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UNIFORMLY GLOBAL ATTRACTVITY IN FUNCTIONAL DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, we discuss the existence as well as uniformly global attractivity results on unbounded intervals of functional differential equation through application of classical hybrid fixed point theorem.

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1. INTRODUCTION

Let R be the real line and let R_+ be set of nonnegative real numbers. Let $I_0 = [-\delta, 0]$ be a closed and bounded interval in R for some real number $\delta > 0$ and let $= I_0 U\, R_+$. Let C denote the Banach Space of continuous real-valued functions $\emptyset\ on\ I_0$ with the supremum norm $\|\,.\,\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_o} |\phi(t)|$$

Clearly, C is a Banach Space with this supremum norm. For a fixed t , $t \in R_+, let\ x_t$ denote the element of C defined by

$$x_t(\theta) = x(t+\theta), \theta \epsilon\ [-\delta, 0].$$

The space C is called the history space of the past interval I_0 for the functional differential equations to describing the past history of the problems.

Let $CRB(R_+)$ denote the class of functions a: $R_+ \to R - \{0\}$ satisfying the following properties:

(i) A is continuous,

(ii) $\lim_{t\to\infty} a(t) = \pm\infty$, and

(iii) a(0) = 1

There do exist functions satisfying the above conditions. In fact, if $a_1(t) = t + 1, a_2(t) = e^t$, then $a_1, a_2 \in CRB\ (R_+)$. Again, the class of continuous and strictly monotone functions $a : R_+ \to R - \{0\}$ with a(0) = 1 satisfy the above criteria. Note that if $a\ \epsilon\ CRB\ (R_+)$, then the reciprocal function $\bar{a} : R_+ \to R$ defined by $\bar{a}(t) = \frac{1}{a(t)}$ is continuous and

$$\lim_{t\to\infty} \bar{a}(t) = 0.$$

Given a function $\emptyset \in C$, we consider the following functional differential equation (FDE),

$$\frac{d}{dt}[a(t)x(t)] = f(t, x(t), x_t) + g(t, x(t), x_t),\ a.e.\ t \in R_+ \qquad (1.1)$$
$$x_0 = \emptyset$$

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Where, $a \epsilon CRB(R_+)$ and $f,\ g : R_+ \times R \times C \to R$.

It is clear that the functional differential equation(1.1) the scalar perturbation of second kind for the following nonlinear first order FDE on unbounded interval,

$$x\,(t) = f(t, x(t), x_t) + g(t, x(t), x_t) a.e. t \in R_+ \qquad (1.2)$$
$$x_0 = \emptyset$$

A different types of perturbations for the nonlinear differential equations appears in a recent paper of Dhage [4]. Some special cases of these FDE with $a = 1$ have already been studied in the literature on closed and bounded intervals for various aspects of the solutions. See Hale [10] Ntouyas [11]. The FDEs (1.1) is not discussed so far in the literature on closed but unbounded intervals of real line. In this paper, we discuss the functional differential equation (1.1) for existence as well as for uniformly global attractivity of the solution. Here we, use application of fixed point theory specially hybrid fixed point theory to formulate functional differential equation (1.1) on unbounded intervals of real line.

2. FIXED POINT THEORY

Let X be a nono-empty set and let T: $X \to X$. An invariant point under T in X is called a fixed point of T, that is, the fixed points are the solutions of the functional equation T $x = x$. Any statement asserting the existence of fixed point of the mapping T is called fixed point theorem for the mapping T in X. We give some fixed point theorems useful in establishing the solution for FDE (1.1).Before results we give some basic definitions.

Let X be an infinite dimensional Banach space with the norm ‖ . ‖. A mapping Q: $X \to X$ is called D-Lipschitz if there is a continuous and nondecreasing function $\emptyset : R_+ \to R_+$ satisfying

$$\|Qx - Qy\| \le \emptyset(||\ x - y\ ||)$$

for all $x, y \in X$, where $\emptyset(0) = 0. If\ \emptyset(r) = kr, k > 0$, then Q is called Lipschitz with the Lipschitz constant k. In particular, if k<1, then Q is called a contraction on X with the contraction constant k. Further, if $\emptyset(r) < r\ for\ r > 0$, then Q is called nonlinear D-contraction and the function $\emptyset$ is called D-function of Q on X.

