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SEPARATION AXIOMS ON $\hat{\mu}\beta$ CLOSED SETS

J. SUBASHINI*
Department of Mathematics,
Rathnavel Subramaniam College of Arts & Science Sulur, Coimbatore - 641 402, India.

K. INDIRANI
Nirmala College for women, Red fields, Coimbatore – 641 018, India.

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ABSTRACT

The aim of this paper is to introduce the concept of $\hat{\mu}\beta$ closed set and their relations. And also we define some new types of separation axioms in topological spaces by using $\hat{\mu}\beta$ open sets. Also the concept of $\hat{\mu}\beta$ R_0 and $\hat{\mu}\beta$ R_1, $\hat{\mu}\beta$ T_i are introduced. Several properties of these spaces are investigated.

Keywords: *$\hat{\mu}$ closed set, $\hat{\mu}\beta$ closed set, β open, $\hat{\mu}\beta$ open set, $\hat{\mu}\beta$ R_0, $\hat{\mu}\beta$ R_1, $\hat{\mu}\beta$ T_i (i=0, 1, 2).*

1. INTRODUCTION

In 1970, Levine [9] introduced the concept of generalized closed set in topological spaces. In 2000, M.K.R.S Veerakumar [19] introduced several generalized closed sets namely g* closed set,*g closed set, α*g closed set, *gs closed set, $\hat{g}$closed set, μ closed set, μs closed set. S.Pious Missier and E.Sucila [16] introduced $\hat{\mu}$ closed set and their continuity. Andrijevic[1] introduced semi preopen set(β open set) in general topology. The aim of this paper is to introduce the some properties of $\hat{\mu}\beta$ closed and new types of separation axiom [5, 7, 8] via $\hat{\mu}\beta$ open sets, and investigate the relations among these concepts. Throughout this paper, (X,τ) and (Y,σ) (or simply X and Y) represents the non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, Cl(A) and Int(A) represents the closure of A and Interior of A respectively.

2. PRELIMINARIES

Definition 2.1: A subset A of X is called generalized closed (briefly g-closed) [9] set if $cl(A) \subseteq U$ whenever $A \subseteq U$ *and* U *is open*.

Definition 2.2: A subset A of X is called regular open (briefly r-open) [11] set if $A = int(cl(A))$ and regular closed (briefly r-closed) [4] set if $A = cl(int(A))$.

Definition 2.3: A subset A of X is called pre open set [13] if $A \subseteq int(cl(A))$ and pre-closed [6] set if $cl(int(A)) \subseteq A$

Definition 2.4: A subset A of X is called α *open* [14] if $A \subseteq int(cl(int(A)))$ *and* $\alpha - closed$ [10] if $cl(int(cl(A))) \subseteq A$.

Definition 2.5: A subset A of X is called θ closed [20] if A= cl_θ(A), where cl_θ(A)={x∈ $X{:}\, cl(U) \cap A \neq U \in \tau\}$

Definition 2.6: A subset A of X is called δclosed [20] if A = cl_δ(A), where cl_δ(A)=$x \in X{:}\, int(cl(U)) \cap A \neq U \in \tau\}$

Definition 2.7: A subset A of X is called Semi generalized closed (briefly sg closed) [2] if scl(A)⊆U whenever A⊆ U and U is semi open in X.

Definition 2.8: A subset A of X is called Generalized α closed (briefly gα closed) [4] if α-cl(A)⊆U whenever A⊆ U and U is α-open in X.

Corresponding Author: J. Subashini, Department of Mathematics,*
Rathnavel Subramaniam College of Arts & Science Sulur, Coimbatore - 641 402, India.

Definition 2.9: A subset A of X is called Generalized semi-preclosed (briefly gsp closed) [14] if spcl(A)⊆U whenever A⊆ U and U is open in X.

Definition 2.10: A subset A of X is called Regular generalized closed (briefly rg closed) [15] if cl(A)⊆U whenever A⊆ U and U is regular open in X.

Definition 2.11: A subset A of X is called θ generalized closed (briefly θg closed) [6] if cl_θ(A)⊆U whenever A⊆ U and U is open in X.

Definition 2.12: A subset A of X is called δ generalized closed (briefly δg closed) [18] if cl_δ(A)⊆U whenever A⊆ U and U is open in X.

Definition 2.13: A subset A of X is called Strongly generalized closed (briefly g* closed) [13] if cl(A)⊆U whenever A⊆ U and U is g-open in X.

Definition 2.14: A subset A of X is called Weakly closed (briefly w closed) [10] if cl(A)⊆U whenever A⊆ U and U is semi open in X.

Definition 2.15: A subset A of X is called Regular weakly closed (briefly rw closed) [4] if cl(A)⊆U whenever A⊆ U and U is regular semi open in X.

Definition 2.16: A subset A of X is called Regular generalized weakly closed (briefly rgw closed) [17] if cl(int(A)) ⊆U whenever A⊆ U and U is regular semi open in X.

Definition 2.17: A subset A of a space (X, τ) is called regular semi open [17] if there is a regular open set U such that U⊂A⊂cl(U)

Definition 2.18: A subset A of X is called $g\alpha$* closed set [16] if αcl(A)⊆ int(U) whenever A⊆ U and U is α open in X.

Definition 2.19: A subset A of X is called μ closed set [16] if cl(A)⊆U whenever A⊆ U and U is $g\alpha$* open in X.

Definition 2.20: A subset A of X is called $\hat{\mu}$ closed set [16] if scl(A)⊆U whenever A⊆ U and U is μ open in X.

3. *On* $\hat{\mu}\beta$ closed set

Definition 3.1: A subset A of a topological space (X,τ) is called β *open* if $A \subseteq cl(int(cl(A)))$,whenever A⊆ U and U is open in X.

Definition 3.2: A subset A of a topological space (X,τ) is called $\hat{\mu}$β closed set if $\hat{\mu}$cl(A)⊆ U, whenever A⊆ U and U is β open in X.

Remark 3.3: $\emptyset$ and X are $\hat{\mu}$β closed subset of X.

Theorem 3.4: Every closed set is $\hat{\mu}$β closed set, but not conversely.

Proof: Let A be closed set such that A⊆ U and U is β open set. A=Cl(A)⊆ U. Every closed set is $\hat{\mu}$ closed. Therefore $\hat{\mu}$cl(A)⊆ U,whenever A⊆ U and U is β-open.Hence A is $\hat{\mu}$β closed set.

Example 3.5: Let X={a, b, c, d}, τ={X,ϕ, {a},{b},{a, b},{a, b, c}} here A={a, d} is $\hat{\mu}$β closed but not closed set in X.

Remarks 3.6: Every θ-closed, π *closed*, δ closed, r closed set is closed. Therefore every θ-closed, π *closed*, δ closed, r closed set is $\hat{\mu}$β closed set.

Theorem 3.7: Every g closed set is $\hat{\mu}$β closed set, but not conversely.

Proof: Let A be g closed set such that cl(A) ⊆U, whenever A⊆U and U is open. Then cl(A) ⊆ $\hat{\mu}$cl(A) ⊆U.Therefore $\hat{\mu}$cl(A) ⊆U, whenever A⊆U and U is open. Since every open set is β-open, therefore **e**very g closed set is $\hat{\mu}$β closed set.

Example 3.8: Let X={a, b, c, d}, τ={X,ϕ,{a},{b},{a, b},{a, b, c}}.Let A={a} is $\hat{\mu}$β closed but not g closed.

Remarks 3.9: Every gr closed, g^* closed set is g closed. Therefore every gr closed, g^* closed set is $\hat{\mu}$β closed set.

Theorem 3.10: Every sg closed set is $\hat{\mu}\beta$ closed set, but not conversely.

