 11]. A new generalisation of contraction mappings called T-contraction mappings was introduced by Beiranvand on metric spaces [4]. Recently Morales and Rojas [7], [8], [9], [10] have extended the concept of T-contraction mappings to the cone metric spaces by proving fixed point theorems for T-Kannan, T-Zamfirescu, T-Weakly contraction mappings.

Motivated by that we generalize the theorems given in [1] by extending the expansive mappings.

The following definitions and results will be needed in the sequel.

Definition 1: *Let E be a real Banach space. A subset P of E is called a cone if and only if*

1. *P is closed, nonempty and* $P \neq \{0\}$;
2. $a,b \in R, a,b \geq 0, x,y \in P$ *imply that* $ax+by \in P$;
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K>0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\| \tag{1}$$

The least positive number satisfying the above inequality is called the normal constant of P. We shall write $x<y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y-x \in$ int P (interior of P).

In the following we always suppose that E is a Banach space, P is a cone in E with int$P \neq 0$ and $\leq$ is partial ordering with respect to P.

Corresponding Author: Sherly George,*
Department of Mathematics, Sam Higginbottom University of Agriculture,
Technology and Sciences, Allahabad-211007, India.

Definition 2: *Let X be a nonempty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies:*

1. *0≤d(x,y) for all x,y∈X and d(x,y)=0 if and only x=y;*
2. *d(x,y)=d(y,x) for all x,y∈X;*
3. *d(x,y)≤d(x,z)+d(z,y) for all x,y,z∈X.*

Then d is called a cone metric on X and (X,d) is called a cone metric space. The concept of cone metric space is more general than that of metric space.

Definition 3: *Let (X,d) be a cone metric space,* $\{x_n\}$ *a sequence in X and x∈X. For every c∈E with 0≪c, we say that* $\{x_n\}$ *is :*

1. *a Cauchy sequence if there is an N such that, for n, m ,>N,* $d(x_n, x_m) \ll c$
2. *a convergent sequence if there is an N such that, for all n, > N,* $d(x_n, x) \ll c$ *for some x in X*

A cone metric space *X* is said to be complete if every Cauchy sequence in *X* is convergent in *X*. It is known that $\{x_n\}$ converges to *x*∈*X* if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided *P* is a normal cone with normal constant *K* [5].

Definition 4: *Let (M,d) be a cone metric space, P be a normal cone with normal constant K and* $T : M \to M$. *Then*

1. *T is said to be continuous if* $\lim_{n\to\infty} x_n = x$, *implies that* $\lim_{n\to\infty} Tx_n = Tx$ *for every* $\{x_n\}$ *in M;*
2. *T is said to be sequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{Tx_n\}$ *is convergent, then* $\{x_n\}$ *is also convergent;*
3. *T is said to be subsequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{Tx_n\}$ *is convergent, then* $\{x_n\}$ *has convergent subsequence.*

Definition 5: *Let (X,d) be a cone metric space and* $T, S : X \to X$ *be two functions. S is said to be T- contraction if there exists a∈[0,1) such that*

$$d(TSx,TSy) \leq a\,d(Tx,Ty),\ \forall x,y \in X.$$

The following two lemmas of Huang and Zhang [5] will be required in the sequel.

Lemma 1 [5,lemma 1]: *Let (M,d) be a cone metric space, P be a normal cone with normal constant K. A sequence* $\{x_n\}$ converges to *x*∈X if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

Lemma 2 [5, lemma 4]: Let (M,d) be a cone metric space, P be a normal cone with normal constant K. A sequence$\{x_n\}$ in X *is Cauchy sequence if and only if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$

2. MAIN RESULTS

Theorem 1: Let (*X*,*d*) be a complete cone metric space, *P* be a normal cone with normal constant *K* and two continuous mappings T,*f* : X→ X satisfies the contractive condition

$$d(fTx,fTy) \geq kd(fx,fy)$$

for all *x*,*y*∈*X*, where *k* >1 is a constant. Assume that *T* is onto, *f* is one-one, subsequentially convergent, and *T*,*f* are commutative. Then *T* has a unique fixed point in X.

