

A FIXED POINT THEOREM ON PRODUCT OF METRIC SPACES

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ABSTRACT

J. Matkowski [19], gave an important generalization of Banach contraction principle for a finite product of metric spaces. This result has been extended and generalized by several mathematicians. Recently Pant [22] gave an important concept of reciprocal continuity for a pair of maps. In this paper, we introduced the coordinatewise reciprocal continuity and proved a fixed point theorem which extend and unify the result of Jungck [13], Matkowski [op. cit.] and some of their generalization, for non continuous systems of maps.

Subject Classifications: 47H10, 54H25.

Key Words: Fixed point, coincidence point, Matkowski contraction, product space, reciprocal continuous maps, coordinatewise commuting, weakly commuting, compatible and R-weakly commuting maps.

1. INTRODUCTION

In the galaxy of contraction principles, two important generalizations of well-known Banach contraction principle were obtained by Gerald Jungck [13] and Janusz Matkowski [18]-[19]. Jungck's result being simple and elegant in nature has led to a massive growth of fixed point theorems for contractive type maps (see, [2], [3], [6]-[7], [12], [15], [24], [27]-[30], [34], [36]-[38]). Matkowski's fixed point theorem (Matkowski contraction principle) being somewhat tedious in nature could draw the attention of only a few researchers in applicable mathematics (see, [4]-[5], [8]-[11], [16], [20], [25]-[26], [31]-[33]). Singh- Gairola [31], [32] extend and unify the result of Jungck [op. cit.] and Matkowski [op. cit.] and some of their generalizations by introducing a new class of maps- coordinatewise commuting and their weaker forms (see, also [8] and [11]).

With a view to generalizing fixed point theorems for non continuous maps Pant [22], introduce reciprocal continuity. If S and T are maps on a metric space (M, d) then the pair (S, T) is said to be reciprocal continuous if and only if $\lim_{n \rightarrow \infty} STx_n = St$ and $\lim_{n \rightarrow \infty} TSx_n = Tt$ whenever $\{x_n\}$ be a sequence in M such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some t in M . Motivated by the work of Singh-Gairola [31] and Pant [22], we extend and generalize the Matkowski contraction for non continuous maps. We do this by introducing a new class of maps- coordinatewise reciprocal continuous maps.

Throughout this paper we shall follow the following notations and definitions. Let (a_{ik}) be a $n \times n$ square matrix with non-negative entries defined in Czerwik [4] and Matkowski [op.cit].

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & i \neq k \\ 1 - a_{ik} & i = k \end{cases} \quad i, k = 1, \dots, n \quad (1.1)$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1, k+1}^{(t)} + c_{i+1, 1}^{(t)} c_{1, k+1}^{(t)}, & i \neq k \\ c_{11}^{(t)} c_{i+1, k+1}^{(t)} - c_{i+1, 1}^{(t)} c_{1, k+1}^{(t)}, & i = k \end{cases} \quad (1.2)$$

$t = 1, \dots, n-1, i, k = 1, \dots, n-t.$

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Let (X_i, d_i) , $i = 1, \dots, n$, be metric spaces,

$$X = X_1 \times X_2 \times \dots \times X_n;$$

$$x = (x_1, \dots, x_n);$$

$$P_i, T_i : X \rightarrow X_i; \quad i = 1, \dots, n \text{ and}$$

$$\{x^m\} = \{(x_1^m, \dots, x_n^m)\}, m \in \mathbb{N} \text{ (set of natural numbers) be a sequence in } X.$$

Definition 1.1 [31]: Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are coordinatewise commuting at a point $x \in X$ if and only if $P_i(T_1x, \dots, T_nx) = T_i(P_1x, \dots, P_nx)$ for all $i = 1, \dots, n$. Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are coordinatewise commuting on X if and only if they are coordinatewise commuting at every point of X .

Definition 1.2 [31]: Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are coordinatewise weakly commuting at a point $x \in X$ if and only if $d_i(P_i(T_1x, \dots, T_nx), T_i(P_1x, \dots, P_nx)) \leq d_i(P_ix, T_ix)$ for all $i = 1, \dots, n$. Two systems of maps are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X .

Remark 1.1: Evidently coordinatewise commuting systems of maps are coordinatewise weakly commuting. However, the weakly commuting systems of maps need not to be commuting (see, [31], [32]).

