STRONG BLOCK DOMINATION IN GRAPHS

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(Received On: 13-10-17; Revised & Accepted On: 09-11-17)

ABSTRACT

For any graph \( G = (V, E) \), the block graph \( B(G) \) is a graph whose set of vertices is the union of the set of blocks of \( G \) in which two vertices are adjacent if and only if the corresponding blocks of \( G \) are adjacent. For any two adjacent vertices \( u \) and \( v \) we say that \( u \) strongly dominates \( v \) if \( \text{deg}(u) \geq \text{deg}(v) \). A dominating set \( D \) of a graph \( B(G) \) is a strong block dominating set of \( G \) if every vertex in \( V(B(G)) - D \) is strongly dominated by at least one vertex in \( D \). Strong block domination number \( \gamma_{SB}(G) \) of \( G \) is the minimum cardinality of strong block dominating set of \( G \). In this paper, we study graph theoretic properties of \( \gamma_{SB}(G) \) and many bounds were obtained in terms of elements of \( G \) and its relationship with other domination parameters were found.

Keywords: Dominating set/ Independent domination/ Block graph/ Line graph/ Roman domination/ Strong split domination/ Strong block domination.

Subject Classification number: AMS - 05C69, 05C70.

1. INTRODUCTION

In this paper, all the graphs considered here are simple and finite. For any undefined terms or notation can be found in Harary [5]. In general, we use \( < X > \) to denote the subgraph induced by the set of vertices \( X \) and \( N(v) \) denote open (closed) neighborhoods of a vertex \( v \). Let \( \text{deg}(v) \) be the degree of vertex \( v \) and \( \delta(G)(\Delta(G)) \) be the minimum (maximum) degree. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. The notation \( \alpha_s(G)(\alpha_t(G)) \) is the minimum number of vertices (edges) in vertex (edge) cover of \( G \). The minimum distance between any two farthest vertices of a connected \( G \) is called the diameter of \( G \) and is denoted by \( \text{diam}(G) \).

A block graph \( B(G) \) is the graph whose vertices corresponds to the blocks of \( G \) and two vertices in \( B(G) \) are adjacent if and only if the corresponding blocks in \( G \) are adjacent.

A set \( S \subseteq V(G) \) is said to be a dominating set of \( G \), if every vertex in \( V - S \) is adjacent to some vertex in \( S \). The minimum cardinality of vertices in such a set is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A set \( S \subseteq V(B(G)) \) is said to be a dominating set of \( B(G) \), if every vertex in \( V - S \) is adjacent to some vertex in \( S \). The minimum cardinality of vertices in such a set is called the domination number of \( B(G) \) and is denoted by \( \gamma(B(G)) \). A dominating set \( S \) is called the total dominating set, if for every vertex \( v \in V \), there exists a vertex \( u \in S \), \( u \neq v \) such that \( u \) is adjacent to \( v \). The total domination number of \( G \) is denoted by \( \gamma_t(G) \) is the minimum cardinality of total dominating set of \( G \). A dominating set \( S \subseteq V(G) \) is a connected dominating set, if the induced subgraph \( < S > \) has no isolated vertices. The connected domination number \( \gamma_c(G) \) of \( G \) is the minimum cardinality of a connected dominating set of \( G \). Also in terms of connected block domination \( \gamma_{cb}(G) \) which is discussed in [13]. Also characterized graphs achieving these bounds.

The concept of Roman domination function (RDF) was introduced by E.J. Cockayne, P.A.Dreyer, S.M.Hedetiniemi and S.T.Hedetiniemi in [1]. A Roman dominating function on a graph \( G = (V, E) \) is a function \( f: V \rightarrow \{0,1,2\} \) satisfying the condition that every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex of \( v \) for which \( f(v) = 2 \). The weight of a Roman dominating function is the value \( f(V) = \sum_{v \in V} f(v) \). The Roman domination number of a graph, denoted by \( \gamma_r(G) \), equals the minimum weight of a Roman dominating function on \( G \). A Roman dominating function \( f = \left(V_0, V_1, V_2\right) \) on a graph \( G \) is a connected Roman dominating function (CRDF) on \( G \) if \( \left(V_0 \cup V_2\right) \neq \emptyset \).

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or \( \{V_2\} \) is connected. The minimum weight of a CRDF is called a connected Roman domination number of \( G \) and is denoted by \( \gamma_{sc}(G) \). (see[12]). A dominating set \( S \subseteq V(G) \) is restrained dominating set of \( G \), if every vertex not in \( S \) is adjacent to a vertex in \( S \) and to a vertex in \( V(G) - S \). The restrained domination number of a graph \( G \) is denoted by \( \gamma_r(G) \) is the minimum cardinality of a restrained dominating set in \( G \). The concept of restrained domination in graphs was introduced by Domke et.al.[2].

