International Journal of Mathematical Archive-8(11), 2017, 47-54
Available online through www.ijma.info ISSN 2229 – 5046

STRONG BLOCK DOMINATION IN GRAPHS

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(Received On: 13-10-17; Revised & Accepted On: 09-11-17)

ABSTRACT

For any graph $G = (V, E)$, the block graph $B(G)$ is a graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. For any two adjacent vertices u and v we say that u strongly dominates v if $deg\ (u) \geq deg\ (v)$. A dominating set D of a graph $B(G)$ is a strong block dominating set of G if every vertex in $V[B(G)] - D$ is strongly dominated by at least one vertex in D. Strong block domination number $\gamma_{SB}(G)$ of G is the minimum cardinality of strong block dominating set of G. In this paper, we study graph theoretic properties of $\gamma_{SB}(G)$ and many bounds were obtain in terms of elements of G and its relationship with other domination parameters were found.

***Keywords**: Dominating set/ Independent domination/ Block graph/ Line graph/ Roman domination/ Strong split domination/ Strong block domination.*

***Subject Classification number:** AMS - 05C69, 05C70.*

1. INTRODUCTION

In this paper, all the graphs consider here are simple and finite. For any undefined terms or notation can be found in Harary [5]. In general, we use $< X >$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N([v])$ denote open (closed) neighborhoods of a vertex v. Let $deg(v)$ is the degree of vertex v and as usual $\delta(G)(\Delta(G))$ is the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The notation $\alpha_o(G)(\alpha_1(G))$ is the minimum number of vertices (edges) in vertex (edge) cover of G. The minimum distance between any two farthest vertices of a connected G is called the diameter of G and is denoted by $diamG$. A block graph $B(G)$ is the graph whose vertices corresponds to the blocks of G and two vertices in $B(G)$ are adjacent if and only if the corresponding blocks in G are adjacent.

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A set $S \subseteq V[B(G)]$ is said to be a dominating set of $B(G)$, if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of $B(G)$ and is denoted by $\gamma[B(G)]$. A dominating set S is called the total dominating set, if for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$ such that u is adjacent to v. The total domination number of G is denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. A dominating set $S \subseteq V(G)$ is a connected dominating set, if the induced subgraph $< S >$ has no isolated vertices. The connected domination number $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set of G. Also in terms of connected block domination $\gamma_{cb}(G)$ which is discussed in [13]. Also characterized graphs achieving these bounds.

The concept of Roman domination function (RDF) was introduced by E.J. Cockayne, P.A.Dreyer, S.M.Hedetiniemi and S.T.Hedetiniemi in [1]. A Roman dominating function on a graph $G = (V, E)$ is a function $f: V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex of v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{v \epsilon V} f(v)$. The Roman domination number of a graph, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G. A Roman dominating function $f = (V_0, V_1, V_2)$ on a graph G is a connected Roman dominating function (CRDF) on G if $\langle V_1 \cup V_2 \rangle$

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or $\langle V_2 \rangle$ is connected. The minimum weight of a CRDF is called a connected Roman domination number of G and is denoted by $\gamma_{RC}(G)$, (see[12]). A dominating set $S \subseteq V(G)$ is restrained dominating set of G, if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) - S$. The restrained domination number of a graph G is denoted by $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set in G. The concept of restrained domination in graphs was introduced by Domke *et.al.*,[2].

The concept of strong split block domination in graphs was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [9]. A dominating set D of a graph G is a strong split block dominating set if the induced subgraph $\langle V[B(G)] - D \rangle$ is totally disconnected with at least two vertices. The strong split block domination number $\gamma_{ssb}(G)$of G is the minimum cardinality of strong split block dominating set of G. The concept of strong nonsplit block domination in graphs was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [10]. A dominating set D of a graph $B(G)$ is a strong nonsplit block dominating set if the induced subgraph $\langle V[B(G)] - D \rangle$ is complete. The strong nonsplit block domination number $\gamma_{snsb}(G)$ of G is the minimum cardinality of strong nonsplit block dominating set of G. Recently we study a variation on the domination which is called strong line domination in graphs, was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [11]. A dominating set D of a graph $L(G)$ is a strong line dominating set if every vertex in $\langle V[L(G)] - D \rangle$ is strongly dominated by at least one vertex in D. Strong line domination number $\gamma_{SL}(G)$ of G is the minimum cardinality of strong line dominating set of G. Analogously, a dominating set $S \subseteq$V(G) is a cototal dominating set, if the induced subgraph $< V - S >$ has no isolated vertices. The cototal domination number, $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set of G. This concept was introduced by Kulli *et.al*, [3]. A dominating set D of a graph G is a split dominating set of G if the induced subgraph <V-D> is disconnected (see [4]). The split domination number $\gamma_s(G)$ is the minimum cardinality of the minimal split dominating set of G.

