On Strongly Generalized $b$-Closed Sets

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ABSTRACT

In this paper, we study a new class of generalized sets called strongly generalized $b$-closed sets, briefly $g^b$-closed sets. We study some of their properties. These sets are placed between the class of gs-closed sets and gp-closed sets.

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1. INTRODUCTION AND PRELIMINARIES:

Generalized closed sets form a strong tool in the characterization of topological spaces satisfying weak separation axioms. The concept of generalization was first initiated by Levine [4] in 1963. Since then this method of generalizing sets was adopted by many topologists. Andrijevic [1] introduced a new class of generalized open sets in a topological space, the so-called b-open sets. The class of b-open sets is contained in the class of semi-preopen sets and contains all semi-open and preopen sets. The class of b-open sets generates the same topology as the class of preopen sets. Extensive research on generalizing closedness was done in recent years as the notions of a generalized closed, generalized semi-closed, $\alpha$-generalized closed, generalized semi-preopen closed sets were investigated in [2,3,5].

The aim of this paper is to continue the study of generalized closed sets. In particular, the notion of strongly generalized b-closed sets and its various characterizations are given in this paper. All through this paper, all spaces $X$ and $Y$ (or $(X, \tau)$ and $(Y, \sigma)$) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of $A$ and the interior of $A$ will be denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively.

Definition: 1.1 [7] A subset $A$ of a space $X$ is said to be:

1. $\alpha$-open if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$;
2. Semi-open if $A \subseteq \text{Cl}(\text{Int}(A))$;
3. Preopen or nearly open if $A \subseteq \text{Int}(\text{Cl}(A))$;
4. $\beta$-open or semi-preopen if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$;
5. b-open or sp-open if $A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A))$.

The complement of a b-open set is said to be b-closed [2]. The intersection of all b-closed sets of $X$ containing $A$ is called the $b$-closure of $A$ and is denoted by $b\text{Cl}(A)$. The union of all b-open sets of $X$ contained in $A$ is called $b$-interior of $A$ and is denoted by $b\text{Int}(A)$. The family of all b-open (resp. $\alpha$-open, semi-open, preopen, $\beta$-open, b-closed, preclosed) subsets of a space $X$ is denoted by $bO(X)$ (resp. $\alpha O(X)$, $SO(X)$, $PO(X)$, $\beta O(X)$, $bC(X)$, $PC(X)$) and the collection of all b-open subsets of $X$ containing a fixed point $x$ is denoted by $bO(X, x)$. The sets $SO(X, x)$, $\alpha O(X, x)$, $PO(X, x)$, $\beta O(X, x)$ are defined analogously.

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Definition: 1.2[6] A subset A of a space (X, τ) is called
(1) a generalized closed set (briefly g-closed) if Cl(A) ⊆ U whenever A ⊆ U and U is open;
(2) a generalized preclosed set (briefly gp-closed) if pCl(A) ⊆ U whenever A ⊆ U and U is open;
(3) a generalized semi-preclosed set (briefly gp-closed) if spCl(A) ⊆ U whenever A ⊆ U and U is open;
(5) a generalized b-closed set (briefly gb-closed) [15] if bCl(A) ⊆ U whenever A ⊆ U and U is open.

Complements of g-closed (resp. gp-closed, etc.) sets are called g-open (resp. gp-open, etc.)

2. STRONGLY GENERALIZED b-CLOSED SETS:

Definition: 2.1 A subset of a topological space (X, τ) is said to be g*b-closed set in (X, τ) if bcl(A) ⊆ G whenever A ⊆ G where G is g-open. The collection of all g*b-closed sets of (X, τ) is denoted by G*bC(X, τ).

Theorem 2.2: If a subset A of a topological space (X, τ) is closed, then it is g*b-closed.

Proof: Let G be a g-open set containing A. Then G ⊇ A = cl(A) as A is closed. Also cl(A) ⊇ bcl(A). Thus G ⊇ bcl(A). Hence A is a g*b-closed set in (X, τ).

The converse of the above theorem need not be true as seen from the following example.