Proof: Let A be sg closed set such that $s$cl(A)⊆U, whenever A⊆U and U is semi open. Then $s$cl(A) ⊆ cl(A) ⊆U. Therefore $\hat{\mu}$cl(A) ⊆U, whenever A⊆U and U is semi open. Since every semi open set is β-open, therefore **e**very sg closed set is $\hat{\mu}\beta$ closed set.

Example 3.11: Let X={a ,b, c, d}, τ={X,ϕ,{a},{b},{a, b},{a, b, c}}. Let A={a, b, d} is $\hat{\mu}\beta$ closed but not sg closed.

Theorem 3.12: Every gs closed, w closed, gα closed, αg closed set is $\hat{\mu}\beta$ closed set, but not conversely.

Example 3.13: Let X={a, b, c, d}, τ={X,ϕ,{a},{b},{a,b},{a,b,c}}.Let A = {a, c} is $\hat{\mu}\beta$ closed but not gs closed.

Theorem 3.14: Every gsp closed set is $\hat{\mu}\beta$ closed set.

Proof: Let A be gsp closed set such that spcl(A) ⊆U, whenever A⊆U and U is open. Then spcl(A) ⊆ cl(A) ⊆U. Therefore $\hat{\mu}$cl(A) ⊆U, whenever A⊆U and U is open. Since every open set is β-open, therefore **e**very gsp closed set is $\hat{\mu}\beta$ closed set.

Theorem 3.15: Let A⊆ B⊆ $\hat{\mu}$cl(A) and A is a $\hat{\mu}\beta$ closed subset of (X,τ) then B is also a $\hat{\mu}\beta$ closed subset of (X,τ).

Proof: Since A is a $\hat{\mu}\beta$ closed subset of (X,τ), So $\hat{\mu}$cl(A)⊆U, whenever A ⊆U and U is β open subset of X. Let A⊆ B⊆ $\hat{\mu}$cl(A) . That is $\hat{\mu}$cl(A)=$\hat{\mu}$cl(B).Let if possible there exists an β open subset V of X such that B ⊆ V.So A ⊆ V and A being $\hat{\mu}\beta$ closed subset of X, $\hat{\mu}$cl(A) ⊆ V. That is $\hat{\mu}$cl(B) ⊆ V. Hence B is also a $\hat{\mu}\beta$ closed subset of X.

Theorem 3.16: Let A⊆ B⊆X, where B is β open in X. If A is $\hat{\mu}\beta$ closed in X, then A is $\hat{\mu}\beta$ closed in B.

Proof: Let A ⊆ U, where U is β open set of X. Since U=V∩B, for Some β open set V of X and B is β open in X. Using assumption A is $\hat{\mu}\beta$ closed in X. We have $\hat{\mu}$cl(A)⊆U and so $\hat{\mu}$cl(A)=cl(A)∩B ⊆ U∩B ⊆ U. Hence A is $\hat{\mu}\beta$ closed in B.

Theorem 3.17: A subset A of X is $\hat{\mu}\beta$ closed sets iff $\hat{\mu}$cl(A) ∩ A^c contains no non-zero closed set in X.

Proof: Let A be a $\hat{\mu}\beta$ closed subset of X. Also if possible let M be closed subset of X such that M ⊆ $\hat{\mu}$cl(A) ∩A^c. That is M ⊆ $\hat{\mu}$cl(A) and M ⊆ A^c. Since M is a closed subset of X, M^c is an open subset of X ⊆ A, and A being $\hat{\mu}\beta$ open subset of X, $\hat{\mu}$cl(A) ⊆ M^c. But M⊆ $\hat{\mu}$cl(A).So we get a contradiction .Therefore M=∅.So the condition is true. Conversely, let A ⊆ N, and N is a open subset of X. Then N^c ⊆A^c, And N^c is a closed subset of X. Let if possible $\hat{\mu}$cl(A) ⊆ N .Then $\hat{\mu}$cl(A) ∩ N^c is a nonzero closed subset of $\hat{\mu}$cl(A) ∩ A^c, which is a contradiction .Hence A is a $\hat{\mu}\beta$ closed subset of X.

Theorem 3.18: A subset A of X is $\hat{\mu}\beta$ closed set in X iff$\hat{\mu}$cl(A)-A contain no non-empty β closed set in X.

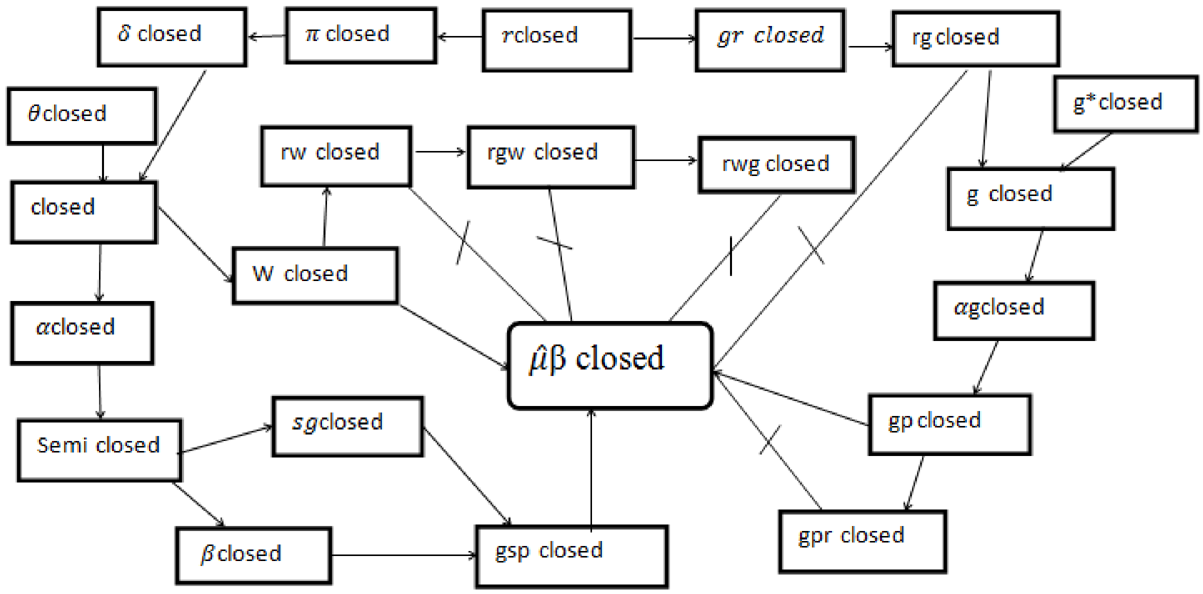
Proof: Suppose that F is a non-empty β closed subset if $\hat{\mu}$cl(A)-A. Now F ⊆ $\hat{\mu}$cl(A)-A. Then F ⊆ $\hat{\mu}$cl(A) ∩ A^c. Therefore F⊆ A^c. Since F^c is β open set and A is $\hat{\mu}\beta$ closed, $\hat{\mu}$cl(A) ⊆ F^c. That is F ⊆ $\hat{\mu}$cl(A)^c. Hence F⊆ $\hat{\mu}$cl(A) ∩[$\hat{\mu}$cl(A)]^c=∅. That is F=∅. Thus $\hat{\mu}$cl(A)-A contains no non empty β closed set. Conversely assume that $\hat{\mu}$cl(A)-A contains no nonempty β closed set. Let A ⊆ U and U is β open.Suppose that $\hat{\mu}$cl(A) is not contained in U. Then $\hat{\mu}$cl(A) ∩ U^c is a non-empty β closed set and contained in $\hat{\mu}$cl(A)-A .which is a contradiction. Therefore $\hat{\mu}$cl(A) ⊆ U and hence A is $\hat{\mu}\beta$ closed set.