The fixed point theorem is

Theorem 2.1 (Granas and Dugundji) [9]: Let S be a non-empty, closed, convex and bounded subset of the Banach Space X and let $Q: S \to S$ be a continuous and compact operator. Then the operator equation

$$\text{Q}x = x \text{ has a solution in S.} \qquad (2.1)$$

We use the following variant of a fixed point theorem of Burton [3] which is a special case of a hybrid fixed point theorem.

Theorem 2.2 (Dhage[4]): Let S be a closed, convex and bounded subset of the Banach Space X and let $A: X \to X\ and\ B: S \to X$ be two operator such that

(a) A is nonlinear D-contraction,
(b) B is completely continuous,
(c) $x = Ax + By\ \ ,\ \ x \epsilon S$

Then the operator equation

$$Ax + Bx = x \qquad (2.2)$$

has a solution in S.

Theorem 2.3 (Dhage[8]): Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and Let $A: X \to X\ and\ B : S \to X$ be two operators such that

(a) A is D-Lipschitz with D-function ψ,
(b) B is completely continuous,
(c) $x = Ax\ By\ \to x \in S\ for\ all\ y\ \in S, and$
(d) $M\psi(r) < r, where\ M = ||\ B(S)\ || = \sup\{||Bx||: x \in S\}$.

Then the operator equation

$$Ax\ Bx = x \qquad (2.3)$$

has a solution in S.

3. CHARACTERZATION OF SOLUTIONS

We seek the solution of the FDE (1.1) in the space BC ($I_0 \cup\ R_+R_+$) of continuous and bounded real-valued functions defined on $I_0 \cup R_+$. Define a standard supremum norm ‖ .‖ and a multiplication " . " in BC ($I_0 \cup R_+, R$) by

$$||x|| = \sup_{t\, \in\, I_0\, \cup\, R_+} |x(t)| and\ (xy)(t) = x\,(t) y\,(t), \qquad t \in R_+.$$

Clearly, BC$(I_0 \cup R_+, R)$ becomes a Banach algebra with respect to the above norm and the multiplication in it. By $L^1(R_+, R)$. we denote the space of lebesgue integrable functions on $R_+$and the norm $||\,.\,||L^1\ in\ L^1(R_+, R)$ is defined by

$$|| \, x \, || \, L^1 = \int_0^\infty |x(t)| ds.$$

Let us assume that $E = BC(I_0 \cup R_+, R)$ and let Ω be a non-empty subset of X. Let Q: $E \to E$ be a operator and consider the following operator equation in E,

$$Qx(t) = x(t) \tag{3.1}$$

for all $t \in I_0 \cup R_+$.

Definition 3.1: We say that solutions of the operator equation (4.1) are locally attractive if there exists a closed ball $\overline{B_r}(x_0)$ in the space $BC(I_0 \cup R_+, R)$ for some $x_0 \in BC(I_0 \cup R_+, R)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation(3.1) belonging to $\overline{B_r}(x_0)$, we have that

$$\lim_{t\to\infty}(x(t) - y(t)) = 0 \tag{3.2}$$

In the case when the limit (3.2) is uniform with respect to the set $\overline{B_r}(x_0)$, i.e., when for each $\in > 0$ there exists T >0 such that

$$|x(t) - y(t)| \leq \in \tag{3.3}$$

for all $x, y \in \overline{B_r}(x_0)$being solutions of (3.1) and for $t \geq T$, we will say that solutions of equation (3.1) are uniformly locally attractive on $I_0 \cup R_+$.

Definition 3.2: A solution $x = x(t)$ of equation (3.1) is said to be globally attractive if (3.2) holds for each solution $y = y(t)$ of (3.1) in BC$(I_0 \cup R_+, R)$.In other words, we may say that solutions of the equation (3.1) are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of (3.1) in $BC(I_0 \cup R_+, R)$. The condition (3.2) is satisfied. In the case when the condition (3.2) is satisfied uniformly with respect to the space BC$(I_0 \cup R_+, R)$ i.e., if for every ∈>0 there exists T > 0 such that the inequality (3.2) is satisfied for al $y(t)$x, y,∈ $BC(I_0 \cup R_+, R)$ being the solutions of (3.1) and for $i \geq T$, we will say that solutions of the equation (3.1) are uniformaly globally attractive on $I_0 \cup R_+$.