Example: 2.3 Let X= {a, b, c} and τ = {X, φ, {a}}, then the subset {c} is g*b-closed but not closed in (X, τ).

Corollary: 2.4 If a subset A of a topological space (X, τ) is regular closed, then it is g*b-closed but not conversely.

Proof: Since every regular closed set is closed but not conversely. By theorem 2.2 every closed set is g*b-closed but not conversely. Hence every regular closed set is g*b-closed but not conversely.

Theorem 2.5 If a subset A of a topological space (X, τ) is g*b-closed, then it is gb-closed.

Proof: Let G be an open set containing A. Then G ⊇ A = bcl(A) as A is gb-closed. Hence A is a gb-closed set in (X, τ).

The converse of theorem 2.5 need not be true as seen from the following example.

Example: 2.6 Let X= {a, b, c} and τ = {X, φ, {a}, {b}, {a, b}, {a, b, c}}, then the subset {b, c} is gb-closed but not g*b-closed set in (X, τ).

Theorem: 2.7 If a subset A of a topological space (X, τ) is b-closed, then it is g*b-closed.

Proof: Let G be an open set containing A. Then G ⊇ A = bcl(A) as A is b-closed. Thus G ⊇ bcl(A). Hence A is g*b-closed in (X, τ).

The converse of the above theorem need not be true and it can be seen from the following example.

Example: 2.8 Let X= {a, b, c} and τ = {X, φ, {a}, {b}, {a, b}}, then the subset {b} is g*b-closed but not b-closed in (X, τ).

Theorem: 2.9 If a subset A of a topological space (X, τ) is g*b-closed, then it is gb-closed.

Proof: Let G be an open set containing A. Then G ⊇ A = bcl(A) as A is g*b-closed. Hence A is gb-closed in (X, τ).

The converse of the above theorem is not true and it can be seen from the following example.

Example 2.10: Let X= {a, b, c} and τ = {X, φ, {a}, {a, b}}, then the subset {a, c} is gb-closed but not g*b-closed.

Theorem: 2.11 Let A be a subset of a topological space (X, τ). If A is g*b-closed, then it is gs-closed.
Proof: Let \( G \) be an open set containing \( A \). Then \( G \supseteq \text{bcl}(A) \), as \( A \) is \( g^*b \)-closed. Thus \( G \supseteq \text{bcl}(A) \supseteq \text{Scl}(A) \).

Therefore \( A \) is \( gs \)-closed in \((X, \tau)\).

The following example shows that the converse of the above theorem need not be true.

Example: 2.12 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then the subset \( \{a\} \) is \( gs \)-closed but is not \( g^*b \)-closed in \((X, \tau)\).

Remark: 2.13 The following example shows that every \( g \)-closed set is \( g^*b \)-closed but not conversely.

Example: 2.14 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{b\}, \{a, b\}\} \), then the subset \( \{a\} \) is \( g^*b \)-closed but not \( g \)-closed.

Remark: 2.15 The following examples shows that the concept of semi-closed and \( g^*b \)-closed sets are independent.

Example: 2.16 (a) Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then the subset \( \{a, c\} \) is \( g^*b \)-closed but not semi-closed.

Example: 2.16 (b) Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\} \), then the subset \( \{b\} \) is semi closed but not \( g^*b \)-closed.

Remark: 2.17 The following examples shows that the concept of pre-closed and \( g^*b \)-closed sets are independent.

Example: 2.18 Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\} \), then the subset \( \{a, c, d\} \) is \( g^*b \)-closed but not pre-closed.

Remark: 2.20 The following examples shows that the concept of \( g \)-closed and \( g^*b \)-closed sets are independent.

Example: 2.21 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}\} \), then the subset \( \{a, b\} \) is \( g \)-closed but not \( g^*b \)-closed.

Example: 2.22 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then the subset \( \{b\} \) is \( g^*b \)-closed but not \( g \)-closed.

Remark: 2.23 The following examples shows that the concept of \( bg^* \)-closed and \( g^*b \)-closed sets are independent.