The concept of a dominating set D of a graph G is a strong split dominating set if the induced subgraph $\langle V - D \rangle$ is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ of graph G is the minimum cardinality of a strong split dominating set of G. The concept of Strong domination was introduced by Sampathkumar and Pushpa Latha in [14] and well studied in [6, 7 and 8]. Given two adjacent vertices u and v we say that u strongly dominates v if deg $(u) \geq$ deg (v). A set $D \subseteq V(G)$ is strong dominating set of G if very vertex in $V - D$ is strongly dominated by at least one vertex in D. The strong domination number $\gamma_s(G)$ is the minimum cardinality of a strong dominating set of G. A dominating set D of a graph $B(G)$ is a strong block dominating set of G if every vertex in $V[B(G)] - D$ is strongly dominated by at least one vertex in D. Strong block domination number $\gamma_{SB}(G)$ of G is the minimum cardinality of strong block dominating set of G. In this paper, many bounds on $\gamma_{SB}(G)$ were obtained in terms of elements of G but not the elements of $B(G)$. Also its relation with other domination parameters were established.

2. MAIN RESULTS

First we obtained necessary and sufficient condition on G for which $\gamma_{SB}(G)$ is connected.

Theorem 1: For any graph G with at least two block, then $\gamma_{SB}(G) \leq q - 1$.

Proof: Suppose block graph $B(G)$ has at least two vertices. Then G has at least two blocks. If two blocks of G are edges, then $\gamma_{SB}(G) = q - 1$. Otherwise the inequality holds. Thus $\gamma_{SB}(G) \leq q - 1$.

Theorem 2: For any connected (p, q) graph G, $\gamma_{SB}(G) \leq \gamma_{bc}(G)$.

Proof: Let $H = \{B_1, B_2, ..., B_n\}$ be the set of blocks of G and $B = \{B_1, B_2, ..., B_i\}$ be the set of all non-end blocks of G. Let $B_1 = \{b_1, b_2, ..., b_i\}$ be the vertices of block graph $B(G)$ corresponding to the elements of B. Since $\forall b_j \in B_1, 1 \leq j \leq i$ is a cutvertex in $B(G)$, then there exists a set $B_1^{'} \subseteq B_1$ such that $\forall b_k \in B_1^{'}$ is adjacent to at least one vertex of $V[B(G)] - B_1^{'}$ and $\langle B_1^{'} \rangle$ is connected clearly $B_1^{'}$ is a $\gamma_{bc}(G) - set$. Let $B_1^{"} = \{b_1, b_2, ..., b_n\} \subseteq B_1^{'}$ and if $\forall v \in B_1^{"}$, $\deg(v) \geq \deg(u)$, $\forall u \in V[B(G)] - B_1^{"}$, $N[B_1^{"}] = V[B(G)]$. Then $B_1^{"}$ is a $\gamma_{SB} - set$. Hence $|B_1^{'}| \geq |B_1^{"}|$ gives $\gamma_{SB}(G) \leq \gamma_{bc}(G)$.

Corollary: For any block graph G with $p \geq 2$ vertices, $\gamma_{SB}(G) \geq \lceil p/3 \rceil$.

Theorem 3: For any non-trivial connected tree T, $\gamma_{SB}(T) \leq \gamma_{Rc}(T)$.