Proof: If *fT*(*x*) = *fT*(*y*), then

$$0 \geq kd(f(x),f(y)) \Rightarrow 0 = d(f(x),f(y))$$
$$i.e, fT(x)=fT(y) \Rightarrow fx=fy$$

Since the maps commute, we get

$$Tf(x)=Tf(y) \Rightarrow f(x)=f(y).$$

Thus T is one-one. Define $G = T^{-1}$

$$\begin{aligned} d(f(x), f(y)) &= d(fTT^{-1}x, fTT^{-1}y) \\ &\geq kd(fT^{-1}x, fT^{-1}y) \\ &\geq kd(fGx, fG(y)). \end{aligned}$$

$$d(fG(x), fGy) \leq \frac{1}{k} d(fx, fy)$$
$$\leq \mathrm{M} d(fx, fy) \text{ where } \mathrm{M} = \frac{1}{k} \leq 1$$

From definition 5 and by Theorem 3.3 in [7], G has a unique fixed point u in X.

$$i.e., G(u) = u \Rightarrow T^{-1}u = u,$$
$$u = Tu.$$

$\Rightarrow$ T has a unique fixed point

When $T=I$ we get the following theorem

Corollary 1: *[1, Theorem 2.1] Let (X,d) be a complete cone metric space and the mapping* $T : X \to X$ *is onto and satisfies the contractive condition*

$$d(Tx,Ty) \geq kd(x,y),$$

for all $x,y \in X$, *where* $k>1$ *is a constant. Then T has a unique fixed point in X.*

Theorem 2: Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K and two commutative continuous mappings $f, T : X \to X$ satisfy the condition

$$d(fTx,fTy) \geq k[d(fTx,fx)+d(fTy,fy)],$$

for all $x,y \in X$ where $\frac{1}{2} < k \leq 1$. Assume that f is subsequentially convergent, injective and T is onto. Then T has fixed point in X.

Proof: Since T is onto, for each $x_0 \in X$, there exist $x_1 \in X$ such that $Tx_1 = x_0$. Similarly for each $n \geq 1$ there exist $x_{n+1} \in X$, such that $x_n = Tx_{n+1}$. If $x_{n-1} = x_n$, then x_n is a fixed point of T. Thus, Suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$. Then

$$d(fx_n, fx_{n-1}) = d(fTx_{n+1}, fTx_n)$$
$$\geq k[d(fTx_{n+1}, fx_{n+1}) + d(fTx_n, fx_n)]$$
$$\geq kd(fx_n, fx_{n+1}) + d(fx_{n-1}, fx_n).$$
$$d(fx_n, fx_{n+1}) \leq \frac{1-k}{k} d(fx_{n-1}, fx_n) = hd(fx_{n-1}, fx_n).$$

From this we get

$$d(fx_n, fx_{n+1}) \leq h^n d(fx_{0,}fx_1) \qquad \text{where } h = \frac{1-k}{k} \text{ and } 0 \leq h < 1$$

Now, for $n<m$

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots .. + d(fx_{m-1}, fx_m)$$
$$\leq (h^n + h^{n+1} + \ldots + h^{m-1}) d(fx_0, fx_1)$$
$$\leq \frac{h^n}{1-h} d(fx_0, fx_1).$$

From (1) we get

$$\|d(fx_n, fx_m)\| \leq \mathrm{K}\frac{h^n}{1-h} \quad \|d(fx_0, fx_1)\| \to 0 \text{ as } n \to \infty$$

$\{fx_n\}$ is Cauchy sequence. Since (X,d) be complete cone metric space there exist $u \in X$ such that $\lim_{n \to \infty} fx_n = u.$

Since f is subsequentially convergent $\{x_n\}$ has a convergent subsequence $\{x_m\}$

$$i.e., \lim_{m \to \infty} x_m = x^*.$$

Since f is continuous

$$\lim_{m\to\infty} f(x_m) = fx^*.$$

By uniqueness of limit,

$$u = fx^*,$$

$$i.e., \lim_{n\to\infty} fx_n = fx^*$$

Since T is continuous

$$\lim_{n\to\infty} Tfx_n = Tfx^*.$$

$$d(fTx^*, fx^*) \leq d(fTx^*, fTx_n) + d(fTx_n, fx^*) = 0 \quad as \quad n\to\infty$$

Then we get

$$d(fTx^*, fx^*) = 0$$

Which implies that

$$fTx^* = fx^*.$$

Since f is injective, $Tx^* = x^*$ Thus T has a fixed point in X.

