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PRIMITIVE WEAKLY STANDARD RINGS

K. SUVARNA[1], M. MANJULA DEVI*[2]

[1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

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ABSTRACT

In this paper, we prove that all commutators and associators are in the center of a prime weakly standard ring. By using these we prove that a primitive weakly standard ring is either commutative or associative.

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1. INTRODUCTION

In [1] paul considered prime ring R satisfying (x, y, z) = (x, z, y) with nucleus N and center C. He proved that if R has commutators in the middle nucleus then either R is associative or N=C. San Soucie [2] proved that a prime ring is weakly standard if and only if it is either associative or commutative. In a weakly standard ring we have the identity (x, y, z) = - (z, y, x) and all commutators in the nucleus. Using these properties in this section we show that all commutators and associators are in the centre of a prime weakly standard ring. By using these we prove that a primitive weakly standard ring is either commutative or associative. At the end of this paper we give an example of a weakly standard ring which is not associative.

2. PRELIMINARIES

In this paper we denote R as a nonassociative weakly standard ring. A nonassociative ring R is a weakly standard ring if it satisfies the following identities

$$(x, y, x) = 0, \tag{1}$$

$$((w, x), y, z) = 0, \tag{2}$$

and

$$(w, (x, y), z) = 0, \tag{3}$$

for all w, x, y, z $\in$ R. Hence all commutators are in the nucleus N of R.

A linearization of flexible law (1) yields the identity (x, y, z) = - (z, y, x).

We know that the nucleus N of R is the set of all elements n in R such that
(n, R, R) = (R, n, R)= (R, R, n) = 0 and the center C of R is the set of all elements c in N such that (c, R) = 0. If we define S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y),

we have the following identities in any ring:

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z, \tag{4}$$

$$(xy, z) – x(y, z) – (x, z)y = (x, y, z) + (z, x, y) – (x, z, y), \tag{5}$$

$$(xy, z) + (yz, x) + (zx, y) = S(x, y, z), \tag{6}$$

and

$$((x, y), z) + ((y, z), x) + ((z, x), y) = S(x, y, z) – S(x, z, y) \tag{7}$$

Putting z=x in (5) gives (xy, x) – x(y, x) = (x, y, x).

$$\text{That is } (xy, x) + x(x, y) = 0. \tag{8}$$

With w=n, where n$\in$N in (4) we obtain (nx, y, z) = n(x, y, z).

Corresponding Author: M. Manjula Devi[2]*
[2]Department of Mathematics, Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

Combining this with (2) yields
$(nx, y, z) = n(x, y, z) = (xn, y, z)$. (9)

A ring R is prime if whenever A and B are ideals of R such that AB=0 then either A=0 or B=0. A ring R is primitive if R contains a regular maximal right ideal E which contains no two sided ideal of R other than the zero ideal. A ring R is n – divisible if nx=0 implies x=0 for all x in R and n a natural number.

3. MAIN RESULTS

Lemma 1: If R be an arbitrary Primitive ring, then R is a prime ring.

Proof: Suppose E is a maximal right ideal of R such that ex-x is in E for all x in R and for some e in R. Let I and J be two ideals of R such that IJ=0 and assume that $I\neq 0$. Then $I\not\subset E$ so R=E+I and $RJ = EJ\subset E$. Hence $ej\in E$ and thus $-j\in E$, $J\subset E$ and J=0.

Hence R is a prime ring.

Lemma 2: If R is a Prime weakly standard ring, then all commutators are in the center.

Proof: Forming associators of (8) and using (2)

We obtain $(x(y, x), r, s) = ((xy, x), r, s) = 0$.

That is $(x(y, x), r, s) = 0$.

This implies $o=(x(y, x), r, s) = (x(x, y), r, s) = ((x, y)x, r, s)$. By using this and (4) we get
$$((x, y)x, r, s) = (x, y) (x, r, s).$$

Therefore $(x, y) (x, r, s) = 0$.

Linearizing the above equation with $x=x+x^1$,

We obtain $(x, y) (x^1, r, s) + (x^1, y) (x, r, s) = 0$. (10)

If we substitute a commutator v for x^1, we see that $(x, y) (v, r, s) + (v, y) (x, r, s) = 0$.