4. ATTRACTIVITY RESULT

We prove the existence and uniformly global attractivity results for the FDE (1.1) on $I_0 \cup R_+$ under some suitable conditions.

We need the following definitions.

Definition 4.1: By a solution for the functional differential equation (1.1) we mean a function $x \in BC(I_0 \cup R_+, R) \cap AC(R_+, R)$ such that

(i) The function $t \to a(t)x(t)$is absolutely continuous on R_+, and
(ii) x satisfies the equations in (1.1),

Where $AC(R_+, R)$ is the space of absolutely continuous real-valued functions on right half real axis R_+.

Definition 4.2: A function f : $R_+ \times R \times C \to R\ is\ called\ caratheodory\ if$

(i) $t \to f(t, x, y)$ is measuable for all $x \in R\ and\ y \in C, and$
(ii) $(x, y) \to f(t, x, y)$ is continuous for all t $\in R_+$

Consider the following hypotheses.

(A) There exists a continuous function h: $R_+ \to R_+\ such\ that$

$$|f(t, x, y) + g(t, x, y)| \leq h(t) a.e. t \in R_+$$

for all $x \in R$ and $y \in C$. Moreover, we assume that $\lim_{t\to\infty} |\bar{a}(t)| \int_0^t h(s)ds = 0$.

Remark: If the hypothesis (A) holds and $a \in CRB\ (R_+)$, then $\bar{a} \in BC\ (R_+, R)$ and the function $W: R_+ \to R$ defined by the expression $W(t) = |\bar{a}(t)| \int_0^t h(s)ds$ is continuous on R_+. Therefore, the number $W = \sup_{t\geq 0} w(t)$ exists.

Theorem 4.1: Assume that the hypotheses (A) holds. Then the FDE (1.1) has a solution and solutions are uniformly globally attractive on $I_0 \cup R_+$.

Proof: Set $X = BC(I_0 \cup R_+, R)$.Define an operator Q on X by

$$Q\,x(t) = \begin{cases} \begin{cases} \phi(0)\bar{a}(t) + \bar{a}(t)\int_0^t f(s, x(s)\, x_s)\, ds + \bar{a}(t)\int_0^t g(s, x(s)\, x_s)\, ds, & t \in R_+ \end{cases} \\ \phi(t),\ if\ \ t \in I_0 \end{cases} \tag{4.1}$$

We show that Q defines a mapping $Q : X \to X. Let\ x \in X$ be arbitrary. Obviously, Q x is a continuous function on $I_0 \cup R_+$. We show that Q x is bounded on $I_0 \cup R_+$. Thus, if $t \in R_+$, then we obtain:$|Qx(t)| \leq$

$$|\emptyset(0)||\bar{a}(t)| + |\bar{a}(t)|\int_0^t |f(s,x(s),x_s)|ds + |\bar{a}(t)|\int_0^t |g(s,x(s),x_s)|ds \leq |\emptyset(0)||| \bar{a} || + |\bar{a}(t)|\int_0^t h(s)ds.$$

Since $\lim_{t\to\infty} |\bar{a}(t)| \int_0^t h(s)ds = 0, and\ the$ function $w: R_+ \to R\ defined\ by\ w(t) = |\bar{a}(t)| \int_0^t h(s)ds$ is continuous, there is constant W > 0 such that

$$\sup_{t\geq 0} w(t) = \sup_{t\geq 0} |\bar{a}(t)| \int_0^t h(s)ds \leq W.$$

Therefore,

$$|Qx(t) \leq |\emptyset(0)| \,||\, \bar{a} \,||\, + W \leq ||\, \bar{a} \,||\, ||\emptyset|| + W$$

for all $t \in R_+$. Similarly, if $t \in I_0$ then $|Qx(t)| \leq ||\ \emptyset\ ||$. As a result, we have that

