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SOME REFINEMENTS OF ENESTROM-KAKEYA THEOREM

M. H. Gulzar*

Department of Mathematics, University of Kashmir, Srinagar, India

E-mail: gulzarmh@gmail.com

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ABSTRACT

In this paper we extend and generalize some known results concerning the Enestrom-Kakeya theorem.

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INTRODUCTION AND STATEMENT OF RESULTS

The following well-known theorem is due to Enestrom and Kakeya [8].

Theorem A: (Enestrom – Kakeya) Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$0 \le a_1 \le a_1 \le \cdots \le a_n$$

then p (z) has all its zeros in the closed unit disk $|z| \le 1$

In the literature, there exist several generalizations of this result, (see [1], [3], [4], [7], [8]). Aziz and Zargar [2] relaxed the hypothesis in several ways and proved:

Theorem B: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0,$$

then all the zeros of P (z) lie in

$$|z + (k - 1)| \le \frac{ka_n + |a_0| - a_0}{|a_n|}$$

For polynomials, whose coefficients are not necessarily real, Govil and Rehamn [6] proved the following generalization of Theorem A.

Theorem C: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with

$$Re(a_j) = \alpha_j \, and \, Im(a_j) = \beta_j \text{ such that}$$
$$\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0 \ge 0$$

where $\alpha_n > 0$, *then* $p(z)$ *has all its zeros in*

$$|z| \le 1 + \left(\frac{2}{\alpha_n}\right)\left(\sum_{j=0}^{n} |\beta_j|\right)$$

***Corresponding author:** *M. H. Gulzar**, ***E-mail:** *gulzarmh@gmail.com*

Recently Govil and Mc-tume [5] proved the following generalization of Theorems B and C

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j, j=1,2,\dots,n. If\ for\ \ some\ k\geq 1,$

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

then P (z) has all its zeros in

$$|z+k-1| \leq \frac{k\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$. If for some $k \geq 1$

$k\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \cdots \geq \beta_0$, then p (z) has all its zeros in

$$|z+k-1| \leq \frac{k\beta_n-\beta_0+|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}$$

In this paper we shall present some interesting generalizations of Theorems D and E, and consequently of Enestrom Kakaye Theorem. We begin with

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j;$ if for some real number $\tau, 0<\tau\leq 1$

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0 > 0$$

then P (z) has all its zeros in

$$|z| \leq 1+\frac{2\left[(1-\tau)\alpha_0+\sum_{j=0}^{n}|\beta_j|\right]}{|\alpha_n|}$$

Remark 1: Taking $\tau=1$, in Theorem 1, we get Theorem C. If we apply Theorem 1 to the polynomial $\{iP(z)\}$, we easily get the following result.

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$. If for some real number $\tau, 0<\tau\leq 1$

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \tau\beta_0 > 0,$$

then p(z) has all its zeros in

$$|z| \leq 1+\frac{2\left[(1-\tau)\beta_0+\sum_{j=0}^{n}|\alpha_j|\right]}{|\beta_n|}$$

Remark 2: If we take $\tau=1$ in Theorem 1 and all the coefficients are real i.e. $\beta_j=0\ for\ all\ j$, it reduces to Enestrom-kakeya Theorem.

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j;$ if for some real numbers $\tau, 0<\tau\leq 1$ and k≥1,

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \tau\alpha_0$$

then P(z) has all its zeros in

$$|z+k-1| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0|+\alpha_0) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Remark 3: Taking $\tau=1$ in theorem 3, we get Theorem D. Applying Theorem 3 to $(-ip(z))$, we get

Theorem 4: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j;$ if for some real numbers $k\geq 1,\ 0<\tau\leq 1$

$$k\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \tau\beta_0$$

then all the zeros of P (z) lie in

$$|z+k-1| \leq \frac{k\beta_n + 2|\beta_0| - \tau(|\beta_0|+\beta_0) + 2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}$$

Theorem 5: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$. If for some real numbers $k\geq 1,\ 0<\lambda\leq n-1$,

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_\lambda, \qquad k\alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$$

then all the zeros of p (z) lie in

$$|z| \leq 1 + \frac{2(k-1)a_\lambda + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Taking λ=n in Theorem 5, we obtain

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$. If for some real numbers $k\geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$$

then all the zeros of p(z) lie in

$$|z| \leq (2k-1) + \frac{2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Remark 4: Taking k=1 in corollary 1, we get Theorem D of Govil and McTume If all the coefficients are real i.e., $\beta_j=0, \forall j$ and k=1, we get Enestrom-Kakeya Theorem.