Example: 2.24 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then the subset \( \{a, c\} \) is \( g^*b \)-closed but not \( bg^* \)-closed in \((X, \tau)\).

Example: 2.25 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), then the subset \( \{a, b\} \) is \( bg^* \)-closed but not \( g^*b \)-closed in \((X, \tau)\).

Remark: 2.26 The following diagram shows the relationship between \( g^*b \)-closed sets with various sets.

![Diagram showing relationships between various types of closed sets]

where \( A ightarrow B \) (resp. \( A \rightarrow B \)) represents \( A \) implies \( B \) and \( B \) need not implies \( A \) (resp. \( A \) & \( B \) are independent of each other).

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3. PROPERTIES OF \( g^*b \)-CLOSED SETS IN TOPOLOGICAL SPACES:

**Theorem 3.1**

(i) Let \( A \subset (X, \tau) \) be \( g^*b \)-closed. Then \( \text{Cl}(A) \backslash A \) contains no non empty \( b \)-closed set.

(ii) If \( A \) is \( g^*b \)-closed \( A \subset B \subset \text{cl}(A) \), then \( \text{cl}(B) \backslash B \) contains no non empty \( b \)-closed sets.

**Proof:**

(i) Let us suppose that \( A \) is \( g^*b \)-closed and \( F \) is any \( b \)-closed subset of \( \text{Cl}(A) \backslash A \). Then \( F \subset X \backslash A \Rightarrow A \subset X \backslash F \) is \( b \)-open. Since \( A \) is \( g^*b \)-closed, \( b\text{cl}(A) \subset X \backslash F \). That is \( F \subset X \backslash b\text{cl}(A) \).

We already have \( F \subset b\text{Cl}(A) \). So \( F \subset b\text{cl}(A) \cap X \backslash b\text{cl}(A) = \emptyset \). Thus \( F = \emptyset \). Hence \( \text{Cl}(A) \backslash A \) contains no non empty \( b \)-closed set.

(ii) Let \( A \) be \( g^*b \)-closed and \( A \subset B \subset \text{cl}(A) \), then we have \( \text{cl}(B) \cap X \backslash B \subset \text{cl}(A) \cap X \backslash A \). That is \( \text{cl}(B) \backslash B \subset \text{cl}(A) \backslash A \).

By (i) \( \text{cl}(A) \backslash A \) has no nonempty \( b \)-closed set. Hence \( \text{cl}(B) \backslash B \) contains no nonempty \( b \)-closed set.

**Theorem 3.2**

A \( g^*b \)-closed set \( A \) is \( b \)-closed if and only if \( b\text{cl}(A) - A \) is \( bg \)-closed.

**Proof:**

Necessity: Since \( A \) is \( b \)-closed, we have \( b\text{cl}(A) = A \). Then \( b\text{cl}(A) - A = \emptyset \) is \( b \)-closed and hence \( bg \)-closed.

Sufficiency: By theorem 3.1, \( b\text{cl}(A) - A \) contains no non empty \( bg \)-closed set. That is \( b\text{cl}(A) - A = \emptyset \).

Therefore \( b\text{cl}(A) = A \). Hence \( A \) is \( b \)-closed.

**Theorem 3.3**

If \( A \) is a \( g^*b \)-closed set and \( B \) is any set such that \( A \subset B \subset b\text{cl}(A) \), then \( B \) is a \( g^*b \)-closed set.

**Proof:**

Let \( B \subset U \) where \( U \) is \( g \)-open set. Since \( A \) is \( g^*b \)-closed set and \( A \subset U \), then \( b\text{cl}(A) \subset U \) and also \( b\text{cl}(A) = b\text{cl}(B) \).Therefore \( b\text{cl}(B) \subset U \) and hence \( B \) is a \( g^*b \)-closed set.

**Theorem 3.4**

(i) The intersection of a \( g^*b \)-closed set and a \( b \)-closed sets is always a \( g^*b \)-closed set.

(ii) If \( A \) is a \( g^*b \)-closed set and \( A \subset B \subset \text{cl}(A) \), then \( B \) is \( g^*b \)-closed set.