The concept of strong split block domination in graphs was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [9]. A dominating set \( D \) of a graph \( G \) is a strong split dominating set if the induced subgraph \( (V[B(G)] - D) \) is totally disconnected with at least two vertices. The strong split block domination number \( \gamma_{sSB}(G) \) of \( G \) is the minimum cardinality of strong split block dominating set of \( G \). The concept of strong nonsplit block domination in graphs was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [10]. A dominating set \( D \) of a graph \( B(G) \) is a strong nonsplit block dominating set if the induced subgraph \( (V[B(G)] - D) \) is complete. The strong nonsplit block domination number \( \gamma_{ssb}(G) \) of \( G \) is the minimum cardinality of strong nonsplit block dominating set of \( G \). Recently we study a variation on the domination which is called strong line domination in graphs, was introduced by M.H.Muddebihal and Nawazoddin U.Patel in [11]. A dominating set \( D \) of a graph \( L(G) \) is a strong line dominating set if every vertex in \( (V[L(G)] - D) \) is strongly dominated by at least one vertex in \( D \). Strong line domination number \( \gamma_{sSL}(G) \) of \( G \) is the minimum cardinality of strong line dominating set of \( G \). The concept of restrained domination in graphs was introduced by Sambanthakumar and Pushpa Latha in [14] and well studied in [6, 7 and 8]. Given two adjacent vertices \( u \) and \( v \) we say that \( u \) strongly dominates \( v \) if \( \deg(u) \geq \deg(v) \). A set \( D \subseteq V(G) \) is a strong dominating set of \( G \) if every vertex in \( V(G) - D \) is strongly dominated by at least one vertex in \( D \). The strong domination number \( \gamma_s(G) \) is the minimum cardinality of a strong dominating set of \( G \). A dominating set \( D \) of a graph \( B(G) \) is a strong block dominating set of \( G \) if every vertex in \( V[B(G)] - D \) is strongly dominated by at least one vertex in \( D \). Strong block domination number \( \gamma_{SB}(G) \) of \( G \) is the minimum cardinality of strong block dominating set of \( G \) in this paper, many bounds on \( \gamma_{SB}(G) \) were obtained in terms of elements of \( G^* \) but not the elements of \( B(G) \). Also its relation with other domination parameters were established.

2. MAIN RESULTS

First we obtained necessary and sufficient condition on \( G \) for which \( \gamma_{SB}(G) \) is connected.

**Theorem 1:** For any graph \( G \) with at least two block, then \( \gamma_{SB}(G) \leq q - 1 \).

**Proof:** Suppose block graph \( B(G) \) has at least two vertices. Then \( G \) has at least two blocks. If two blocks of \( G \) are edges, then \( \gamma_{SB}(G) = q - 1 \). Otherwise the inequality holds. Thus \( \gamma_{SB}(G) \leq q - 1 \).

**Theorem 2:** For any connected \((p,q)\) graph \(G\), \( \gamma_{SB}(G) \leq \gamma_{bc}(G) \).

**Proof:** Let \( H = \{B_1,B_2,\ldots,B_n\} \) be the set of blocks of \( G \) and \( B = \{B_1,B_2,\ldots,B_i\} \) be the set of all non-end blocks of \( G \). Let \( B_i = \{b_1,b_2,\ldots,b_i\} \) be the vertices of block graph \( B(G) \) corresponding to the elements of \( B \). Since \( \forall b_j \in B_i, 1 \leq j \leq i \) is a cutvertex in \( B(G) \), then there exists a set \( B_1 \subseteq B_i \) such that \( \forall b_k \in B_1 \) is adjacent to at least one vertex of \( V[B(G)] - B_1 \) and \( \{B_1\} \) is connected clearly \( B_1 \) is a \( \gamma_{bc}(G) - set \). Let \( B_1^* = \{b_1,b_2,\ldots,b_n\} \subseteq B_i \) and if \( \forall v \in B_1^* \), \( \deg(v) \geq \deg(u) \), \( \forall u \in V[B(G)] - B_1^* \). \( N[B_1^*] = V[B(G)] \). Then \( B_1^* \) is a \( \gamma_{SB} - set \) . Hence \( |B_1^*| \geq |B_1| \) gives \( \gamma_{SB}(G) \leq \gamma_{bc}(G) \).
Corollary: For any block graph $G$ with $p \geq 2$ vertices, $\gamma_{SB}(G) \geq \left\lceil \frac{p}{3} \right\rceil$.