Applying Theorem 5 to $(-iP(z))$, we get the following interesting result.

Theorem 6 Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$. If for some real numbers $k>1,\ 0<\lambda<n-1$,

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_\lambda\ ,\ k\beta_\lambda \geq \beta_{\lambda-1} \geq \cdots \geq \beta_1 \geq \beta_0 \geq 0$$

then p(z) has all its zeros in

$$|z| \leq 1+\frac{2(k-1)\beta_\lambda + 2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}$$

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned} F(z) &= (1 \quad z)p(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0 \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\qquad -i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})\, z^j. \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \tau\alpha_0)z \\ &\quad + (\tau\alpha_0 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\textstyle\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j \end{aligned}$$

Now

$$\begin{aligned} |F(z)| &= |-\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \tau\alpha_0)z + (\tau\alpha_0 - \alpha_0)z + \alpha_0 \\ &\qquad - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j| \end{aligned}$$

$$\begin{aligned} &\geq |z|^{n+1}|\alpha_n| - |z|^n \Big\{ |(\alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})1/z + \cdots + (\alpha_1 - \tau\alpha_0)1/z^{n-1} \\ &\qquad + (\tau - 1)\alpha_0 . 1/z^{n-1} + \alpha_0/z^n - i\beta_n z + i\beta_0 1/z^n \\ &\qquad + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})\, 1/z^{n-j}\,| \Big\} \end{aligned}$$

For |z|>1, we have

$$|F(z)| > |z|^n \Bigg[|\alpha_n||z| - \{|\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}|\ 1/|z| + \cdots + |\alpha_1 - \tau\alpha_0|\ 1/|z|^{n-1} + |(\tau - 1)||\alpha_0|\ 1/|z|^{n-1} + |\alpha_0|/|z|^n + |\beta_n||z| + |\beta_0|\ 1/|z|^n + \sum_{j=1}^{n} (|\beta_j| + |\beta_{j-1}|)\ 1/|z|^{n-j}\} \Bigg]$$

$$> |z|^n \Bigg[|\alpha_n||z| - \{(\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + \alpha_1 - \tau\alpha_0 + (1-\tau)\alpha_0 + \alpha_0 + |\beta_n| + |\beta_0| + \sum_{j=1}^{n} |\beta_j| + |\beta_{j-1}|\} \Bigg]$$

(By hypothesis)

$$= |z|^n [|\alpha_n||z| - \{\alpha_n + 2(1-\tau)\alpha_0 + 2\textstyle\sum_{j=0}^{n} |\beta_j|\}] > 0 \ if$$

$$|z| > 1 + \frac{2(1-\tau)\alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}$$

Hence all the zeros of F (z) whose modulus is greater than 1 lie in

$$|z| \le 1 + \frac{2(1-\tau)\alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}$$

Since all the zeros of F(z) whose modulus is $\le$ 1 already satisfy the inequality, it follows that all the zeros of F(z) and that of P(z) lie in the circle

$$|z| \le 1 + \frac{2(1-\tau)\alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}$$

This completes the proof of the Theorem 1.

Proof of Theorem 3: Consider

$$\begin{aligned} F(z) &= (1-z)p(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + a_1 z + a_0 \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad -i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j \end{aligned}$$

$$= -\alpha_n z^{n+1} - k\alpha_n z^n + \alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \tau\alpha_0)z + (\tau\alpha_0 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j$$

Now for $|z| > 1$

$$|F(z)| = \left| -\alpha_n z^{n+1} - k\alpha_n z^n + \alpha_n z^n + (k\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \tau\alpha_0)z + (\tau\alpha_0 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j \right|$$

$$\geq |z|^n |\alpha_n z + k\alpha_n - \alpha_n| - |z|^n$$

$$\left| (k\alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})1/z + \cdots + (\alpha_1 - \tau\alpha_0)1/z^{n-1} + (\tau\alpha_0 - \alpha_0)1/z^{n-1} + \alpha_0/z^n - i\beta_n z + \frac{i\beta_0}{z^n} + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})1/z^{n-j} \right|$$

$$> |z|^n [|\alpha_n||z + (k-1)| - |z|^n \left(|k\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}|\frac{1}{|z|} + \cdots + |\alpha_1 - \tau\alpha_0|\frac{1}{|z|^{n-1}} + |1-\tau||\alpha_0|\frac{1}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} + |\beta_n||z| + \frac{|\beta_0|}{|z|^n} + \sum_{j=1}^{n}|\beta_j| + |\beta_{j-1}|\frac{1}{|z|^{n-j}} \right]$$