Definition 1.3 [11]: Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are coordinatewise asymptotically commuting or, following the terminology of Jungck [15], coordinatewise compatible, if and only if

$$\lim_{m \rightarrow \infty} d_i(P_i(T_1x^m, \dots, T_nx^m), T_i(P_1x^m, \dots, P_nx^m)) = 0$$

whenever $\lim_{m \rightarrow \infty} P_ix^m = \lim_{m \rightarrow \infty} T_ix^m = u_i$ for some $u_i \in X_i$, $i = 1, \dots, n$.

Definition 1.4 [8]: Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are coordinatewise R-weakly commuting at a point $x \in X$ if and only if $d_i(P_i(T_1x, \dots, T_nx), T_i(P_1x, \dots, P_nx)) \leq Rd_i(P_ix, T_ix)$, for all $i = 1, \dots, n$ and for any positive real number R . Two systems of maps are coordinatewise R-weakly commuting on X if and only if they are coordinatewise R-weakly commuting at every point of X .

Remark 1.2: Coordinatewise weakly commuting maps are coordinatewise R-weakly commuting. However, coordinatewise R-weakly commuting maps need not to be coordinatewise weakly commuting. The following example shows the coordinatewise R-weak commutativity of two systems of maps and illustrates that the coordinatewise R-weak commutativity need not imply coordinatewise weak commutativity.

Example 1.1: Let $X_1 = X_2 = [0, 1]$ be usual metric spaces and $P_i, T_i : X_1 \times X_2 \rightarrow X_i$, $i = 1, 2$ such that

$$P_1x = x_1^2, \quad P_2x = 0,$$

$$T_1x = 2x_1 - 1, \quad T_2x = 0.$$

Since

$$d_1(T_1(P_1x, P_2x), P_1(T_1x, T_2x)) = d_1(2x_1^2 - 1, (2x_1 - 1)^2) = 2(x_1 - 1)^2 = 2d_1(2x_1 - 1, x_1^2) = 2d_1(T_1x, P_1x)$$

and

$$d_2(T_2(P_1x, P_2x), P_2(T_1x, T_2x)) = d_2(0, 0) = 0 \leq d_2(T_2x, P_2x).$$

Then two systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise R-weakly commuting. However, they are not coordinatewise weakly commuting for

$$d_1(T_1(P_1x, P_2x), P_1(T_1x, T_2x)) = d_1(2x_1^2 - 1, (2x_1 - 1)^2) = 2(x_1 - 1)^2 > d_1(T_1x, P_1x), \forall x \in X.$$

Definition 1.5: Two systems of maps $\{P_1, \dots, P_n\}$ and $\{T_1, \dots, T_n\}$ are said to be coordinatewise reciprocal continuous if and only if $\lim_{m \rightarrow \infty} P_i(T_1x^m, \dots, T_nx^m) = P_iz$ and $\lim_{m \rightarrow \infty} T_i(P_1x^m, \dots, P_nx^m) = T_iz$, whenever there exist a sequence $\{x^m\}$ in X such that $\lim_{m \rightarrow \infty} P_ix^m = \lim_{m \rightarrow \infty} T_ix^m = z_i$ for all $i = 1, \dots, n$.

If each member of systems of maps is continuous then systems of maps are coordinatewise reciprocal continuous but the converse need not be true.

Remark 1.3: Notice that definitions above with $n = 1$ are standard ones for commuting, weakly commuting (see [15] and [28]), asymptotically commuting (see, [37]) (also called compatible [14]), R-weakly commuting (see [21]) and reciprocal continuous maps ([22] and see also [23]).

Remark 1.4: Asymptotically commuting (or compatible) class of maps includes commuting and weakly commuting maps. Commuting maps are necessarily weakly and asymptotically commuting both (see, for instance, [14], [28], [31], [37]).

Remark 1.5: The commutativity, weak commutativity and asymptotic commutativity (or compatibility) are equivalent at the point of coincidence of two (or two systems of) maps (see, [1], [14]).

The following example illustrates the coordinatewise reciprocal continuity of systems of maps and shows that coordinatewise reciprocal continuity of systems of maps does not imply continuity of any member of systems of maps.