That is $(v, y) (x, r, s) = 0$ using (2).

This can be restated as $((R, R), R) (R, R, R) = 0$. But now the ideal generated by the double commutator $((R, R), R)$ annihilates the associator ideal. Since R is prime and not associative, we conclude that $((R, R), R) = 0$. Hence all commutators are in the center.

Lemma 3: If R is a 2- divisible weakly standard ring, then $S(x, y, z) = 0$.

Proof: By taking y = x in (1), then $(x, x, x) = 0$.

If we linearize $(x, x, x) = 0$, we get
$$S(x, y, z) + S(x, z, y) = 0.$$

Using lemma (2) in (7) then we get $S(x, y, z) – S(x, z, y) = 0$.

By adding the above two equations, we obtain $2S(x, y, z) = 0$ and then $S(x, y, z) = 0$, since R is 2- divisible. This proves the lemma.

Lemma 4: If R is a 2- and 3- divisible weakly standard ring, then $((w, x, y), z) = 0$ and $(v(x, y, z), w) = 0$ for all $v, w, x, y, z\in R$.

Hence all associators are in the center of R.

Proof: From (1), $(x, z, y) = - (y, z, x)$. Substituting this in (5), we get
$$(xy, z) – x(y, z) – (x, z)y = (x, y, z) + (z, x, y) + (y, z, x),$$
$$(xy, z) – x(y, z) – (x, z)y = 0 \text{ using lemma (3).}$$
That is $(xy, z) = x(y, z) + (x, z)y$. (11)

Now we take $w, x, y, z \in R$, then

$((w, x, y), z) = ((wx \cdot y - w \cdot xy), z)$. Repeated use of the equation (10) we obtain

$((w, x, y), z) = wx(y, z) + (wx, z)y - w(xy, z) - (w, z)xy,$

$((w, x, y), z) = wx(y, z) + w(x, z) \cdot y + (w, z)\, x \cdot y - w(x(y, z) + (x, z)y) - (w, z)xy,$

$((w, x, y), z) = (w, x, (y, z)) + (w, (x, z), y) + ((w, z), x, y).$

From (1), (2) and (3) we get commutator is in the nucleus. Hence using this property we get

$$((w, x, y), z) = 0. \tag{12}$$

By taking $n=(v, x) \in R$ in $(nx, y, z) = n(x, y, z)$ we get

$(v, x)\,(x, y, z) = ((v, x)\, x, y, z)$, using (11) we get $(v, x)x = (vx, x)$.

Therefore $(v, x)\,(x, y, z) = ((vx, x), y, z)$.

Using this and (2) we get

$$(v, x)\,(x, y, z) = 0. \tag{13}$$

By linearization, this identity becomes

$$(v, w)\,(x, y, z) = - (v, x)\,(w, y, z). \tag{14}$$

By using flexibility (14), lemma (3), (13) and (1), we obtain

$$\begin{aligned}(v, w)\,(x, y, y) &= - (v, w)\,(y, y, x),\\ &= (v, y)\,(w, y, x),\\ &= (v, y)\,(-(y, x, w) - (x, w, y)),\\ &= -(v, y)\,(y, x, w) - (v, y)\,(x, w, y),\\ &= - (v, y)\,(y, x, w) + (v, y)\,(y, w, x),\\ &= 0.\end{aligned}$$

That is $(v, w)\,(x, y, y) = 0$. (15)

By linearization of this identity, we get $(v, w)\,((x, y, z) + (x, z, y)) = 0$.

$(v, w)\,((x, y, z) - (y, z, x)) = 0$ using (1).

That is $(v, w)\,(x, y, z) = (v, w)\,(y, z, x)$. (16)

From (15) and (1) we have $(v, w)\,(y, y, x) = 0$.

Again by linearization we get $(v, w)\,((y, z, x) + (z, y, x)) = 0$.

Then $(v, w)\,(y, z, x) = - (v, w)\,(z, y, x)$. Using this and (15) we get

$$(v, w)\,(y, z, x) = (v, w)\,(z, x, y). \tag{17}$$

By using lemma (3), (15), (16) and (17),

We obtain $(v, w)\,((x, y, z) + (y, z, x) + (z, x, y)) = 0$.