$$> |z|^n \left[|\alpha_n|\,|z + (k-1)| - |z|^n \left\{ (k\alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \cdots + (\alpha_1 - \tau\alpha_0) + (1-\tau)|\alpha_0| + |\alpha_0| + |\beta_n| + |\beta_0| + \sum_{j=1}^{n}|\beta_j| + |\beta_{j-1}| \right\} \right]$$

$$= |z|^n \left[|\alpha_n||z + k - 1| - \left(k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j| \right) \right]$$

$$> 0, if$$

$$|z + k - 1| > \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Hence all the zeros of $F(z)$ whose modulus is greater than 1 lie in

$$|z + k - 1| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

But all the zeros of F (z) whose modulus is ≤ 1 already satisfy the inequality. This shows that all the zeros of F (z) and hence of P (z) lie in the disk

$$|z+k-1| \leq \frac{k\alpha_n + 2|\alpha_0| - \tau(|\alpha_0| + \alpha_0) + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

That proves the result.

Proof of Theorem 5: Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)p(z) \\ &= (1-z)\left(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{\lambda+1}z^{\lambda+1} + a_\lambda z^\lambda + a_1 z + a_0\right) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + \\ &\qquad (a_1 - a_0)z + a_0 \\ &= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \cdots \\ &\qquad + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1}) \end{aligned}$$

For $|z| > 1,$ *we have*

$$\begin{aligned} |F(z)| &= \Big| -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} - k\alpha_\lambda z^\lambda + \alpha_\lambda z^\lambda + (k\alpha_n - \alpha_{n-1})z^n \\ &\qquad + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i\beta_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j \Big| \end{aligned}$$

$$\begin{aligned} &\geq |z|^n \Big[|\alpha_n||z| - \Big\{\Big|(\alpha_n - \alpha_{n-1}) + \cdots + (\alpha_{\lambda+1} - \alpha_\lambda)\frac{1}{z^{n-\lambda-1}} - (k-1)\alpha_\lambda \frac{1}{z^{n-\lambda}} + (k\alpha_\lambda - \alpha_{\lambda-1})\frac{1}{z^{n-\lambda}} \\ &\qquad + \cdots + (\alpha_1 - \alpha_0)\frac{1}{z^{n-1}} + \alpha_0 \frac{1}{z^n} - i\beta_n z + i\beta_0 \frac{1}{z^n} + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})\frac{1}{z^{n-j}}\Big|\Big\}\Big] \end{aligned}$$

$$\begin{aligned} &> |z|^n \Big[|\alpha_n||z| - \Big\{|(\alpha_n - \alpha_{n-1})| + \cdots + |(\alpha_{\lambda+1} - \alpha_\lambda)| + (k-1)|\alpha_\lambda| + |(k\alpha_\lambda - \alpha_{\lambda-1})| \\ &\qquad + \cdots + |(\alpha_1 - \alpha_0)| + |\alpha_0| - |\beta_n| + |\beta_0| + \sum_{j=1}^{n}|\beta_j| + |\beta_{j-1}|\Big\}\Big] \end{aligned}$$

$$\begin{aligned} &= |z|^n \Big[|\alpha_n||z| - \{(\alpha_n - \alpha_{n-1}) + \cdots + (\alpha_{\lambda+1} - \alpha_\lambda) + (k-1)\alpha_\lambda + (k\alpha_\lambda - \alpha_{\lambda-1}) \\ &\qquad + \cdots + (\alpha_1 - \alpha_0) + \alpha_0 + 2\sum_{j=0}^{n}|\beta_j|\Big\}\Big] \end{aligned}$$

$$= |z|^n \left[|\alpha_n||z| - \left\{-\alpha_n + 2(k-1)\alpha_\lambda + 2\sum_{j=0}^{n}|\beta_j|\right\}\right] \quad >0, \text{ if}$$

$$|z| > 1 + \frac{2(k-1)\alpha_\lambda + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in

$$|z| \leq 1 + \frac{2(k-1)a_\lambda + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

But all those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Thus, all the zeros of F(z) and therefore P(z) lie in the circle defined

$$|z| \leq 1 + \frac{2(k-1)a_\lambda + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}$$

This completes the proof of Theorem 5.

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