Theorem 3: For any non-trivial connected tree $T$, $\gamma_{SB}(T) \leq \gamma_{Rs}(T)$.

Proof: Let $G$ be any connected graph with a CRDF $f = (V'_0, V'_1, V'_2)$. Suppose $G$ is a non-trivial tree $T$. Let $V_{en} = \{v_1, v_2, ..., v_m\}$ be the set of all end vertices, $V_c = \{v_1, v_2, ..., v_c\}$ be the set of all cutvertices in $T$ such that $V(T) = V_c \cup V_{en}$ and $V_c \subseteq V_c$ be the set of all cutvertices adjacent to end vertices in $T$. Then $\forall v_i \in V_c$, $w(v_i) = 2$ and $\forall v_i \in V_c \subseteq V_c / V_c$, $w(v_i) = 1$ such that $w(N(v_i) \cap N(v_j)) = 1$ or 2. Then $\{v_i,v_j\}$ is connected. Hence $V_c$ forms $\gamma_{Rs} - set$ in $T$ and $|V_c| = |V_c'| + |V_c''| = \gamma_{Rs}(T)$. Next we consider $\{b_1,b_2, ..., b_n\}$ be the set of vertices of $B(T)$ corresponding to the blocks $\{B_1,B_2, ..., B_n\}$ of $T$. Let $D = \{b_1,b_2, ..., b_n\}$ where $m < n$ is a minimal dominating set of $B(T)$ such that $\forall v_i \in N \cup N_j \cup N_k$, $\deg(v_i) \leq \deg(v_j), \forall v_j \in D$. Then $|D| = \gamma_{SB}(T)$. Hence $\gamma_{SB}(T) = |D| \geq |V_c'| = \gamma_{Rs}(T)$ which gives $\gamma_{SB}(T) \leq \gamma_{Rs}(T)$.

Theorem 4: For any connected tree $T$ with $p \geq 4$, then $\gamma_{SB}(T) \geq \gamma(T) - 1$.

Proof: Let $V = \{v_1, v_2, ..., v_p\}$ be the set of all vertices of $T$ and suppose $D = \{v_1, v_2, ..., v_l\}$, $l < p$ be the minimal dominating set of $T$ such that $|D| = \gamma(T)$. Let $A = \{B_1, B_2, ..., B_p\}$ be the set of all blocks of $T$ and $H = \{b_1, b_2, ..., b_{p-1}\}$ be the corresponding block vertices in $B(T)$. $\forall B_j$ adjacent to end blocks containing $v_i \in D$ in $T$, there exists a corresponding blocks vertex set $\{b_i\}$ in $B(T)$ such that $\{b_i\} \in V_c \cup V_i$ and $B_j$ not adjacent to end blocks in $T$ there exist a corresponding block vertex set $\{b_j\}$ in $B(T)$ such that $\{b_j\} \in V_i$. Hence $\{b_i \cup b_j\}$ is strongly dominated by at least one vertex in $D'$ and it forms $\gamma_{SB} - set$ such that $|V'_1| + |V'_2| = |D'| = \gamma_{SB}(T)$. Clearly $|D| \leq |D'| - 1$ gives $\gamma_{SB}(T) \geq \gamma(T) - 1$.

In the following theorem we obtain the relation between for $\gamma_{SB}(G)$ in term of $\gamma_{as}(G)$.

Theorem 5: For any connected $(p,q)$ graph $G$, $\gamma_{SB}(G) \leq \gamma_{as}(G)$.

Proof: let $S'$ be a maximum independent set of vertices in $G$ and $S' \subseteq S'$ be the of all isolated vertices in $< S' >$. Then $(V - S') \cup S'$ is a strong split dominating set of $G$. Since for each vertex $v \in (V - S') \cup S'$ either $v$ is an isolated vertex in $(V - S') \cup S'$ or there exists a vertex $u \in S' - S'$ and $v$ is adjacent to $u$, $(V - S') \cup S'$ is minimal. Since $S'$ is maximum, $(V - S') \cup S'$ is minimum. Thus $|S'| = \gamma_{as}(G)$. Let $E = \{e_1, e_2, e_3, ..., e_n\}$ be set of edges in $G$ and $F \subseteq E(G)$, then in $B(G)$, $D' = \{v_1, v_2, v_3, ..., ..., v_n\}$ which corresponds to $\forall e_i \in F$. Let $\deg(e_i) \forall e_i \in F$ and $\deg(e_i) \forall e_i \in E(G) - F$ such that $\deg(e_i) \geq \deg(e_i)$. Suppose $D' = \{v_1, v_2, v_3, ..., ..., v_i\} \subseteq D'$ and $\forall v_k \in V(B(G)) \forall v_k \in D'$, $1 \leq k \leq i$. Then $D'$ forms a $\gamma_{SB} - set$. It follows that $|D'| \leq |(V - S') \cup S'|$. Hence $\gamma_{SB}(G) \leq \gamma_{as}(G)$.