Proof: Let G be any connected graph with a CRDF $f = (V_0^{'}, V_1^{'}, V_2^{'})$. Suppose G be a non-trival tree T. Let $V_{en} = \{v_1, v_2, ..., v_{en}\}$ be the set of all end vertices, $V_c = \{v_1, v_2, ..., v_c\}$ be the set of all cutvertices in T such that $V(T) = V_c \cup V_{en}$ and $V_c^{'} \subseteq V_c$ be the set of all cutvertices adjacent to end vertices in T. Then $\forall v_i \in V_c^{'}$, $w(v_i) = 2$ and $\forall V_j \in V_c^{'} \subseteq V_c / V_c^{'}$, $w(v_j) = 1$ such that $w(N(v_i) \cap N(v_j)) = 1$ or 2. Then $\langle v_i v_j \rangle$ is connected. Hence V_c forms $\gamma_{Rc} - set$ in T and $|V_c| = |V_1^{'}| + |V_2^{'}| = \gamma_{Rc}(T)$. Next we consider $\{b_1, b_2, ..., b_n\}$ be the set of vertices of $B(T)$ corresponding to the blocks $\{B_1, B_2, ..., B_n\}$ of T. Let $D = \{b_1, b_2, ..., b_m\}$ where $m < n$ is a minimal dominating set of $B(T)$ such that $V[B(T)] - D = N, \forall v_i \in N$, $\deg(v_i) \leq \deg(v_j), \forall v_j \in D$. Then $|D| = \gamma_{SB}(T)$. Hence $\gamma_{SB}(T) = |D| \geq |V_c| = \gamma_{Rc}(T)$ which gives $\gamma_{SB}(T) \leq \gamma_{Rc}(T)$.

Theorem 4: For any connected tree T with $p \geq 4$, then $\gamma_{SB}(T) \geq \gamma(T) - 1$.

Proof: Let $V = \{v_1, v_2, ..., v_p\}$ be the set of all vertices of T and suppose $D = \{v_1, v_2, ..., v_l\}$, $l < p$ be the minimal dominating set of T such that $|D| = \gamma(T)$. Let $A = \{B_1, B_2, ..., B_{p-1}\}$ be the set of all blocks of T and $H = \{b_1, b_2, ..., b_{P-1}\}$ be the corresponding block vertices in $B(T)$. $\forall B_i$ adjacent to end blocks containing $v_i \in D$ in T, there exists a corresponding blocks vertex set $\{b_i\}$ in $B(T)$ such that $\{b_i\} \in V_2 \cup V_1$ and B_j not adjacent to end blocks in T there exist a corresponding block vertex set $\{b_i\}$ in $B(T)$ such that $\{b_j\} \in V_1$. Hence $\langle b_i \cup b_j \rangle$ is strongly dominated by at least one vertex in $D^{'}$ and it forms $\gamma_{SB} - set$ such that $|V_1| + |V_2| = |D^{'}| = \gamma_{SB}(T)$. Clearly $|D| \leq |D^{'}| - 1$ gives $\gamma_{SB}(T) \geq \gamma(T) - 1$.

In the following theorem we obtain the relation between for $\gamma_{SB}(G)$ in term of $\gamma_{ss}(G)$.

Theorem 5: For any connected (p, q) graph G, $\gamma_{SB}(G) \leq \gamma_{ss}(G)$.

Proof: let S' be a maximum independent set of vertices in G and $S'' \subset S'$ be the of all isolated vertices in $< S'' >$. Then $(V - S') \cup S''$ is a strong split dominating set of G. Since for each vertex $v \in (V - S') \cup S''$ either v is an isolated vertex in $< (V - S') \cup S' >$ or there exists a vertex $u \in S' - S''$ and v is adjacent to $uv, (V - S') \cup S''$ is minimal.Since S' is maximum, $(V - S') \cup S''$ is minimum. Thus $|(V - S') \cup S''| = \gamma_{ss}(G)$. Let $F = \{e_1, e_2, e_3, ..., e_n\}$ be set of edges in G and $F \subseteq E(G)$. Then in $B(G), D' = \{v_1, v_2, v_3, ..., v_n\}$ which corresponds to $\forall e_i \in F$. Let deg (e_i),$\forall e_i \in F$ and $\deg(e_j)$ $\forall e_j \in E(G) - F$ such that $\deg(e_i) \geq \deg(e_j)$. Suppose $D'' = \{v_1, v_2, v_3, ..., v_i\} \subseteq D'$ and $N[v_k] = V(B(G))$, $\forall u_k \in D''$, $1 \leq k \leq i$. Then D'' forms a $\gamma_{SB} - set$. It follows that $|D''| \leq |(V - S') \cup S''|$. Hence $\gamma_{SB}(G) \leq \gamma_{ss}(G)$.