Example 1.2: Let $X_1 = [0, 1]$, $X_2 = [0, 1]$ be metric spaces with usual metrics and mappings $P_i, T_i : X_1 \times X_2 \rightarrow X_i$, for $i = 1, 2$ such that

$$P_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0 \\ \frac{1}{2} & \text{if } x_1 > 0 \end{cases} \quad P_2(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 = 0 \\ \frac{1}{2} & \text{if } x_2 > 0 \end{cases}$$

$$T_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0 \\ 1 & \text{if } x_1 > 0 \end{cases} \quad T_2(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 = 0 \\ 1 & \text{if } x_2 > 0 \end{cases}.$$

Suppose $\{x^m\}$ be a sequence in $X_1 \times X_2$ such that $P_i x^m \rightarrow z_i$ and $T_i x^m \rightarrow z_i$ for some z_i , $i = 1, 2$, as $m \rightarrow \infty$. Then for $z = (0, 0)$ and $\{x^m\} = \{(0, 0)\} \in X$ for each m , $P_i(T_1 x^m, T_2 x^m) \rightarrow 0 = P_i z$ and $T_i(P_1 x^m, P_2 x^m) \rightarrow 0 = T_i z$ as $m \rightarrow \infty$. Hence systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise reciprocal continuous at $z = (0, 0)$ but they are not continuous at $z = (0, 0)$. To see this, let $\{x^m\} = \left\{\left(\frac{1}{m}, \frac{1}{m}\right)\right\}$ be a sequence in $X_1 \times X_2$. Since $\{x^m\} \rightarrow (0, 0)$ as $m \rightarrow \infty$ but then $P_i x^m \rightarrow \frac{1}{2} \neq P_i(0, 0)$ and $T_i x^m \rightarrow 1 \neq T_i(0, 0)$ for $i = 1, 2$, as $m \rightarrow \infty$.

The following Lemma is due to Matkowski [19] (see also [4], [33]).

Lemma 1.1: Let $c_{i,k}^{(0)} \geq 0$, $i, k = 1, \dots, n$, $n \geq 2$, then the system of inequalities

$$\sum_{k=1}^n a_{ik} r_k < r_i, \quad i = 1, \dots, n \quad (1.3)$$

has a positive solution r_1, \dots, r_n if and only if the following inequalities hold:

$$c_{ii}^{(t)} > 0, \quad i = 1, \dots, n-t; \quad t = 0, 1, \dots, n-1, \quad n \geq 2. \quad (1.4)$$

Moreover, there exists a positive number $h < 1$ such that

$$\sum_{k=1}^n a_{ik} r_k \leq h r_i, \quad i = 1, \dots, n, \quad (1.5)$$

for some positive number r_1, \dots, r_n . Indeed such an h may be found by

$$h = \max_i \left\{ r_i^{-1} \sum_{k=1}^n a_{ik} r_k \right\}. \quad (1.6)$$

Now we will state our main results.

2. RESULTS

Theorem 2.1: Let (X_i, d_i) , $i = 1, \dots, n$, be complete metric spaces and $P_i, Q_i, S_i, T_i : X \rightarrow X_i$, for all $i = 1, \dots, n$, be such that

$$P_i(X) \subset T_i(X), \quad Q_i(X) \subset S_i(X), \quad i = 1, \dots, n. \quad (2.1)$$

The system (P_1, \dots, P_n) coordinatewise R-weakly commutes with the system (S_1, \dots, S_n) and the system (Q_1, \dots, Q_n)

coordinatewise R-weakly commutes with the system (T_1, \dots, T_n) . (2.2)

The system of maps (P_1, \dots, P_n) coordinatewise reciprocal continuous with the system (S_1, \dots, S_n) or the system of maps

(Q_1, \dots, Q_n) coordinatewise reciprocal continuous with the system (T_1, \dots, T_n) . (2.3)

If there exist non- negative numbers b and a_{ik} , $i, k = 1, \dots, n$ such that (1.1), (1.2), (1.4) and the following hold:

$$0 \leq b < 1 - h, \text{ where } h \text{ is defined in (1.6) and} \quad (2.4)$$

$$d_i(P_i x, Q_i y) \leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k x, T_k y), b \max \left\{ \frac{d_i(S_i x, P_i x), d_i(T_i y, Q_i y),}{2} \right\} \right\} \quad (2.5)$$

for all $x, y \in X$, then the system of equations

$$P_i x = Q_i x = x_i = S_i x = T_i x \quad (2.6)$$

has a unique common solution (x_1, \dots, x_n) such that $x_i \in X_i$, $i = 1, \dots, n$.