Theorem 6: For any connected $(p,q)$ graph $G$, $\gamma_{SB}(G) \leq p - \Delta(G)$.

Proof: Suppose $G$ is a connected graph with $n$ blocks in which at least one block has maximum number of vertices with $\Delta(G) \geq 3$. Then in $B(G)$, $\gamma_{SB} - set$ is always less than $p - \Delta(G)$. Now we consider the graph $G$ such that each block of $G$ is an end. Let $B = \{B_1,B_2,B_3, ..., B_n\}$, be the set of blocks in $G$. Suppose $F = \{v_1, v_2, ..., v_i\} \subseteq V(B(G))$ be the set of vertices with $\deg(v_j) \geq 2$. Suppose there exists a vertex set...
\[D \subseteq F\] with \(N[D] = V(B(G))\) and if \(|\deg(x) - \deg(y)| \leq 1, \forall x \in D, y \in V(B(G)) - D\). Then \(D\) forms a strong block dominating set in \(B(G)\). Otherwise there exists at least one vertex \(\{w\} \subseteq F\) where \(\{w\} \notin D\) such that \(D \cup \{w\}\) forms a minimal \(\gamma_{SB} - \text{set}\) in \(B(G)\). Since for any graph \(G\), there exists at least one vertex \(v \in V(G)\) of maximum degree \(\Delta(G)\), it follows that \(|D \cup \{w\}| \leq p \cup |\deg(v)|\). Clearly, \(\gamma_{SB}(G) \leq p - \Delta(G)\).

**Theorem 7:** For any connected \((p,q)\) graph \(G\), \(\gamma_{SB}(G) \leq \text{diam}(G) - 1\).

**Proof:** Suppose \(A = \{e_1, e_2, \ldots, e_j\} \subseteq E(G)\) be the minimal set of edges which constitute the longest path between any two distinct vertices \(u, v \in V(G)\) with \(\text{dist}(u, v) = \text{diam}(G)\). Let \(\{u_1, u_2, u_3, \ldots, u_n\} \subseteq V[B(G)]\) be the set of vertices. Suppose \(D \subseteq k\) be the set of vertices with \(\deg(w) \geq 3\) for every \(w \in D\). Assume there exists \(D' \subseteq D\) such that \(\forall u_j \in D', \deg(u_j) \geq \deg(u_k), \forall u_k \in V[B(G)] - D'\). Clearly \(D'\) forms a strong block dominating set. Since each block in \(G\) is either an edge of at least one block contain more than two edges, then \(|D'| \leq \text{diam}(G) - 1\) which gives \(\gamma_{SB}(G) \leq \text{diam}(G) - 1\).

The following theorem we obtain upper bound for \(\gamma_{SB}(G)\) in terms of Roman domination number \(\gamma_{R}(G)\).

**Theorem 8:** For any connected \((p,q)\) graph \(G\), \(\gamma_{SB}(G) \leq \gamma_{R}(G) + \Delta(G) - 3\).

**Proof:** Let \(f = (V_0, V_1, V_2)\) be any \(\gamma_{R}\) - function of \(G\). Suppose \(V_1 \cup V_2\) or \(V_2\) form a \(\gamma_{R}\) - set of \(G\) such that \(|H| = \gamma_{R}(G)\). Next we consider \(\{b_1,b_2,b_3,\ldots,b_m\}\) be the set of vertices of \(B(G)\) corresponding to the blocks \(\{b_1,b_2,b_3,\ldots,b_m\}\) of \(G\). Let \(D' = \{b_1,b_2,b_3,\ldots,b_m\}\) where \(m < n\) is a minimal dominating set of \(B(G)\) such that \(V[B(G)] - D' = N, \forall v \in N\) is a strongly dominated by at least one vertex of \(D'\). Suppose there exists at least one vertex \(v\) of \(G\) with maximum degree. Then \(\Delta(G) = \deg(v)\), which gives \(|D'| = \gamma_{SB}(G)\). Hence \(\gamma_{SB}(G) = |D'| \leq |H| = \gamma_{R}(G)\), clearly \(\gamma_{SB}(G) \leq \gamma_{R}(G) + \Delta(G) - 3\).