When $f=I$ we get the following result as corollary.

Corollary 2: *[1,Theorem 2.3]Let (X,d) be a complete cone metric space and continuous, onto mapping* $T : X \to \infty$ *satisfies the condition*

$$d(Tx,Ty)\geq k[d(Tx,x)+d(Ty,y)],$$

for all x,y∈X where $\frac{1}{2} < k \leq 1$. *Then T has fixed point in X.*

Theorem 3: Let (X,d) be a complete cone metric space with normal constant K and two commutative continuous mappings f,T satisfies the contractive condition

$$d(fTx,fTy)\geq kd(fx,fy)+ld(fTx,fy),$$

for all $x,y,\in X$, where $l \geq 0$, $k > 1$. Assume that T is onto, f is injective and subsequentially convergent. Then T has a fixed point in X.

Proof: Since T is onto, for each $x_0 \in X$ there exist $x_1 \in X$ such that $Tx_1 = x_0$, Similarly for each $n\geq 1$ there exist $x_{n+1} \in X$ such that

$x_n = Tx_{n+1}$. If $x_{n-1} = x_n$, then x_n is a fixed point of T. Suppose that $x_{n-1} \neq x_n$ for all $n\geq 1$.

$$\begin{aligned} d(fx_n, fx_{n-1}) &= d(fTx_{n+1}, fTx_n) \\ &\geq kd(fx_{n+1}, fx_n) + ld(fTx_{n+1}, fx_n) \\ &= kd(fx_{n+1}, fx_n) + ld(fx_n, fx_n) \\ &\leq kd(fx_n, fx_{n+1}). \end{aligned}$$

$$d(fx_n, fx_{n+1}) \leq \frac{1}{k} d(fx_{n-1}, fx_n).$$

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq hd(fx_{n-1}, fx_n) \quad where \quad h = \frac{1}{k} < 1. \\ &\leq h^2 d(fx_{n-2}, fx_{n-1}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \\ &\leq h^n d(fx_0, fx_1). \end{aligned}$$

Consider $n<m$

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots .. + d(fx_{m-1}, fx_m)$$
$$\leq (h^n + h^{n+1} + \ldots + h^{m-1}) d(fx_0, fx_1)$$
$$\leq \frac{h^n}{1-h} d(fx_0, fx_1).$$

From (1) we get

$$\|d(fx_n, fx_m)\| \leq \mathrm{K}\frac{h^n}{1-h}\|d(fx_0, fx_1)\| \to 0 \text{ as } n \to \infty$$

This shows that $\{fx_n\}_{n=1}$ is Cauchy sequence.

Since (X,d) complete cone metric space,

$$\lim_{n\to\infty} fx_n = u.$$

Since f is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$. i.e., $\lim_{m\to\infty} x_m = x^*$

Since f is continuous,

$$\lim_{m\to\infty} fx_m = fx^*.$$

By uniqeness of limit

$$u = fx^*.$$
$$i.e., \lim_{n\to\infty} fx_n = fx^*.$$

Since T is continuous,

$$\lim_{n\to\infty} Tfx_n = Tfx^*.$$

Now, $$d(fTx^*, fx^*) \leq d(fTx^*, fTx_n) + d(fTx_n, fx^*)$$

Which implies that

$$d(fTx^*, fx^*) = 0 \text{ as } n \to \infty$$

Heance, $fTx^* = fx^*$

Since f is injective, $Tx^* = x^*$. Thus T has a fixed point in X.

When $f=I$ we get the following result as corollary.

Corollary 3: *[1,Theorem 2.5] Let (X,d) be a complete cone metric sace and continuous, onto mapping T satisfies the contractive condition*

$$d(Tx,Ty) \geq kd(x,y) + ld(Tx,y),$$

for all $x,y,\in X$, where $l\geq 0$ $k>0$. Then T has a fixed point in X.