$$||Qx|| \leq (||\, \bar{a} \,|| + 1)||\emptyset|| + W \tag{4.2}$$

for all $x \in X$ and therefore, Q maps X into X itself. Define a closed ball $\overline{B_r}(0)$ centered at origin of radius r, where $r = (||\, \bar{a} \,|| + 1)||\emptyset|| + W. C$learly defines a mapping$Q : X \to \bar{B}_r(0)$ and in particular $Q : \overline{B_r}(0) \to \overline{B_r}(0)$. We show that Q satisfies all the conditions of Theorem 2.1. First, we show that Q is continuous on $\overline{B_r}(0)$. To do this, let us fix arbitrarily $\in > 0\ and\ let\ \{x_n\}$ be a sequence of points in $\overline{B_r}(0)$ converging to a point $x \in \overline{B_r}(0)$. Then we get:

$$\begin{aligned}|(Qx_n)(t) - (Qx)(t)| \leq |\bar{a}(t)| &\int_0^t |\, f(s,x_n(s),x_n(\theta+s) - f(s,x(s),x(\theta+s)|ds \\ &+ \bar{a}(t)| \int_0^t |\, g(s,x_n(s),x_n(\theta+s) - g(s,x(s),x(\theta+s)|ds \\ &\leq |\bar{a}(t)| \int_0^t [\,|f(s,x_n(s),x_n(\theta+s)| + |f(s,x(s),x(\theta+s))|]ds \\ &+ |\bar{a}(t)| \int_0^t [\,|g(s,x_n(s),x_n(\theta+s)| + |g(s,x(s),x(\theta+s))|]ds \\ &\leq 2|\bar{a}(t)| \int_0^t h(s)ds \\ &\leq 2w(t)\end{aligned} \tag{4.3}$$

Hence, by hypothesis (A), we infer that there exists a T >0 such that $w(t) \leq \in for\ t \geq T.$Thus,$for\ t \geq T$ from the estimate (4.2) we derive that

$$|(Qx_n)(t) - (Qx)(t)| \leq 2 \in as\ n \to \infty.$$

Furthermore, let us assume that $t \in |0,T]$. then by Lebesgue dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}\lim_{n\to\infty} Qx_n(t) &= \lim_{n\to\infty}[\emptyset(0)\bar{a}(t) + \bar{a}(t)\int_0^t f(s,x_n(s),x_n(\theta+s)ds + \bar{a}(t)\int_0^t g(s,x_n(s),x_n(\theta+s)ds] \\ &= \emptyset(0)\bar{a}(t) + \bar{a}(t)\int_0^t [\lim_{n\to\infty} f(s,x_n(s),x_n(s),x_n(\theta+s)]ds \\ &\quad + \bar{a}(t)\int_0^t [\lim_{n\to\infty} g(s,x_n(s),x_n(s),x_n(\theta+s)]ds \\ &= Qx(t)\end{aligned}$$

for all $t \in [0,T]$.similarly, if $t \in I_0$ then

$$\lim_{n\to\infty} Qx_n(t) = \emptyset(t) = Qx(t).$$

Thus, $Qx_n \to Qx$ as $n \to \infty$ uniformly on R_+ and hence Q is a continuous operator on $\overline{B_r}(0)$ into $\overline{B_r}(0)$.

Next, we show that Q is compact operator on $\overline{B_r}(0)$. to finish this, it is enough to show that every sequence $\{Qx_n\}$ in Q $(\overline{B_r}(0)$ has a cauchy subsequence. Now,

$$\begin{aligned}|Qx_n(t)| &\leq |\emptyset(0)||\bar{a}(t)| + |\bar{a}(t)| \int_0^t |f(s,x_n(s),x_n(\theta+s)|ds + |\bar{a}(t)| \int_0^t |g(s,x_n(s),x_n(\theta+s)|ds \\ &\leq (||\bar{a}|| + 1)|\,\emptyset(0)| + w(t) \\ &\leq (||\bar{a}|| + 1)||\emptyset|| + w(t)\end{aligned}$$

For all $t \in R_+$. Taking supremum over t, we obtain

$$||Qx_n|| \leq (||\bar{a}|| + 1)||\emptyset|| + W$$

For all $n \in N$. this shows that $\{Qx_n\}$ is a uniformly bounded sequence in $Q(\bar{B}_r(0)$.