In the following theorem we establish the relation between \(\gamma_{SB}(G)\) and \(\gamma_{c}(G)\).

**Theorem 9:** For any connected \((p,q)\) graph \(G\), \(\gamma_{SB}(G) \leq \gamma(G) + \gamma_{c}(G) - 1\).

**Proof:** Suppose \(G\) has at least one block which is not an edge. Then \(G\) has at least one block with maximum number of vertices. Hence one can easily verify the inequality. Now we consider every block of \(G\) is an edge. Let \(D = \{v_1,v_2,\ldots,v_k\} \subseteq V(G)\) which are non end vertices. Suppose \(D' \subseteq D\), where \(\forall v_i \in D', N[v_i] = V[G]\). Then \(D'\) is a minimal dominating set of \(G\). Further if \(\{D'\}\) is connected then \(D'\) is also a \(\gamma_{c}\) - set \(\). Otherwise there exists \(H = \{v_1,v_2,\ldots,v_k\}, H \subseteq D\) which forms a dominating set which is minimal and \(\{D' \cup H\}\) is connected. Then \(\{D' \cup H\}\) is a \(\gamma_{c}\) - set of \(G\). Let \(M = \{e_1,e_2,\ldots,e_n\} \subseteq E(G)\) be the set of all non end edges of \(G\). Since \(M \subseteq V[B(G)]\) and each block of \(B(G)\) is complete, \(\forall v_j \in M\) is a cutvertex of \(B(G)\). Now we consider \(M' \subseteq M\). Suppose \(\forall v \in M'\) has \(\deg(v) \geq \deg(v_i)\), where \(v_i \in V[B(G)] - M'\). Then \(M'\) is a \(\gamma_{SB}\) - set of \(G\). Thus \(|M'| \leq |D'| + |D| - 1\) or \(|M'| \leq |D'| + |D' \cup H| - 1\) which gives \(\gamma_{SB}(G) \leq \gamma(G) + \gamma_{c}(G) - 1\).

**Theorem 10:** For any connected \((p,q)\) graph \(G\), \(\gamma_{SB}(G) \leq 2C + \Delta(G) - 3\).

Where \(C\) is the number of cutvertices of \(G\).
Proof: Let $C = \{v_1, v_2, \ldots, v_n\}$ where $n < p$ be the number of cutvertices of $G$. Then there exists a vertex $v \in V$ such that $\deg(v) = \Delta(G)$. Assume $\{b_1, b_2, b_3, \ldots, b_n\}$ be the number of vertices of $B(G)$ corresponding to the blocks $\{B_1, B_2, B_3, \ldots, B_n\}$ of $G$. Then we prove the result by induction on the number of blocks of $G$.

Assume that the result is true for $n = 2$. Then $\gamma_{SB}(G) = 1, v \geq 1$ and $C \neq \emptyset$. If $v \neq 1$, then $\gamma_{SB}(G) = 2 + \Delta(G)$. If $v > 1$, then $\gamma_{SB}(G) \leq 2C + \Delta(G)$. Assume the result for $n = k$. Then $\gamma_{SB}(G) \leq 2C + \Delta(G)$. Let $D_{SB} = \{b_1, b_2, b_3, \ldots, b_j\}$ with $j \leq n$ be the minimal strong dominating set of $B(G)$ such that $|D_{SB}| = \gamma_{SB}(G)$. Suppose $G$ has $(k + 1)$ blocks. Then $v' \geq v$ and $C \geq C$. If $v' \geq v$, then $C = C'$, if follows that $\gamma_{SB}(G) = |D_{SB}| \leq 2|C| + |v| < 2|C| + \gamma = 2C + \Delta(G) - 3$, clearly $\gamma_{SB}(G) \leq 2C + \Delta(G) - 3$.

Theorem 11: For any non-trivial tree $T$ with $n \geq 3$ blocks, $\gamma_{SB}(G) \leq \gamma_{sbd}(G) - 1$.

Proof: We consider only those graphs which are not $n = 1$. Let $H = \{v_1, v_2, \ldots, v_p\}, H_1 = \{v_1, v_2, \ldots, v_i\}$. Then $H
1 \leq i \leq p$ be a subset of $V(G) = H$ which are end vertices in $G$. Let $J = \{v_1, v_2, \ldots, v_j\} \subseteq V(G)$ with $1 \leq j \leq p$ such that $\forall v_j \in J, N(v_j) \cap N(v_j) = \emptyset$ and $\langle V(G) - (H_1 \cup J) \rangle$ has no isolates, then $|H_1 \cup J| = \gamma_{sbd}(G)$.