**Proof:**

(i) Let \( A \) be \( g^*b \)-closed set and let \( F \) be a \( b \)-closed set. Suppose \( G \) is a \( g \)-open set with \( A \cap F \subset G \), then \( A \subset G \cup F \) where \( G \cup F \) is \( b \)-open.

Therefore \( b\text{cl}(A) \subset G \cup F \). Now \( b\text{cl}(A \cap F) \subset b\text{cl}(A) \cap b\text{cl}(F) = b\text{cl}(A) \cap F \subset G \).

Hence \( A \cap F \) is a \( g^*b \)-closed set.

(ii) Let \( A \) be \( g^*b \)-closed and \( B \subset G \) where \( G \) is a \( g \)-open set. Then \( A \subset G \). Since \( A \) is \( g^*b \)-closed, \( b\text{cl}(A) \subset G \).

Hence by assumption \( b\text{cl}(B) \subset b\text{cl}(A) \subset G \). Thus \( b\text{cl}(B) \subset G \) implies that \( B \) is \( g^*b \)-closed.

**Theorem 3.5**

Let \( \{ A_i : i \in I \} \) be a locally finite family of \( g^*b \)-closed sets. Then \( A = \bigcup A_i \) is \( g^*b \)-closed for every \( i \in I \).

**Proof:**

Since \( \{ A_i : i \in I \} \) is locally finite, \( cl(\bigcup A_i) = \bigcup cl(A_i) \). Assume that for some \( b \)-open set we have \( A = \bigcup A_i \subset U \). Then \( cl(\bigcup A_i) = \bigcup cl(A_i) \subset U \), since each \( A_i \) is \( g^*b \)-closed. Thus \( A \) is \( g^*b \)-closed.

**Remark 3.6**

The spaces \( g^*bTg^* \) and space \( g^*bTb \) are independent as seen from the following examples:

**Example 3.7**

Let \( X = \{ a, b, c \} \) with topology \( \tau = \{ X, \phi, \{ a, b \} \} \). Then \( (X, \tau) \) is a \( g^*bTg^* \)-space but not a \( g^*bTb \)-space, since \( \{ a,c \} \) is \( g^*-closed \) but not \( b \)-closed in \( (X, \tau) \).

**Example 3.8**

Let \( X = \{ a, b, c \} \) with topology \( \tau = \{ X, \phi, \{ a \}, \{ a, b \} \} \). Then \( (X, \tau) \) is a \( g^*bTb \)-space but not a \( g^*bTg^* \)-space, since \( \{ b \} \) is \( g^*b \)-closed but not \( g^*-closed \) in \( (X, \tau) \).

**Theorem 3.9**

If \( (X, \tau) \) is both \( b \)-space and \( g^*bTb \)-space, then it is a \( g^*bTg^* \)-space.

**Proof:**

Let \( A \) be a \( g^*b \)-closed set in \( (X, \tau) \). Since \( (X, \tau) \) is a \( g^*bTb \)-space, \( A \) is \( b \)-closed in \( (X, \tau) \). Since \( (X, \tau) \) is a \( b \)-space, every \( b \)-closed set is closed and hence \( A \) is closed in \( (X, \tau) \). We know that every closed set is \( g^*-closed \) in
Theorem 3.10: If $(X, \tau)$ is both $T^*_{1/2}$ –space and $g^*bTg^*$ –space, then it is a $g^*bTb$ –space.

Proof: Let $A$ be a $g^*b$-closed set in $(X, \tau).$ Since $(X, \tau)$ is a $g^*bTg^*$ –space, $A$ is $g^*$-closed. Since $(X, \tau)$ is a $T^*_{1/2}$-space, $A$ is closed in $(X, \tau)$. Since every closed set is $b$-closed, $A$ is $b$-closed in $(X, \tau)$. Hence it is a $g^*bTb$ –space.

Remark: 3.11 In a semiregular space $T1/2$ –space, the concepts of $g^*b$ closed, $g$-closed and closed sets coincide.

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