Theorem 6: For any connected (p, q) graph G, $\gamma_{SB}(G) \leq p - \Delta(G)$.

Proof: Suppose G is a connected graph with $n-$ blocks in which at least one block has maximum number of vertices with $\Delta(G) \geq 3$. Then in $B(G)$, $|\gamma_{SB}| - set$ is always less than $p - \Delta(G)$. Now we consider the graph G such that each block of G is an edge. Let $B = \{B_1, B_2, B_3, ..., B_k\}$, be the set of blocks in G. Suppose $F = \{v_1, v_2, ..., v_k\} \subseteq V(B(G))$ be the set of vertices with $\deg(v_j) \geq 2$. Suppose there exists a vertex set

$D \subseteq F$ with $N[D] = V(B(G))$ and if $|\deg(x) - \deg(y)| \leq 1, \forall x \in D$, $y \in V(B(G)) - D$. Then D forms a strong block dominating set in $B(G)$. Otherwise there exists at least one vertex $\{w\} \subseteq F$ where $\{w\} \notin D$ such that $D \cup \{w\}$ forms a minimal $\gamma_{SB} - set$ in $B(G)$. Since for any graph G, there exists at least one vertex $v \in V(G)$ of maximum degree $\Delta(G)$, it follows that $|D \cup \{w\}| \leq p \cup |\deg(v)|$. Clearly, $\gamma_{SB}(G) \leq p - \Delta(G)$.

Theorem 7: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq diam(G) - 1$.

Proof: Suppose $A = \{e_1, e_2, ..., e_i\} \subseteq E(G)$ be the minimal set of edges which constitute the longest path between any two distinct vertices $u, v \in V(G)$ with $dist(u,v) = diam(G)$. Let $\{u_1, u_2, u_3, \dots \dots \dots, u_n\} \subseteq V[B(G)]$ be the set of vertices. Suppose $D \subseteq k$ be the set of vertices with $\deg(w) \geq 3$ for every $w \in D$. Assume there exists $D' \subseteq D$ such that $\forall u_j \in D'$ $\deg(u_j) \geq \deg(u_k), \forall u_k \in V[B(G)] - D'$. Clearly D' forms a strong block dominating set. Since each block in G is either an edge of at least one block contain more than two edges, then $|D'| \leq diam(G) - 1$ which gives $\gamma_{SB}(G) \leq diam(G) - 1$.

The following theorem we obtain upper bound for $\gamma_{SB}(G)$ in terms of Roman domination number $\gamma_R(G)$.

Theorem 8: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq \gamma_R(G) + \Delta(G) - 3$.

Proof: Let $f = (V_0, V_1, V_2)$ be any γ_R −function of G. Suppose $V_1 \cup V_2$ or V_2 form a $\gamma_R - set$ of G such that $|H| = \gamma_R(G)$. Next we consider $\{b_1, b_2, b_3, \dots \dots \dots, b_n\}$ be the set of vertices of $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \dots \dots \dots, B_n\}$ of G. Let $D' = \{b_1, b_2, b_3, \dots \dots \dots, b_m\}$ where $m < n$ is a minimal dominating set of $B(G)$ such that $V[B(G)] - D' = N, \forall v_i \epsilon N$ is a strongly dominated by at least one vertex of D'. Suppose there exists at least one vertex v of G with maximum degree. Then $\Delta(G) = \deg(v)$, which gives $|D'| = \gamma_{SB}(G)$. Hence $\gamma_{Sb}(G) = |D'| \leq |H| = \gamma_R(G)$, clearly $\gamma_{SB}(G) \leq \gamma_R(G) + \Delta(G) - 3$.

In the following theorem we establish the relation between $\gamma_{SB}(G)$ and $\gamma_c(G)$.

Theorem 9: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq \gamma(G) + \gamma_c(G) - 1$.