Example 3.19: The figure 1 is justified with the following examples.

Let X= {a, b, c, d}, be with the topology τ={X, ϕ, {a}, {b}, {a, b}, {a, b, c}} then

1. Closed sets in X are X, ϕ, {d}, {c, d}, {a, c, d},{b, c, d}
2. $\hat{\mu}\beta$ closed sets in X are X, ϕ, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b,d},{a, c, d}, {b, c, d}.
3. αclosed sets in X are X, ϕ,{c},{d},{c, d},{a, c, d},{b, c, d}
4. Pre closed sets in X are X, ϕ,{c},{d},{c, d},{a, c, d},{b, c, d}
5. Semi closed sets in X are X, ϕ,{a},{b},{c},{d},{a, c},{a, d},{b, c},{b, d}{c, d},{a, c, d},{b, c, d}
6. Regular closed sets in X are X, ϕ, {a, c, d},{b, c, d}
7. g closed sets in X are X, ϕ,{d},{a, d} ,{b, d},{c, d} {a, b, d},{a, c, d},{b, c, d}.
8. g* closed sets in X are X, ϕ,{d},{a, d}, {b, d},{c, d} {a, b, d},{a, c, d},{b, c, d}.
9. gα closed sets in X are X, ϕ,{c},{d},{c, d},{a, c, d},{b, c, d}.
10. gsp closed sets in X are X, ϕ,{a},{b},{c},{d},{a, c},{a, d},{b, c},{b, d},{c, d},{a, b, d},{a, c, d},{b, c, d}.
11. sg closed sets in X are X, ϕ,{a},{b},{c},{d},{a, c},{a, d},{b, c},{b, d},{c, d},{a, b, d},{a, c, d},{b, c, d}.
12. rg closed sets in X are X, ϕ,{c},{d},{a, b},{a. c},{a, d},{b, c},{b, d},{c, d},{a, b, d},{a, c, d},{b, c, d}.

13. gr closed sets in X are X, ϕ,{d},{a, d} ,{b, d},{c, d},{a, b, d},{a, c, d},{b, c, d}.
14. w closed sets in X are X, ϕ,{d},{c, d},{a, c, d},{b, c, d}.
15. gα* closed sets in X are X, ϕ,{c},{d},{c, d},{a, c, d},{b, c, d}.
16. µ closed sets in X are X, ϕ,{d},{c, d},{a, c, d},{b, c, d}.
17. $\hat{\mu}$closed sets in X are X, ϕ,{a},{b},{c},{d},{a, c},{a, d},{b, d},{c, d} {a, b, d},{a, c, d},{b, c, d}.
18. rw closed sets in X are X, ϕ,{d},{a, b},{c, d},{a, b, c},{a, b, d},{a, c, d},{b, c, d}.
19. rwg closed sets in X are X, ϕ,{c},{d},{a, b},{a. c},{a, d}, {b, c},{b, d}, {c, d},{a, b, c}, {a, b, d},{a, c, d}, {b, c, d}.
20. gpr closed sets in X are X, ϕ,{c},{d},{a, b},{a,c},{a, d},{b, c},{b, d},{c, d},{a, b, c},{a, b, d}, {a, c, d}, {b, c, d}.
21. rgw closed sets in X are X, ϕ,{c},{d},{a, b},{c, d},{a, b, c} {a, b, d},{a, c, d},{b, c, d}.



A ⟶	B Means A implies B but not conversely
A —/—	B means A and B are independent of each other

Figure-1

4. µ̂β T_k Space (k = 0, $^1/_2$, 1, 2)

In this section, some new types of separation axioms are defined and studied in topological spaces called µ̂β T_k for k = 0, $^1/_2$, 1, 2 and µ̂β D_k for k = 0, 1, 2 and some properties of these spaces are also explained. The following definitions are introduced via µ̂β open sets.

Definition 4.1: A subset A of a topological space X is called a µ̂β difference set (briey, µ̂βD set) if there exist two µ̂β open sets U and V such that U ≠ X and A = U \ V.

Definition 4.2: A space X is said to be:
1. µ̂β T_0 if for each pair of distinct points x and y in X, there exists a µ̂β open set A containing x but not y or a µ̂β open set B containing y but not x.
2. µ̂β T_1 if for each pair x, y in X, x ≠ y, there exists a µ̂β open set G containing x but not y and a µ̂β open set B containing y but not x.
3. A space X is said to be µ̂β T_2 if for any pair of distinct points x and y in X, there exist U ∈ µ̂βO(X, x) and V ∈ µ̂βO(X, y) such that U ∩ V = ∅.
4. µ̂β D_0 (resp., µ̂β D_1) if for any pair of distinct points x and y of X there exists a µ̂βD set of X containing x but not y or (resp., and) a µ̂βD set of X containing y but not x.
5. µ̂β D_2 if for any pair of distinct points x and y of X, there exist disjoint µ̂βD sets G and H of X containing x and y, respectively.

Definition 4.3: A topological space X is called µ̂β $T_{1/2}$ if every µ̂β closed set is µ̂ closed.

Theorem 4.4: A topological space (X, τ) is μ̂β T_0 if and only if for each pair of distinct points x, y of X, μ̂βCl({x}) ≠ μ̂βCl({y}).

Proof:
Necessity: Let (X,τ) be a μ̂β T_0 space and x, y be any two distinct points of X. There exists a μ̂β open set U containing x or y, say x but not y. Then X \ U is a μ̂β closed set which does not contain x but contains y. Since μ̂βCl({y}) is the smallest μ̂β closed set containing y, μ̂β Cl({y}) ⊆ X\U and therefore x ∉ μ̂β Cl({y}). Consequently μ̂βCl({x}) ≠ μ̂βCl({y}).

Sufficiency: Suppose that x, y ∈ X, x ≠ y and μ̂βCl({x}) ≠ μ̂β Cl({y}). Let z be a point of X such that z ∈ μ̂βCl({x}) but z ∉ μ̂βCl({y}). We claim that x ∉ μ̂βCl({y}). For, if x ∈ μ̂βCl({y}) then μ̂βCl({x}) ⊆ μ̂βCl({y}). This contradicts the fact that z ∉ μ̂βCl({y}). Consequently x belongs to the μ̂β open set X \ μ̂βCl({y}) to which y does not belong. Hence (X, τ) is a μ̂β T_0 space.

Theorem 4.5: A topological space (X, τ) is μ̂β T_1 if and only if the singletons are μ̂β closed sets.

Proof: Let (X, τ) be μ̂β T_1 space and x any point of X. Suppose y ∈ X \ {x}, then x ≠ y and so there exists a μ̂β open set U such that y ∈ U but x ∉ U. Consequently y ∈ U ⊆ X \{x}, that is X \{x} = ∪{U: y ∈ X \{x}} which is μ̂β -open.

Conversely, suppose {p} is μ̂β closed for every p ∈ X. Let x, y ∈ X with x ≠ y. Now x ≠ y implies y ∈ X \ {x}. Hence X \ {x} is a μ̂β open set containing y but not x. Similarly X \ {y} is a μ̂β open set containing x but not y. Therefore X is a μ̂β T_1 space.

Theorem 4.6: A topological space (X, τ) is μ̂β $T_{1/2}$ if each singleton {x} of X is either μ̂ open or μ̂ closed.

Proof: Suppose {x} is μ̂β open, then it is obvious that (X \ {x}) is μ̂β closed. Since (X, τ) is μ̂β $T_{1/2}$, so (X \ {x}) is μ̂ closed, that is {x} is μ̂ open.