Next, we show that $Q(\overline{B_r}(0)$ is also an equicontinuous set in X. Let $\in > 0$ be given. Since. $\mathrm{Lim}_{t\to\infty} w(t) = 0$, there is a real number $T_1 > 0$ such that $|w(t)| < \frac{\in}{8}$ for all $t \geq T_1$. Similarly, since $\lim_{n\to\infty} \bar{a}(t) = 0,\ for\ above\ \in > 0,$ $there\ is\ real\ muber\ T_2 > 0$ such that $|\bar{a}(t)| < \frac{\in}{8|\emptyset(0)|}$ for all $t \geq T_2$. Thus, if $T = \max\{T_1, T_2\}, then\ |w(t)| < \frac{\in}{8}$ and $|\bar{a}(t)| < \frac{\in}{8|\emptyset(0)|}$ for all $t \geq T. Let\ t, \tau \in I_0 \cup R_+$ be arbitrary. $If\ t, \tau \in I_0$ then by uniform continuity of $\emptyset\ on$ I_0, for above $\in$ we have a $\delta_1 > 0$ which is a function of only $\in$ such that

$$|t-\tau| < \delta_1 \Rightarrow |Qx_n(t) - Qx_n(\tau)| = |\emptyset(t) - \emptyset(\tau)| < \frac{\in}{4}$$

For all $n \in N$. if $t, \tau \in [0, T]$, then we have

$$\begin{aligned}
|Qx_n(t) - Qx_n(\tau)| &\leq |\emptyset(0)||\bar{a}(t) - \bar{a}(\tau)\\
&\quad + ||\bar{a}(t)| \textstyle\int_0^t f(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^\tau f(s, x_n(s), x_n(\theta+s)ds|\\
&\quad + ||\bar{a}(t)| \textstyle\int_0^t g(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^\tau g(s, x_n(s), x_n(\theta+s)ds|\\
&\leq |\emptyset(0)||\bar{a}(t) - \bar{a}(\tau)| + |\bar{a}(t)\textstyle\int_0^t f(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^\tau f(s, x_n(s), x_n\\
&\quad (\theta+s)ds| + |\bar{a}(\tau)\textstyle\int_0^t f(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^t f\big(s, x_n(s), x_n(\theta+s)\big)ds|\\
&\quad + |\bar{a}(t)\textstyle\int_0^t g(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^t g(s, x_n(s), x_n(\theta+s)ds|\\
&\quad + |\bar{a}(\tau)\textstyle\int_0^t g(s, x_n(s), x_n(\theta+s)ds - \bar{a}(\tau)\int_0^t g\big(s, x_n(s), x_n(\theta+s)\big)ds|\\
&\leq |\emptyset(0)||\bar{a}(t) - \bar{a}(r)| + |\bar{a}(t) - \bar{a}(\tau)||\textstyle\int_0^t f(s, x_n(s), x_n(\theta+s)ds|)\\
&\quad + |\bar{a}(\tau)||\textstyle\int_0^t f\big(s, x_n(s), x_n(\theta+s)\big)ds|\\
&\quad + |\bar{a}(t) - \bar{a}(\tau)||\textstyle\int_0^t g(s, x_n(s), x_n(\theta+s)ds|)\\
&\quad + |\bar{a}(\tau)||\textstyle\int_0^t g\big(s, x_n(s), x_n(\theta+s)\big)ds|\\
&\leq |\emptyset(0)||\bar{a}(t) - \bar{a}(\tau)| + |\bar{a}(t) - \bar{a}(\tau)||\textstyle\int_0^T h(s)ds + ||\bar{a}||\int_\tau^t h(s)ds|\\
&\leq |\emptyset(0)|\bar{a}(t) - \bar{a}(\tau)\Big| + |\bar{a}(t) - \bar{a}(\tau)|\textstyle\int_0^T h(s)ds + ||\bar{a}||\Big|p(t) - p(\tau)|\\
&\leq \big[|\emptyset(0)| + ||h||L^1\big]|\bar{a}(t) - \bar{a}(\tau)| + ||\bar{a}|||p(t) - p(\tau)|
\end{aligned}$$