Proof: Suppose G has at least one block which is not an edge. Then G has at least one block with maximum number of vertices. Hence one can easily verify the inequality. Now we consider every block of G is an edge. Let $D = \{v_1, v_2, ..., v_n\} \subseteq V(G)$ which are non end vertices. Suppose $D' \subseteq D$, where $\forall v_i \in D', N[v_i] = V[G]$. Then D' is a minimal dominating set of G. Further if $\langle D' \rangle$ is connected then D' is also a $\gamma_c - set$. Otherwise there exists $H = \{v_1, v_2, ..., v_k\}, H \subseteq D'$ which forms a dominating set which is minimal and $\langle \{D' \cup H\} \rangle$ is connected. Then $\{D' \cup H\}$ is $\gamma_c - set$ of G. Let $M = \{e_1, e_2, ..., e_n\} \subseteq E(G)$ be the set of all non end edges of G. Since $M \subseteq V[B(G)]$ and each block of $B(G)$ is complete, $\forall v_i \in M$ is a cutvertex of $B(G)$. Now we consider $M' \subseteq M$. Suppose $\forall v_k \in M'$ has $\deg(v_k) \geq \deg(v_n)$, where $v_n \in V[B(G)] - M'$. Then M' is a $\gamma_{SB} - set$ of G. Thus $|M'| \leq |D'| + |D'| - 1$ or $|M'| \leq |D'| + |\{D' \cup H\}| - 1$ which gives $\gamma_{SB}(G) \leq \gamma(G) + \gamma_c(G) - 1$.

Theorem 10: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq 2C + \Delta(G) - 3$.
Where C is the number of cutvertices of G.

Proof: Let $C = \{v_1, v_2, ..., v_n\}$ where $n < p$ be the number of cutvertices of G. Then there exists a vertex $v \in V$ such that $\deg(v) = \Delta(G)$. Assume $\{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the number of vertices of $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ of G. Then we prove the result by induction on the number of blocks of G.

Assume that the result is true for $n = 2$. Then $\gamma_{SB}(G) = 1, v \geq 1$ and $C \neq \phi$. If $v \neq 1$, then $\gamma_{SB}(G) = 2C + \Delta(G)$. If $v > 1$, then $\gamma_{SB}(G) \leq 2C + \Delta(G)$. Assume the result for $n = k$. Then $\gamma_{SB}(G) \leq 2C + \Delta(G)$. Let $D_{SB} = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}$ with $j \leq n$ be the minimal strong dominating set of $B(G)$ such that $|D_{SB}| = \gamma_{SB}(G)$. Suppose $G^{'}$ has $(k+1)$ blocks. Then $v^{'} \geq v$ and $C^{'} \geq C$. If $v^{'} \geq v$, then $C^{'} = C$, if follows that $\gamma_{SB}(G) = |D_{SB}| \leq 2|C^{'}| + |v| < 2|C^{'}| + |v^{'}| = 2C + \Delta(G) - 3$, clearly $\gamma_{SB}(G) \leq 2C + \Delta(G) - 3$.

Theorem 11: For any non-trivial tree T with $n \geq 3$ blocks, $\gamma_{SB}(G) \leq \gamma_{\cot}(G) - 1$.

Proof: We consider only those graphs which are not $n = 1$. Let $H = \{v_1, v_2, ..., v_p\}$, $H_1 = \{v_1, v_2, ..., v_i\}$, $1 \leq i \leq p$ be a subset of $V(G) = H$ which are end vertices in G. Let $J = \{v_1, v_2, ..., v_j\} \subseteq V(G)$ with $1 \leq j \leq p$ such that $\forall v_j \in J$, $N(v_i) \cap N(v_j) = \phi$ and $\langle V(G) - (H_1 \cup J) \rangle$ has no isolates, then $|H_1 \cup J| = \gamma_{\cot}(G)$.

Let $V = \{v_1, v_2, ..., v_n\}$ be the vertices in $B(G)$. Consider $D = \{v_1, v_2, ..., v_t\} = V_1 \cup V_2 \cup V_3$ be the set of all vertices of $B(G)$. Where $\forall v_s \in V_1$ and $\forall v_t \in V_2$ with the property $(v_s) \cap N(v_t) = \phi$, $\forall v_l \in V_3$ is a set of all end vertices in $B(G)$. The $\langle D \rangle$ is strongly dominated by at least one vertex in D such that $|D| = \gamma_{SB}(G)$. Clearly $|H_1 \cup J| - 1 \leq |D|$ which gives $\gamma_{SB}(G) \leq \gamma_{\cot}(G) - 1$.