Theorem 4: Let (X,d) be a complete cone metric space with normal constant K and two continuous commutative mappings $T, f : X \to X$ satisfies the condition

$$d(fTx,fTy) \geq kd(fx,fy) + ld(fx,fTx) + pfy,fTy),$$

for all $x,y\in X$ where $k \geq -1$, $p < 1$ and $l > 1$ with $k+l+p>1$. Assume that T is onto and f is injective and subsequentially convergent. Then T has a fixed point in X.

Proof: Since T is onto, for each $x_0 \in X$, there exist $x_1 \in X$ such that $Tx_1 = x_0$. Similarly, for each $n \geq 1$ there exist $x_{n+1} \in X$, such that

$x_n = Tx_{n+1}$. If $x_{n-1} = x_n$, then x_n is a fixed point of T. Thus, suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$. Then

$$\begin{aligned} d(fx_n, fx_{n-1}) &= d(fTx_{n+1}, fTx_n) \\ &\geq kd(fx_{n+1}, fx_n) + ld(fx_{n+1}, fTx_{n+1}) + pd(fx_n, fTx_n) \\ &= kd(fx_{n+1}, fx_n) + ld(fx_{n+1}, fx_n) + pd(fx_n, fx_{n-1}), \end{aligned}$$

$$\begin{aligned} d(fx_{n+1}, fx_n) &\leq \frac{1-p}{k+l} d(fx_n, fx_{n-1}) \\ &= hd(fx_n, fx_{n-1}) where \quad h = \frac{1-p}{k+l} and \quad 0 < h < 1 \\ &\leq h^2 d(fx_{n-1}, fx_{n-2}) \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots. \\ &\leq h^n d(fx_1 f, x_0). \end{aligned}$$

From this we get

$$d(fx_n, fx_{n+1}) \leq h^n d(fx_0, fx_1) where \quad h = \frac{1-p}{k+l} and \quad 0 < h < 1.$$

Consider $m>n$

$$\begin{aligned} d(fx_n, fx_m) &\leq h^n d(fx_0, fx_1) where \quad h = \frac{1-p}{k+l} and \quad 0 < h < 1. \\ &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \ldots + d(fx_{m-1}, fx_m) \\ &\leq (h^n + h^{n+1} + \ldots + h^{m-1}) d(fx_1, fx_0). \\ &\leq \frac{h^n}{1-h} d(fx_1, fx_0) \end{aligned}$$

From (1), we get

$$\|d(fx_n, fx_m)\| \leq \mathrm{K}\frac{h^n}{1-h} \|d(fx_0, fx_1)\| \to 0 \ as \ n \to \infty$$

Hence $\|d(fx_n, fx_m)\| \to 0$

Thus $\{fx_n\}_{n=1}$ is a Cauchy sequence.

Since (X,d) is a complete cone metric space, $\lim\limits_{n \to \infty} fx_n = u$.

Since f is subsequentially convergent $\{x_n\}$ has a convergent subsequence $\{x_m\}$ i.e., $\lim\limits_{m \to \infty} x_m = x^*$

Since f is continuous,

$$\lim_{m \to \infty} fx_m = fx^*,$$

By uniqueness of limit

$$u = fx^*.$$

$$i.e., \lim_{n \to \infty} fx_n = fx^*$$

Since T is continuous

$$\lim_{n\to\infty} Tfx_n = Tfx^*$$

Now $$d(fTx^*, fx^*) \le d(fTx^*, Tfx_n) + d(fTx_n, fx^*)$$

Which implies that

$$d(fTx^*, fx^*) = 0 \ \text{as} \ \ n \to \infty$$

Since f is injective, $Tx^* = x^*$. Then T has a fixed point.

When f=I we get the following result as corollary.

Corollary 4: *[1,Theorem 2.6] Let (X,d) be complete cone metric space and continous, onto mapping* $T: X \to X$ *satisfies the condition*

$$d(Tx,Ty) \ge kd(x,y)+ld(x,Tx)+pd(y,Ty),$$

for all $x,y \in X$ *where* $k \ge -1$, $p<1$ *and* $l>$ *with* $k+l+p>1$. *Then T has a fixed point in X.*

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