Theorem 4.7: The following statements are equivalent for a topological space (X, τ)
1. X is μ̂β T_2.
2. Let x ∈ X. For each y ≠ x, there exists a μ̂β open set U containing x such that y ∉ μ̂βCl(U).
3. For each x ∈ X, ∩{μ̂βCl(U) : U ∈ μ̂βO(X) and x ∈ U} = {x}.

Proof:
(1) ⇒ (2): Since X is μ̂β T_2, there exist disjoint μ̂β open sets U and V containing x and y respectively. So, U ⊆ X\V. Therefore, μ̂βCl(U) ⊆ X\V. So y ∉ μ̂βCl(U).

(2) ⇒ (3): If possible for some y ≠ x, we have y∈ μ̂βCl(U) for every μ̂β open set U Containing x,which contradicts (2).

(3) ⇒ (1): Let x, y ∈ X and x ≠ y. Then there exists a μ̂β open set U containing x such that y ∉ μ̂βCl(U). Let V = X \ μ̂βCl(U), then y ∈ V and x ∈ U and also U ∩V = ∅. Therefore X is μ̂β T_2 space.

Theorem 4.8: Let (X, τ) be a topological space, then the following statements are true:
1. Every μ̂β T_2 space is μ̂β T_1.
2. Every μ̂β T_1 space is μ̂β $T_{1/2}$

Proof: The proof is straightforward from the definitions and theorem 4.5.

Remark 4.9: Every proper μ̂β open set is a μ̂βD set. But, the converse is not true in general as the next example shows.

Example 4.10: Consider X = {a, b, c, d} with the topology τ = {∅, {a}, {b},{a, b},{a, b, c}, X}. So, μ̂βO(X,τ) = {∅,{a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X}, then U = {a, b, d} ≠ X and V = {a,b,c} are μ̂β open sets in X and A = U \ V = {a, b,d}\ {a,b,c} = {d}, then we have A = {d} is a μ̂βD set but it is not μ̂β open. Now we define another set of separation axioms called μ̂βD_k, for k = 0, 1, 2, by using the μ̂βD -sets.

Remark 4.11: For a topological space (X, τ), the following properties hold:
1. If (X, τ) is μ̂β T_k, then it is μ̂β D_k, for k = 0, 1, 2.
2. If (X, τ) is μ̂βD_k, then it is μ̂βD_{k-1}, for k = 1, 2.

Proof: Obvious.

Theorem 4.12: A space X is $\hat{\mu}\beta D_0$ if and only if it is $\hat{\mu}\beta T_0$.

Proof: Suppose that X is $\hat{\mu}\beta D_0$. Then for each distinct pair x, y ∈ X, at least one of x, y, say x, belongs to a $\hat{\mu}\beta$D set G but y ∉ G. Let G = $U_1 \setminus U_2$ where $U_1 \neq$ X and $U_1, U_2 \in \hat{\mu}\beta$O(X, τ). Then x ∈ U_1, and for y ∉ G we have two cases: (a) y ∉ U_1, (b) y ∈ U_1 and y ∈ U_2.

In case (a), x ∈ U_1 but y ∉ U_1.

In case (b), y ∈ U_2 but x ∉ U_2.

Thus in both the cases, we obtain that X is $\hat{\mu}\beta T_0$. Conversely, if X is $\hat{\mu}\beta T_0$, by Remark 4.11 (1), X is $\hat{\mu}\beta D_0$.

Theorem 4.13: A space X is $\hat{\mu}\beta D_1$ if and only if it is $\hat{\mu}\beta D_2$.

Proof: Necessity: Let x, y ∈ X, x ≠ y. Then there exist $\hat{\mu}\beta$D sets G_1, G_2 in X such that x ∈ G_1, y ∉ G_1 and y ∈ G_2, x ∉ G_2. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are $\hat{\mu}\beta$ open sets in X. From x ∉ G_2, it follows that either x ∉ U_3 or x ∈ U_3 and x ∈ U_4. We discuss the two cases separately.
(i) x ∉ U_3. By y ∉ G_1 we have two sub-cases:
(a) y ∉ U_1. Since x ∈ $U_1 \setminus U_2$, it follows that x ∈ $U_1 \setminus (U_2 \cup U_3)$, and since y ∈ $U_3 \setminus U_4$ we have y ∈ $U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \emptyset$.
(b) y ∈ U_1 and y ∈ U_2. We have x ∈ $U_1 \setminus U_2$, and y ∈ U_2. Therefore $(U_1 \setminus U_2) \cap U_2 = \emptyset$.
(ii) x ∈ U_3 and x ∈ U_4. We have y ∈ $U_3 \setminus U_4$ and x ∈ U_4. Hence $(U_3 \setminus U_4) \cap U_4 = \emptyset$. Therefore X is $\hat{\mu}\beta D_2$.

Sufficiency: Follows from Remark 4.11 (2).

Corollary 4.14: If (X, τ) is $\hat{\mu}\beta D_1$, then it is $\hat{\mu}\beta T_0$.

Proof: Follows from Remark 4.11 (2) and theorem 4.12. Here is an example which shows that the converse of Corollary 4.14 is not true in general.

Definition 4.15: A point x ∈ X which has only X as the $\hat{\mu}\beta$ neighbourhood is called a $\hat{\mu}\beta$ neat point.

Proposition 4.16: For a $\hat{\mu}\beta T_0$ topological space (X, τ) the following are equivalent:
1. (X, τ) is $\hat{\mu}\beta D_1$.
2. (X, τ) has no $\hat{\mu}\beta$ neat point.

Proof:
(1) ⇒ (2): Since (X, τ) is $\hat{\mu}\beta D_1$, then each point x of X is contained in a $\hat{\mu}\beta$D set A = U \ V and thus in U. By definition U ≠ X. This implies that x is not a $\hat{\mu}\beta$ neat point.

(2) ⇒ (1): If X is $\hat{\mu}\beta T_0$, then for each distinct pair of points x, y ∈ X, at least one of them, x (say) has a $\hat{\mu}\beta$ neighbourhood U containing x and not y. Thus U which is different from X is a $\hat{\mu}\beta$ D set. If X has no $\hat{\mu}\beta$ neat point, then y is not a $\hat{\mu}\beta$ neat point. This means that there exists a $\hat{\mu}\beta$ neighbourhood V of y such that V ≠ X. Thus y ∈ V \ U but not x and V \ U is a $\hat{\mu}\beta$D set. Hence X is $\hat{\mu}\beta D_1$.

Corollary 4.17: A $\hat{\mu}\beta T_0$ space X is not $\hat{\mu}\beta D_1$ if and only if there is a unique $\hat{\mu}\beta$ neat point in X.

Proof: We only prove the uniqueness of the $\hat{\mu}\beta$ neat point. If x and y are two $\hat{\mu}\beta$ neat points in X, then since X is $\hat{\mu}\beta T_0$, at least one of x and y, say x, has a $\hat{\mu}\beta$ neighbourhood U containing x but not y. Hence U ≠ X. Therefore x is not a $\hat{\mu}\beta$ neat point which is a contradiction.

Definition 4.18: A topological space (X, τ) is said to be $\hat{\mu}\beta$ symmetric if for x and y in X, x ∈ $\hat{\mu}\beta$Cl({y}) implies y ∈ $\hat{\mu}\beta$Cl({x}).

Theorem 4.19: If (X, τ) is a topological space, then the following are equivalent:
1. (X, τ) is a $\hat{\mu}\beta$ symmetric space.
2. {x} is $\hat{\mu}\beta$ closed, for each x ∈ X.