Proof: For any connected $G$ with $n - \text{blocks}$, then $\gamma_{SB}(G) \leq n + \gamma(G) - 3$.

Proof: Suppose $S = \{B_1, B_2, B_3, \ldots, B_n\}$ be the blocks of $G$. Then $M = \{b_1, b_2, b_3, \ldots, b_n\}$ be the corresponding blocks vertices in $B(G)$ with respect to the set $S$. Let $H = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices in $G$. $V(G) = H$. If $J = \{v_1, v_2, \ldots, v_m\}$ where $1 \leq m \leq n$ such that $J \subset H$ and suppose there exists at set $J_1 \subset J$ then $\{v_j\} \in J_1$ which gives $H - J_1$ is a disconnected graph. Suppose $J_1 \cup J$ has the minimum number of vertices, such that $N(J_1 \cup J) = V(G) - (J_1 \cup J)$ gives a minimal split domination set in $G$. Hence $|J_1 \cup J| = \gamma(G)$.

Suppose $D = \{b_1, b_2, b_3, \ldots, b_j\}$ where $1 \leq j \leq n$ such that $D \subset M$ then $\forall b_i \in M$ are cutvertices in $B(G)$, since they are non end blocks in $B(G)$ is strongly dominated by at least one vertex in $D$. Hence $D$ is a $\gamma_{SB} - \text{set}$ of $B(G)$. Clearly $|D| = \gamma_{SB}(G)$. Now $|D| \leq n + |J_1 \cup J| - 3$, gives the required result.

Next, the following theorem establishes the upper bound for $\gamma_{smp}(G)$ and $\gamma_{SB}(G)$.

Theorem 13: For any connected $(p, q)$ graph $G$, $\gamma_{SB}(G) \leq \gamma_{smp}(G)$.

Proof: Assume every block of $G$ is an edge, let $A' = \{B_1, B_2, B_3, \ldots, B_m\}$ be the blocks of $G$ and $M_i = \{b_1, b_2, b_3, \ldots, b_n\}$ be the block vertices in $B(G)$. Again we consider a subset $\{b_i\}$ such that $\{b_i\} \subseteq V(B(G))$. Then $V(B(G)) - \{b_i\} = \{b_i\}$. If $i = 1$, then $\{b_i\}$ is a $\gamma_{smp} - \text{set}$ of $G$. Otherwise if there exists $i > 1$ for $\{b_i\}$, we choose $\forall v_i \in N(b_i)$ such that $V(B(G)) - \{b_i\} \cup \{v_i\} = b_i$ for $i > 1$. Hence $\{b_i\}$ is complete. Thus $\gamma_{smp}(G) = \gamma_{smp}(G)$.
Let $H = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V[B(G)]$ be the set of vertices such that $\{u_i\} \subseteq E(G), 1 \leq i \leq n$, where $\{u_i\}$ are incident with the vertices of $A$. Further let $D \subseteq H$ be the set of vertices with deg $(w) \geq 3$ for every $w \in D$ and $N[D] = V[B(G)]$ and if $\forall v_i \in V[B(G)]$ has degree at most 2 and $v_i \in V[B(G)] - D$. Then $D \cup \{v_i\}$ forms a strong block dominating set. Clearly it follows that $|D \cup \{v_i\}| \leq V[B(G)] - \{v_i\}$ which gives $\gamma_{SB}(G) \leq \gamma_{sub}(G)$.

In the following theorem we establish the relation between with strong split block domination of $G$ and strong block domination $\gamma_{sb}(G)$ of $G$.

If $G$ is a block then $\gamma_{sb}(G)$ does not exist. Hence we consider, $G$ must have at least two blocks.

**Theorem 14:** For any connected $(p,q)$ graph $G$ with $p \geq 4$, then $\gamma_{SB}(G) \leq \gamma_{sb}(G)$.

**Proof:** Suppose $G$ has $p \leq 3$. Then $\gamma_{sb}$ does not exists. Hence we consider $p \geq 4$, for $p = 4$ and each block is an edge then equality holds. Let $B = \{b_1, b_2, b_3, \ldots, b_n\}$ be the set of blocks of $G$, and $H = \{b_1, b_2, b_3, \ldots, b_n\}$ be the vertices of $B(G)$ corresponding to the blocks of $B$.