Theorem 12: For any graph G with $n - blocks$, then $\gamma_{SB}(G) \leq n + \gamma_s(G) - 3$.

Proof: Suppose $S = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ be the blocks of G. Then $M = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the corresponding blocks vertices in $B(G)$ with respect to the set S. Let $H = \{v_1, v_2, ..., v_n\}$ be the set of vertices in G, $V(G) = H$. If $J = \{v_1, v_2, ..., v_m\}$ where $1 \leq m \leq n$ such that $J \subset H$ and suppose there exists at set $J_1 \subset J$ then $\{v_i\} \in J_1$ which gives $H - J_1$ is a disconnected graph. Suppose $J_1 \cup J$ has the minimum number of vertices, such that $N(J_1 \cup J) = V(G) - (J_1 \cup J)$ gives a minimal split domination set in G. Hence $|J_1 \cup J| = \gamma_s(G)$.

Suppose $D = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}$ where $1 \leq j \leq n$ such that $D \subset m$ then $\forall b_i \in M$ are cutvertices in $B(G)$, since they are non end blocks in $B(G)$ is strongly dominated by at least one vertex in D. Hence D is a $\gamma_{SB} - set$ of $B(G)$. Clearly $|D| = \gamma_{SB}(G)$. Now $|D| \leq n + |J_1 \cup J| - 3$, gives the required result.

Next, the following theorem establish the upper bound for $\gamma_{snsb}(G)$ and $\gamma_{SB}(G)$.

Theorem 13: For any connected (p, q) graph G, $\gamma_{SB}(G) \leq \gamma_{snsb}(G)$.

Proof: Assume every block of $G$is an edge, let $A^{'} = \{B_1, B_2, B_3, \dots\dots\dots, B_m\}$ be the blocks of G and $M_1 = \{b_1, b_2, b_3, \dots\dots\dots, b_m\}$ be the block vertices in $B(G)$. Again we consider a subset $\{b_i^{'}\}$ such that $\{b_i^{'}\} \subset V[B(G)] - \{b_i\}$. Then $V[B(G)] - \{b_i^{'}\} = \{b_i\}$. If $i = 1$, then $\{b_i\}$ is a $\gamma_{snsb} - set$ of G. Otherwise if there exists $i > 1$ for $\{b_i\}$, we choose $\forall v_i \epsilon N[b_i]$ such that $V[B(G)] - \{b_i^{'}\} \cup \{v_i\} = b_i$ gives for $i > 1$. Hence $< b_i >$ is complete. Thus $|V[B(G)] - \{b_i^{'}\} \cup \{v_i\}| = \gamma_{snsb}(G)$.

Let $H = \{u_1, u_2, u_3, \dots\dots\dots, u_n\} \subseteq V[B(G)]$ be the set of vertices such that $\{u_i\} = \{e_i\} \in E(G)$, $1 \le i \le n$,where $\{e_i\}$ are incident with the vertices of A'. Further let $D' \subseteq H$ be the set of vertices with $\deg(w) \ge 3$ for every $w \in D'$ and $N[D'] = V[B(G)]$ and if $\forall v_i \in V[B(G)]$ has degree at most 2 and $v_i \in V[B(G)] - D'$. Then $\{D'\} \cup \{v_i\}$ forms a strong block dominating set. Clearly it follows that $\left|D' \cup \{v_i\}\right| \le \left|V[B(G)] - \{b_i'\} \cup \{v_i\}\right|$ which gives $\gamma_{SB}(G) \le \gamma_{snsb}(G)$.

In the following theorem we establish the relation between with strong split block domination of G and strong split block domination $\gamma_{ssb}(G)$ of G.

If G is a block then $\gamma_{ssb}(G)$ does not exist. Hence we consider, G must have at least two blocks.

Theorem 14: For any connected (p,q) graph G with $p \ge 4$, then $\gamma_{SB}(G) \le \gamma_{ssb}(G)$.

Proof: Suppose G has $p \le 3$. Then γ_{ssb} does not exists. Hence we consider $p \ge 4$, for $p = 4$ and each block is an edge then equality holds. Let $B = \{B_1, B_2, B_3, \dots\dots\dots, B_n\}$ be the set of blocks of G, and $H = \{b_1, b_2, b_3, \dots\dots\dots, b_n\}$ be the vertices of $B(G)$ corresponding to the blocks of B.