Proof:
(1) ⇒ (2): Assume that {x} ⊆ U ∈ $\hat{\mu}\beta$O(X), but $\hat{\mu}\beta$Cl({x}) ⊄ U. Then $\hat{\mu}\beta$Cl({x} ∩ X \ U ≠ ∅. Now, we take y ∈ $\hat{\mu}\beta$Cl({x} ∩ X \U, then by hypothesis x ∈ $\hat{\mu}\beta$Cl({y}) ⊆ X \ U and x ∉ U, which is a contradiction. Therefore {x} is $\hat{\mu}\beta$ closed, for each x ∈ X.

(2) ⇒ (1): Assume that $x \in \hat{\mu}\beta Cl(\{y\})$, but $y \notin \hat{\mu}\beta Cl(\{x\})$. Then $\{y\} \subseteq X \setminus \hat{\mu}\beta Cl(\{x\})$ and hence $\hat{\mu}\beta Cl(\{y\} \subseteq X \setminus \hat{\mu}\beta Cl(\{x\})$. Therefore $x \in X \setminus \hat{\mu}\beta Cl(\{x\})$, which is a contradiction and hence $y \in \hat{\mu}\beta Cl(\{x\})$.

Corollary 4.20: If a topological space (X, τ) is a $\hat{\mu}\beta T_1$ space, then it is $\hat{\mu}\beta$ symmetric.

Proof: In a $\hat{\mu}\beta T_1$ space, every singleton is $\hat{\mu}\beta$ closed and therefore is by theorem 4.19, (X, τ) is $\hat{\mu}\beta$ symmetric.

Corollary 4.21: If a topological space (X, τ) is $\hat{\mu}\beta$ symmetric and $\hat{\mu}\beta T_0$, then (X, τ) is $\hat{\mu}\beta T_1$.

Proof: Let $x \neq y$ and as (X, τ) is $\hat{\mu}\beta T_0$, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in \hat{\mu}\beta O(X)$. Then $x \notin \hat{\mu}\beta Cl(\{y\})$ and hence $y \notin \hat{\mu}\beta Cl(\{x\})$. There exists a $\hat{\mu}\beta$ open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus (X, τ) is a $\hat{\mu}\beta T_1$ space.

Corollary 4.22: If a topological space (X, τ) is $\hat{\mu}\beta T_1$, then (X, τ) is $\hat{\mu}\beta$ symmetric and $\hat{\mu}\beta T_{1/2}$

Proof: By Corollary 4.21 and Proposition 4.8, it is true.

Corollary 4.23: For a rg*b-symmetric space (X, τ), the following are equivalent:
1. (X, τ) is $\hat{\mu}\beta T_0$.
2. (X, τ) is $\hat{\mu}\beta D_1$.
3. (X, τ) is $\hat{\mu}\beta T_1$.

Definition 4.24: Let A be a subset of a topological space $((X, \tau)$. The $\hat{\mu}\beta$ kernel of A, denoted by $\hat{\mu}\beta ker(A)$ is defined to be the set $\hat{\mu}\beta ker(A) = \cap\{U \in \hat{\mu}\beta O(X): A \subseteq U\}$.

Theorem 4.25: Let (X, τ) be a topological space and $x \in X$. Then $y \in \hat{\mu}\beta ker(\{x\})$ if and only if $x \in \hat{\mu}\beta Cl(\{y\})$.

Proof: Suppose that $y \notin \hat{\mu}\beta ker(\{x\})$. Then there exists a $\hat{\mu}\beta$ open set V containing x such that $y \notin V$. Therefore, we have $x \notin \hat{\mu}\beta Cl(\{y\})$. The proof of the converse case can be done similarly.

Theorem 4.26: Let (X, τ) be a topological space and A be a subset of X. Then, $\hat{\mu}\beta ker(A) = \{x \in X: \hat{\mu}\beta Cl(\{x\}) \cap A \neq \emptyset\}$.

Proof: Let $x \in \hat{\mu}\beta ker(A)$ and suppose $\hat{\mu}\beta Cl(\{x\}) \cap A = \emptyset$. Hence $x \notin X \setminus \hat{\mu}\beta Cl(\{x\})$ which is a $\hat{\mu}\beta$ open set containing A. This is impossible, since $x \in \hat{\mu}\beta ker(A)$. Consequently, $\hat{\mu}\beta Cl(\{x\}) \cap A \neq \emptyset$. Next, let $x \in X$ such that $\hat{\mu}\beta Cl(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin \hat{\mu}\beta ker(A)$. Then, there exists a $\hat{\mu}\beta$ open set V containing A and $x \notin V$. Let $y \in \hat{\mu}\beta Cl(\{x\}) \cap A$. Hence, V is a $\hat{\mu}\beta$ neighbourhood of y which does not contain x. By this contradiction $x \in \hat{\mu}\beta ker(A)$ and the claim.

Theorem 4.27: The following properties hold for the subsets A, B of a topological space(X, τ)
1. $A \subseteq \hat{\mu}\beta ker(A)$.
2. $A \subseteq B$ implies that $\hat{\mu}\beta ker(A) \subseteq \hat{\mu}\beta ker(B)$.
3. If A is $\hat{\mu}\beta$ open in (X,τ), then $A = \hat{\mu}\beta ker(A)$.
4. $\hat{\mu}\beta ker(\hat{\mu}\beta ker(A)) = \hat{\mu}\beta ker(A)$.

Proof: (1), (2) and (3) are immediate consequences of Definition 4.24. To prove (4), first observe that by (1) and (2), we have $\hat{\mu}\beta ker(A) \subseteq \hat{\mu}\beta ker(\hat{\mu}\beta ker(A))$. If $x \notin \hat{\mu}\beta ker(A)$, then there exists $U \in \hat{\mu}\beta O(X,\tau)$ such that $A \subseteq U$ and $x \notin U$. Hence $\hat{\mu}\beta ker(A) \subseteq U$, and so we have $x \notin \hat{\mu}\beta ker(\hat{\mu}\beta ker(A))$. Thus $\hat{\mu}\beta ker(\hat{\mu}\beta ker(A)) = \hat{\mu}\beta ker(A)$.

Proposition 4.28: If a singleton $\{x\}$ is a $\hat{\mu}\beta D$ set of (X, τ), then $\hat{\mu}\beta ker(\{x\}) \neq X$.

Proof: Since $\{x\}$ is a $\hat{\mu}\beta D$ set of (X, τ), then there exist two subsets $U_1, U_2 \in \hat{\mu}\beta O(X,\tau)$ such that $\{x\} = U_1 \setminus U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that $\hat{\mu}\beta ker(\{x\}) \subseteq U_1 \neq X$ and so $\hat{\mu}\beta ker(\{x\}) \neq X$.

5. $\hat{\mu}\beta$ R_k Space (k = 0, 1)

In this section, new classes of topological spaces called $\hat{\mu}\beta$ R_0 and $\hat{\mu}\beta$ R_1spaces are introduced.

Definition 5.1: A topological space (X, τ) is said to be $\hat{\mu}\beta$ R_0 if U is a $\hat{\mu}\beta$ open set and $x \in U$ then $\hat{\mu}\beta Cl(\{x\}) \subseteq U$.

Theorem 5.2: For a topological space (X, τ) the following properties are equivalent:
1. (X, τ) is $\hat{\mu}\beta$ R_0.
2. For any $F \in \hat{\mu}\beta C(X)$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in \hat{\mu}\beta O(X)$.
3. For any $F \in \hat{\mu}\beta C(X)$, $x \notin F$ implies $F \cap \hat{\mu}\beta Cl(\{x\}) = \emptyset$.
4. For any distinct points x and y of X, either $\hat{\mu}\beta Cl(\{x\}) = \hat{\mu}\beta Cl(\{y\})$ or $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$.