Suppose $S_1 = \{b_1, b_2, b_3, \ldots, b_1\}$ and $S_2 = \{b_1, b_2, b_3, \ldots, b_1\}$ in which $\forall b \in S_1, \text{deg}(b) > 2$ and $\forall b \in S_2, \text{deg}(b) \leq 2$. Since each block in $B(G)$ is complete, then there exists $S_1' \subseteq S_1$ such that $\text{deg}(b_1) \geq \text{deg}(b_n) \forall b \in S_1'$ and $\forall b \in V[B(G)] - S_1'$. Thus $S_1'$ is a $\gamma_{SB}$-set of $G$. Further in case of $\gamma_{SB} - \text{set}$, we have $S_2' \subseteq S_2$ and $J = V[B(G)] - \{S_1' \cup S_2\}$ in which $\forall b \in J$ is an isolate and $|J| \geq 2$ which is a $\gamma_{SB} - \text{set}$. Hence $|S_1'| \leq |S_1' \cup S_2'|$ and gives $\gamma_{SB}(G) \leq \gamma_{sb}(G)$.

**Theorem 15:** For any connected $(p,q)$ graph $G$, $\gamma_{SB}(G) \leq \gamma_{SL}(G)$.

**Proof:** Suppose $A = \{b_1, b_2, b_3, \ldots, b_j\}$ where $1 \leq j \leq n$ such that $A \subseteq A'$ be a set of vertices in $B(G)$. Further $A \subseteq A'$ be a set of vertices in $B(G)$ such that $V[B(G)] - \{A \cup A\}' = N(A)'$ where $\forall v_i \in N(A)'$ is a strongly dominated by at least one vertex in $N$. Hence $|N(A)'| = \gamma_{SB}(G)$. Let $D = \{v_1, v_2, v_3, \ldots, v_k\} \subseteq V[L(G)]$ be the minimal dominating set of $L(G)$ and $\text{deg}(v_i) \geq 2 \forall v_i \in D$ with $\text{deg}(v_k) \leq 2 \forall v_k \in V[L(G)] - D$. Then $D$ is a Strong dominating set of $L(G)$. It follows that $|D| \geq |N(A)'|$ which gives $\gamma_{SB}(G) \leq \gamma_{SL}(G)$.

**Theorem 16:** For any connected $(p,q)$ graph $G$ with $p \geq 4$, then $\gamma_{SB}(G) \leq 3q - 2p$.

**Proof:** suppose $G$ has a block say $B$ with maximum number of vertices and edges. Then $3q - 2p > |\gamma_{SB}(G)|$. Hence we require to get the sharp bound. For this we consider the graph $G$ which is a non-trivial tree with at least 3 blocks. We consider the following cases.

**Case-1:** Suppose $G$ is a path $P_n$, $n \geq 4$ vertices. Then $B(G) = P_{n-1}$. Since the path $P_n$ has $p$ vertices and $q$ edges, then $3q - 2p = 3(p - 1) - 2p = 3 - 3 = 3q - 2p$. One can easily verify that $\gamma_{SB}(G) \leq p - 3 = 3q - 2p$.

**Case-2:** Suppose $G$ is a path. Then there exists at least one vertices $v, \text{deg}(v) \geq 3$. Let $C = \{v_1, v_2, v_3, \ldots, v_n\}$ be the number cut vertices and $D$ be a dominating set of $B(G)$. Suppose each block of $B(G)$ is complete with $P - \text{vertices}$. Then $D = \{v_1, v_2, v_3, \ldots, v_{p-1}\}$ where $D$ consists of $P - 1$ vertices from each block $B(G)$ such that $C \subseteq D$ and $V[B(G)] - D = H$ where $v_i \in H$ is strongly dominated by at least one vertex in $D$. Clearly $|D| = \gamma_{SB}(G) \leq p - 3 = 3q - 2p$.

**Theorem 17:** For any connected $(p,q)$ graph $G$ with $p \geq 3$, then $\gamma_{SB}(G) \leq \gamma_i(G) + 2\gamma(G) - 2$.