So $3(v, w)\,(x, y, z) = 0$.

Since R is 3- divisible, $(v, w)\,(x, y, z) = 0$. (18)

Now from (11), (12) and (18) we have

$(v(x, y, z), w) = v\,((x, y, z), w) + (v, w)\,(x, y, z) = 0.$

Therefore $(v(x, y, z), w) = 0$. (19)

If we substitute an associator u for x^1 in (10) there we get

$(x, y)\,(u, r, s) + (u, y)\,(x, r, s) = 0.$

Using (12) in the above equation we obtain $(x, y)\,(u, r, s) = 0$, as in the proof of lemma (2) $(x, y) \neq 0$, hence $(u,r,s) = 0$. Therefore an associator u is in the left nucleus of R. Using (1) and (3), u is in the nucleus of R. From (12) and (19) it follows that associators are in the center of R.

Lemma 5: If R is a weakly standard ring, then $S = \{s \in R/(s, R) = 0 = (sR, R)\}$ is an ideal of R.

Proof: From (12), we have $((w, x, y), z) = 0$.

If we put w=s in the above equation, then

$((s, x, y), z) = 0,$

$((sx \cdot y - s \cdot xy), z) = 0,$

$(sx \cdot y, z) - (s \cdot xy, z) = 0$. By the definition of S, we obtain

$(s.xy, z) = 0$. So $(sx.y, z) = 0$. Then $sx \in S$. So S is a right ideal of R.

Since $(s, R) = 0$, $(s, x) = 0$. That is $sx\text{-}xs = 0$.

Thus $sx = xs$. Then $xs \in S$. So S is a left ideal of R.

Hence S is an ideal of R.

Let A consists of all finite sums of elements of the form (x, y, z) or of the form w(x, y, z). This is an ideal in any arbitrary ring and is the smallest ideal modulo which the ring is associative. From (12), for any element a in A we have $(a, R) = 0$.

Let B consists of all finite sums of elements of the form (x, y) or of the form (x, y)z. In any arbitrary ring this set need not be an ideal. But by virtue of (2) and (11), it is an ideal. In addition it is also true that B is contained in the nucleus N. B is also the smallest ideal modulo which R is commutative.

From (18), for any element a in A and any element b in B we must have ab = 0.

Therefore AB=0. Suppose that x is an element of $A \cap B$. Then since AB=0, $x^2=0$ implies that x=0.

Theorem 1: A 2- and 3- divisible weakly standard ring R is isomorphic to a subdirect sum of an associative ring and a commutative ring.

Proof: Consider the natural homomorphism from R into $R/A \oplus R/B$. The Kernel of this homomorphism is $A \cap B = 0$.

Hence R is a subdirect sum of R/A and R/B. We know that R/A is associative and R/B is commutative.

This completes the proof of this theorem.

Theorem 2: A 2- and 3- divisible primitive weakly standard ring R is either commutative or associative.

Proof: If R is a primitive weakly standard ring, then it contains regular maximal right ideal E which contains no two-sided ideal of R other than zero ideal.

From lemma (1), we know that if R is a primitive ring, then R is a prime ring. From (18), the ideals A, B of R have the property AB=0. Since R is prime, then either A=0 or B=0. If A=0 then R is associative or if B=0 then R is commutative.

This completes the proof of the theorem.

Now we give an example of a weakly standard ring which is not associative.

Example: Consider the algebra with basis elements 1, a, b, c, d, e over a 6- divisible field, where 1 is the unit element, $e^2=1$, ea=b, ae=d, be=-b, de=-d, eb=b, ed=-b, $b^2=c$ and all other product of basis elements equal to zero. It is easily seen that this is a weakly standard ring. That is

(i) $(e, a, e) = ea.e – e.ae = be – ed = -b + b = 0.$

(ii) $(e, (a, e), e) = (e, ae, e) – (e, ea, e) = 0$

and (iii) $((a, e),e, e) = (ae, e, e) – (ea, e, e) = 0.$

But this ring is not associative, since $(e, a, b) = c$.

4. REFERENCES

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