Proof:

(1) ⇒ (2): Let F ∈ $\hat{\mu}\beta$C(X) and x ∉ F. Then by (1), $\hat{\mu}\beta$Cl({x}) ⊆ X \ F. Set U = X \ $\hat{\mu}\beta$Cl({x}), then U is a $\hat{\mu}\beta$ open set such that F ⊆ U and x ∉ U.

(2) ⇒ (3): Let F ∈ $\hat{\mu}\beta$C(X) and x ∉ F. There exists U ∈ $\hat{\mu}\beta$O(X) such that F ⊆ U and x ∉ U. Since U ∈ $\hat{\mu}\beta$O(X), U ∩ $\hat{\mu}\beta$Cl({x}) = ∅ and F ∩ $\hat{\mu}\beta$Cl({x}) = ∅.

(3) ⇒ (4): Suppose that $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}) for distinct points x, y ∈ X. There exists z ∈ $\hat{\mu}\beta$Cl({x}) such that z ∉ $\hat{\mu}\beta$Cl({y}) (or z ∈ $\hat{\mu}\beta$Cl({y}) such that z ∉ rg*bCl({x})). There exists V ∈ $\hat{\mu}\beta$O(X) such that y ≠ V and z ∈ V; hence x ∈ V. Therefore, we have x ∉ $\hat{\mu}\beta$Cl({y}). By (3), we obtain $\hat{\mu}\beta$Cl({x}) ∩ $\hat{\mu}\beta$Cl({y}) = ∅.

(4) ⇒ (1): let V ∈ $\hat{\mu}\beta$O(X) and x ∈ V. For each y ∉ V, x ≠ y and x ∉ $\hat{\mu}\beta$Cl({y}). This shows that $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}). By (4), $\hat{\mu}\beta$Cl({x}) ∩ $\hat{\mu}\beta$Cl({y}) = ∅ for each y ∈ X\V and hence $\hat{\mu}\beta$Cl({x}) ∩ ($\bigcup_{y\in X\setminus V}$ $\hat{\mu}\beta$Cl(y)) = ∅. On other hand, since V ∈ $\hat{\mu}\beta$O(X) and y ∈ X \ V, we have $\hat{\mu}\beta$Cl({y}) ⊆ X \ V and hence X \ V =. $\bigcup_{y\in X\setminus V}$ $\hat{\mu}\beta$Cl(y). Therefore, we obtain (X \ V) ∩ $\hat{\mu}\beta$Cl({x}) = ∅ and $\hat{\mu}\beta$Cl({x}) ⊆ V . This shows that (X, τ) is a $\hat{\mu}\beta$ R_0 space.

Theorem 5.3: If a topological space (X, τ) is $\hat{\mu}\beta$ T_0 and a $\hat{\mu}\beta$ R_0 space then it is $\hat{\mu}\beta$ T_1.

Proof: Let x and y be any distinct points of X. Since X is $\hat{\mu}\beta T_0$, there exists a $\hat{\mu}\beta$ open set U such that x ∈ U and y ∉ U. As x ∈ U implies that $\hat{\mu}\beta$Cl({x}) ⊆ U. Since y ∉ U, so y ∉ $\hat{\mu}\beta$Cl({x}). Hence y ∈ V = X \ $\hat{\mu}\beta$Cl({x}) and it is clear that x ∉ V. Hence it follows that there exist $\hat{\mu}\beta$ open sets U and V containing x and y respectively, such that y ∉ U and x ∉ V. This implies that X is $\hat{\mu}\beta$ T_1.

Theorem 5.4: For a topological space (X, τ) the following properties are equivalent:

1. (X, τ) is $\hat{\mu}\beta$ R_0.
2. x ∈ $\hat{\mu}\beta$Cl({y}) if and only if y ∈ $\hat{\mu}\beta$Cl({x}), for any points x and y in X.

Proof:

(1) ⇒ (2): Assume that X is $\hat{\mu}\beta$ R_0. Let x ∈ $\hat{\mu}\beta$Cl({y}) and V be any $\hat{\mu}\beta$ open set such that y ∈ V. Now by hypothesis, x ∈ V. Therefore, every $\hat{\mu}\beta$ open set which contain y contains x. Hence y ∈ $\hat{\mu}\beta$Cl({x}).

(2) ⇒ (1): Let U be a $\hat{\mu}\beta$ open set and x ∈ U. If y ∉ U, then x ∉ $\hat{\mu}\beta$Cl({y}) and hence y ∉ $\hat{\mu}\beta$ Cl({x}). This implies that $\hat{\mu}\beta$Cl({x}) ⊆ U. Hence (X, τ) is $\hat{\mu}\beta$ R_0. From Definition 4.18 and theorem 5.4, the notions of $\hat{\mu}\beta$ symmetric and $\hat{\mu}\beta$ R_0 are equivalent.

Theorem 5.5: The following statements are equivalent for any points x and y in a topological space (X, τ):

1. $\hat{\mu}\beta$ker({x}) ≠ $\hat{\mu}\beta$ker({y}).
2. $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}).

Proof:

(1) ⇒ (2): Suppose that $\hat{\mu}\beta$ker({x}) ≠ $\hat{\mu}\beta$ker({y}), then there exists a point z in X such that z ∈ $\hat{\mu}\beta$ker({x}) and z ∉ $\hat{\mu}\beta$ker({y}). From z ∈ $\hat{\mu}\beta$ker({x}) it follows that {x}∩ $\hat{\mu}\beta$Cl({z}) ≠ ∅ which implies x ∈ $\hat{\mu}\beta$Cl({z}). By z ∉ $\hat{\mu}\beta$ker({y}), we have {y}∩ $\hat{\mu}\beta$Cl({z}) = ∅. Since x ∈ $\hat{\mu}\beta$Cl({z}), $\hat{\mu}\beta$Cl({x}) ⊆ $\hat{\mu}\beta$Cl({z}) and {y}∩ $\hat{\mu}\beta$Cl({x}) = ∅. Therefore, it follows that $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}). Now $\hat{\mu}\beta$ker({x}) ≠ $\hat{\mu}\beta$ker({y}) implies that $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}).

(2) ⇒ (1): Suppose that $\hat{\mu}\beta$Cl({x}) ≠ $\hat{\mu}\beta$Cl({y}). Then there exists a point z in X such that z ∈ $\hat{\mu}\beta$Cl({x}) and z ∉ $\hat{\mu}\beta$Cl({y}). Then, there exists a $\hat{\mu}\beta$ open set containing z and therefore x but not y, namely, y ∉ $\hat{\mu}\beta$ker({x}) and thus $\hat{\mu}\beta$ker({x}) ≠ $\hat{\mu}\beta$ker({y}).

Theorem 5.6: Let (X,τ) be a topological space. Then ∩{$\hat{\mu}\beta$Cl({x}) : x ∈ X} = ∅ if and only if $\hat{\mu}\beta$ker({x}) ≠ X for every x ∈ X.

Proof:

Necessity: Suppose that ∩{$\hat{\mu}\beta$Cl({x}) : x ∈ X} = ∅. Assume that there is a point y in X such that $\hat{\mu}\beta$ker({y}) = X. Let x be any point of X. Then x ∈ V for every $\hat{\mu}\beta$ open set V containing y and hence y ∈ $\hat{\mu}\beta$Cl({x}) for any x ∈ X. This implies that y ∈ ∩{$\hat{\mu}\beta$Cl({x}) : x ∈ X}. But this is a contradiction.