**Proof:** Let $A' = \{v_1, v_2, \ldots, v_i\} \subseteq V(G)$ be the set of all non end vertices in $G$. Suppose $A' \subseteq A'$ and $v_i \in V(G) - A'$ are adjacent to at least one vertex of $A'$. Then $A'$ forms a $\gamma$-set of $G$. Let $S \subseteq A'$ be the $\gamma_i$-set of $G$. 
By the minimality for every vertex \( v \in S \), the induced subgraph \( S - v \) contains an isolated vertex. Let \( S_1 = \{ v : v \in S \} \) and \( A_i \) be the set of isolated vertices in \( S_1 \), \( B = S_1 - A_i \), further let \( A \) be the minimum set of vertices of \( S - S_1 \) and each vertex of \( A_i \) is adjacent to some vertex of \( A \). Clearly \( |A| \leq |A_i| \). Suppose \( S' \subseteq S \) \( \{ S_1 \cup A \} \) and every \( u_i v_j \in S' \), \( 1 \leq i \leq k \), clearly \( |S'| = \gamma_i \left( S' \right) \). Then \( S' \) forms a minimal total dominating set of \( G \). Let \( H = \{ u_1, u_2, u_3, \ldots, u_n \} \subseteq V[B(G)] \), suppose \( D \subseteq H \) be the set of vertices with \( \deg(w) \geq 3 \) for every \( w \in D' \) such that \( N[D] = V[B(G)] \) and if \( v_i \in V[B(G)] \) has degree at most 2 and \( v_i \in V[B(G)] - D' \). Then \( D' \) forms a strong block dominating set. Clearly it follows that \( |D| \leq |A| \cup |S'| - 2 \) and hence \( \gamma_{SB}(G) \leq \gamma_r(G) + 2\gamma(G) - 2 \).

**Theorem 18:** For any connected \((p,q)\) graph \( G \), \( \gamma(B(G)) \leq \gamma_{SB}(G) \).

**Proof:** Suppose \( G \) is path with \( P \geq 3 \) vertices. Then \( B(G) \) is also a path with \( p - 1 \) vertices. Since this path has exactly two vertices of degree and remaining \( p - 2 \) vertices are of degree two. Then every minimal dominating set of \( B(G) \) is also a strong dominating set of \( B(G) \). Thus \( \gamma(B(G)) = \gamma_{SB}(G) \). Suppose \( G \) is not a path. Then in \( B(G) \) every block is complete and there exists at least one block with at least three vertices. Now assume let \( B(G) \) has two vertices \( v_i \) and \( v_2 \) with maximum degree. Let \( D \) be strong dominating set, then \( \{ v_i, v_2 \} \subseteq D \), if \( N(v_i) = v_2 \) or \( N(v_2) = v_1 \). Where as in case of \( \gamma(B(G)) \), either \( v_i \) or \( v_2 \) belongs to \( \gamma(B(G)) \). Hence \( \gamma(B(G)) \leq \gamma_{SB}(G) \).

**Theorem 19:** For any connected \((p,q)\) graph \( G \), \( \gamma_{SB}(G) \leq \Delta(G) + \alpha_0(G) - 2 \).

**Proof:** Let \( A \) be the vertex cover of \( G \) with \( |A| = \alpha_0(G) \). Suppose \( V = \{ v_1, v_2, \ldots, v_p \} \) be the set of vertices in \( G \) then there exists at least one vertex \( v \in V \) such that \( \deg(v) = \Delta(G) \). Now without loss of generality in \( B(G) \), suppose there is a set \( D \subseteq V[B(G)] \), consists of at most \( \Delta(G) + |A| \) elements. Hence \( \gamma_{SB}(G) = |D| \leq \Delta(G) + |A| = \Delta(G) + \alpha_0(G) - 2 \). Clearly \( \gamma_{SB}(G) \leq \Delta(G) + \alpha_0(G) - 2 \).

Now obtain the following result on restrained domination number of \( G \).

**Theorem 20:** For any connected \((p,q)\) graph \( G \), \( \gamma_{SB}(G) \leq \gamma(G) + \gamma_{r_0}(G) - 1 \).

**Proof:** Let \( D \) be any \( \gamma - set \) of \( G \) with \( \gamma(G) = |D| \). Suppose \( D_8 \) be the restrained dominating set of \( G \) such that \( \gamma_{r_0}(G) = |D_8| \). If \( D' \) be the strong block dominating set of \( B(G) \), then \( \gamma_{SB}(G) = |D'| \). By the definition of domination number and restrained domination number of \( G \) one can easily verify that, \( |D| \leq |D \cup D_8| - 1 \). Hence \( \gamma_{SB}(G) = |D| \leq |D \cup D_8| = \gamma(G) + \gamma_{r_0}(G) - 1 \) gives \( \gamma_{SB}(G) \leq \gamma(G) + \gamma_{r_0}(G) - 1 \).

3. **REFERENCES**

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Source of support: Nil, Conflict of interest: None Declared.

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