Suppose $S_1 = \{b_1, b_2, b_3, \dots\dots\dots, b_i\}$ and $S_2 = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}$ in which $\forall b \in S_1, \deg(b) > 2$ and $\forall b \in S_2, \deg(b) \le 2$. Since each block in $B(G)$ is complete, then there exists $S_1' \subseteq S_1$ such that $\deg(b_k) \ge \deg(b_m)$ $\forall b_k \in S_1'$ and $\forall b_m \in V[B(G)] - S_1'$. Thus S_1' is a $\gamma_{SB} - set$ of G. Further in case of $\gamma_{SSb} - set$, we have $S_2' \subseteq S_2$ and $J = V[B(G)] - \{S_1' \cup S_2'\}$ in which $\forall b \in J$ is an isolate and $|J| \ge 2$ which is a $\gamma_{SSb} - set$. Hence $|S_1'| \le |S_1' \cup S_2'|$ and gives $\gamma_{SB}(G) \le \gamma_{ssb}(G)$.

Theorem 15: For any connected (p,q) graph G, $\gamma_{SB}(G) \le \gamma_{SL}(G)$.

Proof: Suppose $A = \{b_1, b_2, b_3, \dots\dots\dots, b_j\}$ where $1 \le j \le n$ such that $A \subset A^1$ then $\forall b_i \in A$ are cutvertices in $B(G)$. Further $A_1 \subset A^1$ be a set of vertices in $B(G)$ such that $V[B(G)] - \{A \cup A_1\} = N^1$ where $\forall v_i \in N'$ is a strongly dominated by at least one vertex in N. Hence $|N| = \gamma_{SB}(G)$. Let $D = \{v_1, v_2, v_3, \dots\dots\dots, v_n\} \subseteq V(L(G))$ be the minimal dominating set of $L(G)$ and $\deg(v_i) \ge 2$ $\forall v_i \in D$ with $\deg(v_k) \le 2$ $\forall v_k \in V[L(G)] - D$. Then D is a Strong dominating set of $L(G)$. It follows that $|D| \ge |N|$ which gives $\gamma_{SB}(G) \le \gamma_{SL}(G)$.

Theorem 16: For any connected (p,q) graph G with $p \ge 4$, then $\gamma_{SB}(G) \le 3q - 2p$.

Proof: suppose G has a block say B with maximum number of vertices and edges. Then $3q - 2p > |\gamma_{SB}(G)|$. Hence we require to get the sharp bound. For this we consider the graph G which is a non-trivial tree with at least 3-blocks.We consider the following cases.

Case-1: Suppose G is a path $P_n, n \ge 4$ vertices. Then $B(G) = P_{n-1}$. Since the path P_n has $p -$ vertices and $q -$ edges, then $3q - 2p = 3(p-1) - 2p = p - 3$ for $P \ge 4$. One can easily verify that $\gamma_{SB}(G) \le p - 3 = 3q - 2p$.

Case-2: Suppose G is not a path. Then there exists at least one vertices v, $degv \ge 3$. Let $C = \{v_1, v_2, v_3, \dots\dots\dots, v_i\}$ be the number cutvertices and D be a dominating set of $B(G)$. Suppose each block of $B(G)$ is complete with P –vertices. Then $D = \{v_1, v_2, v_3, \dots\dots\dots, v_{p-1}\}$ where D consists of $P - 1$ vertices from each block $B(G)$ such that $C \subseteq D$ and $V[B(G)] - D = H$,where $v_i \in H$ is strongly dominating by at least one vertex in D. Clearly $|D| = \gamma_{SB}(G) \le p - 3 = 3q - 2p$.

Theorem 17: For any connected (p,q) graph G with $p \ge 3$, then $\gamma_{SB}(G) \le \gamma_t(G) + 2\gamma(G) - 2$.

Proof: Let $A' = \{v_1, v_2, \dots, v_n\} \subseteq V(G)$ be the set of all non end vertices in G. Suppose $A'' \subseteq A'$ and $\forall v_i \in V(G) - A''$ are adjacent to at least one vertex of A''. Then A'' forms a γ - set of G. Let $S \subseteq A''$ be the γ_t - set of G.