Sufficiency: Assume that $\hat{\mu}\beta$ker({x}) ≠ X for every x ∈ X. If there exists a point y in X such that y ∈∩{$\hat{\mu}\beta$Cl({x}) : x ∈ X}, then every $\hat{\mu}\beta$ open set containing y must contain every point of X. This implies that the space X is the unique $\hat{\mu}\beta$ open set containing y. Hence $\hat{\mu}\beta$ker({y}) = X which is a contradiction. Therefore, ∩{$\hat{\mu}\beta$Cl({x}) : x ∈ X} = ∅.

Theorem 5.7: A topological space (X,τ) is $\hat{\mu}\beta\, R_0$ if and only if for every x and y in X, $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$ implies $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$.

Proof:
Necessity: Suppose that (X, τ) is $\hat{\mu}\beta\, R_0$ and $x, y \in X$ such that $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$. Then, there exists $z \in \hat{\mu}\beta Cl(\{x\})$ such that $z \notin \hat{\mu}\beta Cl(\{y\})$ (or $z \in \hat{\mu}\beta Cl(\{y\})$ such that $z \notin rg^*bCl(\{x\})$). There exists $V \in \hat{\mu}\beta O(X)$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \hat{\mu}\beta Cl(\{y\})$. Thus $x \in [X \setminus \hat{\mu}\beta Cl(\{y\})] \in \hat{\mu}\beta O(X)$, which implies $\hat{\mu}\beta Cl(\{x\}) \subseteq [X \setminus \hat{\mu}\beta Cl(\{y\})]$ and $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$.

Sufficiency: Let $V \in \hat{\mu}\beta O(X)$ and let $x \in V$. We still show that $\hat{\mu}\beta Cl(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X \setminus V$. Then $x \neq y$ and $x \notin \hat{\mu}\beta Cl(\{y\})$. This shows that $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$. By assumption, $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$. Hence $y \notin \hat{\mu}\beta Cl(\{x\})$ and therefore $\hat{\mu}\beta Cl(\{x\}) \subseteq V$.

Theorem 5.8: A topological space (X, τ) is $\hat{\mu}\beta\, R_0$ if and only if for any points x and y in X, $\hat{\mu}\beta ker(\{x\}) \neq \hat{\mu}\beta ker(\{y\})$ implies $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\}) = \emptyset$.

Proof: Suppose that (X,τ) is a $\hat{\mu}\beta\, R_0$ space. Thus by Theorem 3.5, for any points x and y in X if $\hat{\mu}\beta ker(\{x\}) \neq \hat{\mu}\beta ker(\{y\})$ then $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$. Now we prove that $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\}) = \emptyset$. Assume that $z \in \hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\})$. By $z \in \hat{\mu}\beta ker(\{x\})$ and theorem 4.24, it follows that $x \in \hat{\mu}\beta Cl(\{z\})$. Since $x \in rg^*bCl(\{x\})$, by theorem 5.2, $\hat{\mu}\beta Cl(\{x\}) = \hat{\mu}\beta Cl(\{z\})$. Similarly, we have $\hat{\mu}\beta Cl(\{y\}) = \hat{\mu}\beta Cl(\{z\}) = \hat{\mu}\beta Cl(\{x\})$. This is a contradiction. Therefore, we have $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\}) = \emptyset$.

Conversely, let (X,τ) be a topological space such that for any points x and y in X, $\hat{\mu}\beta ker(\{x\}) \neq \hat{\mu}\beta ker(\{y\})$ implies $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\}) = \emptyset$. If $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$, then by Proposition 3.4, $\hat{\mu}\beta ker(\{x\}) \neq \hat{\mu}\beta ker(\{y\})$. Hence, $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{y\}) = \emptyset$ which implies $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$. Because $z \in \hat{\mu}\beta Cl(\{x\})$ implies that $x \in \hat{\mu}\beta ker(\{z\})$ and therefore $\hat{\mu}\beta ker(\{x\}) \cap \hat{\mu}\beta ker(\{z\}) \neq \emptyset$. By hypothesis, we have $\hat{\mu}\beta ker(\{x\}) = \hat{\mu}\beta ker(\{z\})$. Then $z \in \hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\})$ implies that $\hat{\mu}\beta ker(\{x\}) = \hat{\mu}\beta ker(\{z\}) = \hat{\mu}\beta ker(\{y\})$. This is a contradiction. Therefore, $\hat{\mu}\beta Cl(\{x\}) \cap \hat{\mu}\beta Cl(\{y\}) = \emptyset$ and by theorem 5.2, (X, τ) is a $\hat{\mu}\beta\, R_0$ space

Theorem 5.9: For a topological space (X,τ) the following properties are equivalent:
1. (X, τ) is a $\hat{\mu}\beta\, R_0$ space.
2. For any non-empty set A and $G \in \hat{\mu}\beta O(X)$ such that $A \cap G \neq \emptyset$, there exists $F \in \hat{\mu}\beta C(X)$ such that $A \cap F \neq \emptyset$ and $F \subseteq G$.
3. For any $G \in \hat{\mu}\beta O(X)$, we have $G = \cup\{F \in \hat{\mu}\beta C(X): F \subseteq G\}$.
4. For any $F \in \hat{\mu}\beta C(X)$, we have $F = \cap\{G \in \hat{\mu}\beta O(X): F \subseteq G\}$.
5. For every $x \in X$, $\hat{\mu}\beta Cl(\{x\}) \subseteq \hat{\mu}\beta ker(\{x\})$.

Proof:
(1) ⇒ (2): Let A be a non-empty subset of X and $G \in \hat{\mu}\beta O(X)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in \hat{\mu}\beta O(X)$, $\hat{\mu}\beta Cl(\{x\}) \subseteq G$. Set $F = \hat{\mu}\beta Cl(\{x\})$, then $F \in \hat{\mu}\beta C(X)$, $F \subseteq G$ and $A \cap F \neq \emptyset$.

(2) ⇒ (3): Let $G \in \hat{\mu}\beta O(X)$, then $G \supseteq \cup\{F \in \hat{\mu}\beta C(X): F \subseteq G\}$. Let x be any point of G. There exists $F \in \hat{\mu}\beta C(X)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in \hat{\mu}\beta C(X): F \subseteq G\}$ and hence $G = \cup\{F \in \hat{\mu}\beta C(X): F \subseteq G\}$.

(3) ⇒ (4): Obvious.

(4) ⇒ (5): Let x be any point of X and $y \notin \hat{\mu}\beta ker(\{x\})$. There exists $V \in \hat{\mu}\beta O(X)$ such that $x \in V$ and $y \notin V$, hence $\hat{\mu}\beta Cl(\{y\}) \cap V = \emptyset$. By (4), $(\cap\{G \in \hat{\mu}\beta O(X): \hat{\mu}\beta Cl(\{y\}) \subseteq G\}) \cap V = \emptyset$ and there exists $G \in \hat{\mu}\beta O(X)$ such that $x \notin G$ and $\hat{\mu}\beta Cl(\{y\}) \subseteq G$. Therefore $\hat{\mu}\beta Cl(\{x\}) \cap G = \emptyset$ and $y \notin \hat{\mu}\beta Cl(\{x\})$. Consequently, we obtain $\hat{\mu}\beta Cl(\{x\}) \subseteq \hat{\mu}\beta ker(\{x\})$.

(5) ⇒ (1): Let $G \in \hat{\mu}\beta O(X)$ and $x \in G$. Let $y \in \hat{\mu}\beta ker(\{x\})$, then $x \in \hat{\mu}\beta Cl(\{y\})$ and $y \in G$. This implies that $\hat{\mu}\beta ker(\{x\}) \subseteq G$. Therefore, we obtain $x \in \hat{\mu}\beta Cl(\{x\}) \subseteq \hat{\mu}\beta ker(\{x\}) \subseteq G$. This shows that (X, τ) is a $\hat{\mu}\beta\, R_0$ space.