Where, $p(t) = \int_0^t h(s)ds\ and\ ||h||L^1 = \int_0^\infty h(s)ds.$

By the uniform continuity of the functions $\bar{a}$ and p on [0,T], for above $\in$ we have the real numbers $\delta_2 > 0\ and\ \delta_3 > 0$ which are the functions of only $\in$ such that

$$|t-\tau| < \delta_2 \Rightarrow |\bar{a}(t) - \bar{a}(\tau) < \frac{\in}{8[|\emptyset(0)| + ||h||L^1]}$$

and

$$|t-\tau| < \delta_3 \Rightarrow |p(t) - p(\tau)| < \frac{\in}{8||\bar{a}||}$$

$Let\ \delta_4 = \min\{\delta_2, \delta_3\}$.Then

$$|t-\tau| < \delta_4 \Rightarrow |Qx_n(t) - Qx_n(\tau)| < \frac{\in}{4}$$

for all $n \in N$. similarly, if $t \in I_0 and\ \tau \in [0, T], then$

$$|Qx_n(t) - Qx_n(\tau)| \leq |Qx_n(t) - Qx_n(0)| + |Qx_n(0) - Qx_n(\tau)|$$

Take $\delta_5 = \min\{\delta_1, \delta_4\} > 0$ which is again a function of only $\in$. Hence by above estimated facts it follows that

$$|t-\tau| < \delta_5 \Rightarrow |Qx_n(t) - Qx_n(\tau)| < \frac{\in}{2}$$

for all $n \in N$Again, if $t, \tau > T,$ then we have a real number $\delta_6 > 0$ which is a function of only $\in$ such that

$$\begin{aligned}
|Qx_n(t) - Qx_n(\tau)| &\leq |\emptyset(0)||a(t) - a(\tau)|\\
&\quad + |\bar{a}(t)\textstyle\int_0^t f\big(s, x_n(s), x_n(\theta+s)\big)ds - \bar{a}(\tau)\int_0^\tau f(s, x_n(s), x_n(\theta+s)ds|\\
&\quad + |\bar{a}(t)\textstyle\int_0^t g\big(s, x_n(s), x_n(\theta+s)\big)ds - \bar{a}(\tau)\int_0^\tau g(s, x_n(s), x_n(\theta+s)ds|\\
&\leq |\emptyset(0)||a(t)| + |\emptyset(0)||a(\tau) + w(t) + w(\tau)\\
&< \frac{\in}{4} + \frac{\in}{4}
\end{aligned}$$

For all $n \in N,$ whenever $t - \tau| < \delta_6$. similarly, $if\ t, \tau \in I_0 \cup R_+$ with $t < T < \tau$, then we have

$$|Qx_n(t) - Qx_n(\tau)| \leq |Qx_n(t) - Qx_n(T)| + |Qx_n(T) - Qx_n(\tau)|$$

Take $\delta = \min\{\delta_5, \delta_6\} > 0$ which is again a function of only $\in$. Therefore, from the above obtained estimates, it follows that

$$|Qx_n(t) - Qx_n(T)| < \frac{\in}{2} \; and \; |Qx_n(T) - Qx_n(\tau)| < \frac{\in}{2}$$

For all $n \in N$, whenever $|\ t-\tau| < \delta$. As a result, $Qx_n(t) - Qx_n(\tau)| < \in$ for all $t, \tau \in I_0 \cup R_+$ and for all n $\in$ N, whenever $|t-\tau| < \delta$. As a result, $|Qx_n(T) - Qx_n(\tau)| < \in \; for \; all \; t, \tau \in$ $I_0 \cup R_+$ and for all n $\in N$, whenever $|t-\tau| < \delta$ This shows that $\{Qx_n\}$ is equicontinuous sequence in X. Now an application of Arzela-Ascoli theorem yields that $\{Qx_n\}$ has a uniformly convergent subsequence on the compact subset $I_0 \cup [0, T]$ of $I_0 \cup R$.Without loss of generality, call the subsequence to be the sequence itself.