By the minimality for every vertex $v \in S$, the induced subgraph $\langle S - v \rangle$ contains an isolated vertex. Let $S_1 = \{v : v \in S\}$ and A_1 be the set of isolated vertices in $\langle S_1 \rangle$, $B = S_1 - A_1$, further let A be the minimum set of vertices of $S - S_1$ and each vertex of A_1 is adjacent to some vertex of A. Clearly $|A| \leq |A_1|$. Suppose $S^{'} = S - \{S_1 \cup A\}$ and every $u_i v_i \in \langle S^{'} \rangle$, $1 \leq i \leq k$, clearly $|S^{'}| = \gamma_t(\langle S^{'} \rangle)$. Then $S^{'}$ forms a minimal total dominating set of G. Let $H = \{u_1, u_2, u_3, \dots \dots \dots, u_n\} \subseteq V[B(G)]$, suppose $D^{'} \subseteq H$ be the set of vertices with $\deg(w) \geq 3$ for every $w \in D^{'}$ such that $N[D^{'}] = V[B(G)]$ and if $\forall v_i \in V[B(G)]$ has degree at most 2 and $v_i \in V[B(G)] - D^{'}$. Then $D^{'}$ forms a strong block dominating set. Clearly it follows that $|D^{'}| \leq |A^{''}| \cup |S^{'}| - 2$ and hence $\gamma_{SB}(G) \leq \gamma_t(G) + 2\gamma(G) - 2$.

Theorem 18: For any connected (p,q) graph G, $\gamma[B(G)] \leq \gamma_{SB}(G)$.

Proof: Suppose G is path with $P \geq 3$ vertices. Then $B(G)$ is also a path with $p-1$ vertices. Since this path has exactly two vertices of degree and remaining $p-2$ vertices are of degree two. Then every minimal dominating set of $B(G)$ is also a strong dominating set of $B(G)$. Thus $\gamma[B(G)] = \gamma_{SB}(G)$. Suppose G is not a path. Then in $B(G)$ every block is complete and there exists at least one block with at least three vertices. Now assume let $B(G)$ has two vertices v_1 and v_2 with maximum degree. Let D be strong dominating set, then $\{v_1, v_2\} \subseteq D$, if $N(v_1) = v_2$ or $N(v_2) = v_1$. Where as in case of $\gamma[B(G)]$, either v_1 or v_2 belongs to $\gamma[B(G)]$. Hence $\gamma[B(G)] \leq \gamma_{SB}(G)$.

Theorem 19: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq \Delta(G) + \alpha_0(G) - 2$.

Proof: Let A be the vertex cover of G with $|A| = \alpha_0(G)$. Suppose $V = \{v_1, v_2, \dots, v_p\}$ be the set of vertices in G then there exists at least one vertex $v \in V$ such that $\deg(v) = \Delta(G)$. Now without loss of generality in $B(G)$, suppose there is a set $D \subseteq V[B(G)]$, consists of at most $\Delta(G) + |A|$ elements. Hence $\gamma_{SB}(G) = |D| \leq \Delta(G) + |A| = \Delta(G) + \alpha_0(G) - 2$, clearly $\gamma_{SB}(G) \leq \Delta(G) + \alpha_0(G) - 2$.

Now obtain the following result on restrained domination number of G.

Theorem 20: For any connected (p,q) graph G, $\gamma_{SB}(G) \leq \gamma(G) + \gamma_{re}(G) - 1$.

Proof: Let D be any $\gamma - set$ of G with $\gamma(G) = |D|$. Suppose D_R be the restrained dominating set of G such that $\gamma_{re}(G) = |D_R|$. If $D^{'}$ be the strong block dominating set of $B(G)$, then $\gamma_{SB}(G) = |D^{'}|$. By the definition of domination number and restrained domination number of G one can easily verify that, $|D^{'}| \leq |D \cup D_R| - 1$. Hence $\gamma_{SB}(G) = |D^{'}| \leq |D \cup D_R| = \gamma(G) + \gamma_{re}(G) - 1$ gives $\gamma_{SB}(G) \leq \gamma(G) + \gamma_{re}(G) - 1$.

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Source of support: Nil, Conflict of interest: None Declared.