Corollary 5.10: For a topological space (X, τ) the following properties are equivalent:
1. (X, τ) is a $\hat{\mu}\beta\, R_0$ space.
2. $\hat{\mu}\beta Cl(\{x\}) = \hat{\mu}\beta ker(\{x\})$ for all $x \in X$.

Proof:
(1) ⇒ (2): Suppose that (X,τ) is a $\hat{\mu}\beta\, R_0$ space. By theorem 5.9, $\hat{\mu}\beta Cl(\{x\}) \subseteq \hat{\mu}\beta ker(\{x\})$ for each $x \in X$. Let $y \in \hat{\mu}\beta ker(\{x\})$, then $x \in \hat{\mu}\beta Cl(\{y\})$ and $\hat{\mu}\beta Cl(\{x\}) = \hat{\mu}\beta Cl(\{y\})$. Therefore, $y \in \hat{\mu}\beta Cl(\{x\})$ and hence $\hat{\mu}\beta ker(\{x\}) \subseteq \hat{\mu}\beta Cl(\{x\})$. This shows that $\hat{\mu}\beta Cl(\{x\}) = \hat{\mu}\beta ker(\{x\})$.

(2) ⇒ (1): Follows from theorem 5.9.

Theorem 5.11: For a topological space (X, τ) the following properties are equivalent:
1. (X, τ) is a µ̂β R_0 space.
2. If F is µ̂β closed, then F = µ̂βker(F).
3. If F is µ̂β closed and x ∈ F, then µ̂βker({x}) ⊆ F.
4. If x ∈ X, then µ̂βker({x}) ⊆ µ̂βCl({x}).

Proof:
(1) ⇒ (2): Let F be a µ̂β closed and x ∉ F. Thus (X \ F) is a µ̂β open set containing x. Since (X, τ) is µ̂β R_0, µ̂βCl({x}) ⊆ (X \ F). Thus µ̂βCl({x}) ∩ F = ∅ and by theorem 2.33, x ∉ µ̂βker(F). Therefore µ̂βker(F) = F.

(2) ⇒ (3): In general, A ⊆ B implies µ̂βker(A) ⊆ µ̂βker(B). Therefore, it follows from (2), that µ̂βker({x}) ⊆ µ̂βker(F) = F.

(3) ⇒ (4): Since x ∈ µ̂βCl({x}) and µ̂βCl({x}) is µ̂β closed, by (3), µ̂βker({x}) ⊆ µ̂βCl({x}).

(4) ⇒ (1): We show the implication by using theorem 5.4. Let x ∈ µ̂βCl({y}). Then by theorem 4.25, y ∈ µ̂βker({x}). Since x ∈ µ̂βCl({x}) and µ̂βCl({x}) is µ̂β closed, by (4), we obtain y ∈ µ̂βker({x}) ⊆ µ̂βCl({x}). Therefore x ∈ µ̂βCl({y}) implies y ∈ µ̂βCl({x}). The converse is obvious and (X,τ) is µ̂β R_0.

Definition 5.12: A topological space (X,τ) is said to be µ̂β R_1 if for x, y in X with µ̂βCl({x}) ≠ µ̂βCl({y}), there exist disjoint µ̂β open sets U and V such that µ̂βCl({x}) ⊆ U and µ̂βCl({y}) ⊆ V

Theorem 5.13: A topological space (X,τ) is µ̂β R_1 if it is µ̂β T_2.

Proof: Let x and y be any points of X such that µ̂βCl({x}) ≠ µ̂βCl({y}). By theorem 4.8 (1), every µ̂β T_2 space is µ̂β T_1. Therefore, by theorem 4.5, µ̂βCl({x}) = {x}, µ̂βCl({y}) = {y} and hence {x} ≠ {y}. Since (X, τ) is µ̂β T_2, there exist disjoint µ̂β open sets U and V such that µ̂βCl({x}) = {x} ⊆ U and µ̂βCl({y}) = {y} ⊆ V . This shows that (X, τ) is µ̂β R_1.

Theorem 5.14: If a topological space (X, τ) is µ̂β symmetric, then the following are equivalent:
1. (X, τ) is µ̂β T_2.
2. (X, τ) is µ̂β R_1 and µ̂β T_1.
3. (X, τ) is µ̂β R_1 and µ̂β T_0.

Proof: Straightforward.

Theorem 5.15: For a topological space (X, τ) the following statements are equivalent:
1. (X, τ) is µ̂β R_1.
2. If x, y ∈ X such that µ̂βCl({x}) ≠ µ̂βCl({y}), then there exist µ̂β closed sets F_1 and F_2 such that x ∈ F_1, y ∉ F_1, y ∈ F_2, x ∉ F_2 and X = F_1 ∪ F_2.

Proof: Obvious.

Theorem 5.16: If (X, τ) is µ̂β R_1, then (X, τ) is µ̂β R_0.

Proof: Let U be µ̂β open such that x ∈ U. If y ∉ U, since x ∉ µ̂βCl({y}), we have µ̂βCl({x}) ≠ µ̂βCl({y}). So, there exists a µ̂β open set V such that µ̂βCl({y}) ⊆ V and x ∉ V , which implies y ∉ µ̂βCl({x}). Hence µ̂βCl({x}) ⊆ U. Therefore, (X,τ) is µ̂β R_0.

Corollary 5.17: A topological space (X, τ) is µ̂β R_1 if and only if for x, y ∈ X, µ̂βker({x}) ≠ µ̂βker({y}), there exist disjoint µ̂β open sets U and V such that µ̂βCl({x}) ⊆ U and µ̂βCl({y}) ⊆ V .

Proof: Follows from Theorem 5.5.

Theorem 5.18: A topological space (X, τ) is µ̂β R_1 if and only if x ∈ X \ µ̂βCl({y}) implies that x and y have disjoint µ̂β open neighbourhoods.

Proof:
Necessity: Let x ∈ X \ µ̂βCl({y}). Then µ̂βCl({x}) ≠ µ̂βCl({y}), so, x and y have disjoint µ̂β open neighbourhoods.

Sufficiency: First, we show that (X, τ) is µ̂β R_0. Let U be a µ̂β open set and x ∈ U. Suppose that y ∉ U. Then, µ̂βCl({y}) ∩ U = ∅ and x ∉ µ̂βCl({y}). There exist µ̂β open sets U_x and U_y such that x ∈ U_x, y ∈ U_y and U_x∩U_y = ∅. Hence, µ̂βCl({x}) ⊆ µ̂βCl(U_x) and µ̂βCl({x}) ∩ U_y ⊆ µ̂βCl(U_x) ∩ U_y = ∅. Therefore, y ∉ µ̂βCl({x}). Consequently,

$\hat{\mu}\beta Cl(\{x\}) \subseteq U$ and (X, τ) is $\hat{\mu}\beta$ R_0. Next, we show that (X, τ) is $\hat{\mu}\beta$ R_1. Suppose that $\hat{\mu}\beta Cl(\{x\}) \neq \hat{\mu}\beta Cl(\{y\})$. Then, we can assume that there exists $z \in \hat{\mu}\beta Cl(\{x\})$ such that $z \notin \hat{\mu}\beta Cl(\{y\})$. There exist $\hat{\mu}\beta$ open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \emptyset$. Since $z \in \hat{\mu}\beta Cl(\{x\})$, $x \in V_z$. Since (X, τ) is $\hat{\mu}\beta$ R_0, we obtain $\hat{\mu}\beta Cl(\{x\}) \subseteq V_z$, $\hat{\mu}\beta Cl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \emptyset$. This shows that (X, τ) is $\hat{\mu}\beta$ R_1.

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