We show that $\{Qx_n\}$ si Cauchy in X. Now $|Qx_n(t) - Qx(t)| \to 0 as \; n \to \infty \; for \; all \; t \in I_0 \cup [0, T]$. Then for given $\in > 0$ there exits an $n_0 \in N$ such that

$$\sup_{-\delta \leq p \leq T} |\bar{a}(p)| \int_0^p |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))| ds < \frac{\in}{2},$$

$$\sup_{-\delta \leq p \leq T} |\bar{a}(p)| \int_0^p |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))| ds < \frac{\in}{2}$$

for all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$, then we have

$$\begin{aligned} || Qx_m - Qx_n|| &= \{\sup_{-\delta \leq t < \infty} ||\bar{a}(t)| \int_0^t |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))| ds\} \\ &\quad + \{\sup_{-\delta \leq t < \infty} \left||\bar{a}(t)| \int_0^t |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))\right| ds\} \\ &\leq \sup_{-\delta \leq p \leq T} ||\bar{a}(p)| \int_0^p |f(s, x_n(s), x_n(\theta+s)) - f(s, x_n(s), x_n(\theta+s))| ds| \\ &\quad + \sup_{p \geq T} |\bar{a}(p)| \int_0^p [\,|f(s, x_n(s), x_n(\theta+s))| + |f(s, x_n(s), x_n(\theta+s))|] ds \\ &\quad + \sup_{-\delta \leq p \leq T} ||\bar{a}(p)| \int_0^p |g(s, x_n(s), x_n(\theta+s)) - g(s, x_n(s), x_n(\theta+s))| ds| \\ &\quad + \sup_{p \geq T} |\bar{a}(p)| \int_0^p [\,|g(s, x_n(s), x_n(\theta+s))| + |g(s, x_n(s), x_n(\theta+s))|] ds \; < \; \in. \end{aligned}$$

This shows that $\{Qx_n\} \subset Q(\overline{B_r(0)}) \subset X$ is Cauchy. Since X is complete, $\{Qx_n\}$ converges to a point in X. As $Q(\overline{B_r}(0))$ is closed $\{Qx_n\}$ converges to a point in $Q(\bar{B}_r(0))$. Hence $Q(\bar{B}_r(0))$ is relatively compact and consequently Q is a continuous and compact operator on $\bar{B}_r(0)$ into itself, Now an application of theorem 2.1 to the operator Q on $\overline{B_r}(0)$ `yields that Q has a fixed point in $\overline{B_r}(0)$ which further implies that the FDE (1.1) has a solution defined on $I_0 \cup R_+$.

Finally, we show that the solutions are uniformly attractive on $I_0 \cup R_+$. Let $x, y \in \bar{B}_r(0)$ be any two solutions the FDE (1.1) defined on $I_0 \cup R_+$. Then,

$$\begin{aligned} |x(t) - y(t)| &\leq |\bar{a}(t) \int_0^t f(s, x(s), x_s) ds - \bar{a}(t) \int_0^t f(s, y(s), y_s) ds| \\ &\quad + |\bar{a}(t) \int_0^t g(s, x(s), x_s) ds - \bar{a}(t) \int_0^t g(s, y(s), y_s) ds| \\ &\leq |\bar{a}(t)| \int_0^t |f\, g(s, x(s), x_s)|\, ds + |\bar{a}(t)| \int_0^t |f(s, y(s), y_s) ds \\ &\quad + |\bar{a}(t)| \int_0^t |g(s, x(s), x_s)|\, ds + |\bar{a}(t)| \int_0^t |g(s, y(s), y_s) ds \\ &\leq 2w(t) \end{aligned} \tag{4.4}$$

For all $t \in I_0 \cup R_+$. Since $\lim_{t\to\infty} w(t) = 0$, there is a real number T>0 such that $w(t) < \frac{\in}{2}$ for all t $\geq$ T.

therefore, $|x(t) - y(t)| \leq \in \; for \; all \; t \geq T$, and so all the solutions of the FDE (1.1) are uniformly globally attractive on $I_0 \cup R_+$.

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