Proof: First we note that in view of the system of inequality (1.5), h defined in (1.6) exists and $0 < h < 1$ (c.f. Czerwik [4]). From the Lemma 1.1 and (1.6) we may choose positive numbers r_1, \dots, r_n such that

$$\sum_{k=1}^n a_{ik} r_k \leq h r_i, \quad i = 1, \dots, n.$$

Pick $x_i^0 \in X_i$, $i = 1, \dots, n$. We in view of (2.1), construct sequences $\{x_i^m\}$ and $\{y_i^m\}$ in X_i such that

$$\begin{aligned} y_i^{2m+1} &= P_i x^{2m} = T_i x^{2m+1} \\ y_i^{2m+2} &= Q_i x^{2m+1} = S_i x^{2m+2}, \quad m = 0, 1, 2, \dots \end{aligned}$$

We may assume that $d_i(y_i^2, y_i^1) \leq r_i$, $i = 1, \dots, n$. From condition (2.5), we have

$$\begin{aligned} d_i(y_i^3, y_i^2) &= d_i(P_i x^2, Q_i x^1) \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k x^2, T_k x^1), b \max \left\{ \frac{d_i(S_i x^2, P_i x^2), d_i(T_i x^1, Q_i x^1),}{2} \right\} \right\} \\ &= \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^1), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^1, y_i^2), \frac{d_i(y_i^2, y_i^2) + d_i(y_i^1, y_i^3)}{2} \right\} \right\} \\ &= \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^1), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^1, y_i^2), \frac{d_i(y_i^1, y_i^3)}{2} \right\} \right\}, \end{aligned}$$

Since $\frac{d_i(y_i^1, y_i^3)}{2} \leq \frac{d_i(y_i^1, y_i^2) + d_i(y_i^2, y_i^3)}{2}$ implies that

$$d_i(y_i^3, y_i^2) \leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^1), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^1, y_i^2) \right\} \right\}.$$

If $d_i(y_i^2, y_i^3) > d_i(y_i^1, y_i^2)$, then

$$\begin{aligned} d_i(y_i^3, y_i^2) &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^1), b d_i(y_i^2, y_i^3) \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} r_k, b d_i(y_i^2, y_i^3) \right\} \leq \max \left\{ h r_i, b d_i(y_i^2, y_i^3) \right\} \leq h r_i. \end{aligned}$$

Since otherwise, we get a contradiction.

If $d_i(y_i^2, y_i^3) < d_i(y_i^1, y_i^2)$, then

$$\begin{aligned} d_i(y_i^3, y_i^2) &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^1), b d_i(y_i^1, y_i^2) \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} r_k, b d_i(y_i^1, y_i^2) \right\}, \\ &\leq \max \left\{ h r_i, b r_i \right\} = c r_i, \quad i = 1, \dots, n, \text{ where } c = \max\{h, b\}. \end{aligned}$$

Again from condition (2.5), we get

$$\begin{aligned} d_i(y_i^4, y_i^3) &= d_i(P_i x^2, Q_i x^3) \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k x^2, T_k x^3), b \max \left\{ d_i(S_i x^2, P_i x^2), d_i(T_i x^3, Q_i x^3), \frac{d_i(S_i x^2, Q_i x^3) + d_i(T_i x^3, P_i x^2)}{2} \right\} \right\} \\ &= \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^3), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4), \frac{d_i(y_i^2, y_i^4) + d_i(y_i^3, y_i^3)}{2} \right\} \right\} \\ &= \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^3), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4), \frac{d_i(y_i^2, y_i^4)}{2} \right\} \right\}. \end{aligned}$$

Since $\frac{d_i(y_i^2, y_i^4)}{2} \leq \frac{d_i(y_i^2, y_i^3) + d_i(y_i^3, y_i^4)}{2}$ implies that

$$\begin{aligned} d_i(y_i^4, y_i^3) &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(y_k^2, y_k^3), b \max \left\{ d_i(y_i^2, y_i^3), d_i(y_i^3, y_i^4) \right\} \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} c r_k, b \max \left\{ d_i(y_i^3, y_i^4), c r_i \right\} \right\} \end{aligned}$$

and arguing same as before this implies

$$\begin{aligned} d_i(y_i^4, y_i^3) &\leq \max \left\{ \sum_{k=1}^n a_{ik} c r_k, b c r_i \right\} \leq \max \{ h c r_i, b c r_i \} \\ &= c r_i \max \{ h, b \} = c^2 r_i, \text{ where } c = \max \{ h, b \}. \end{aligned}$$

Inductively

$$d_i(y_i^{m+2}, y_i^{m+1}) \leq c^m r_i, \quad m = 0, 1, 2, \dots$$

Since $0 \leq c < 1$, hence $\{y_i^m\}_{m=1}^\infty$ is a Cauchy sequence in X_i for all $i = 1, \dots, n$. As X_i is a complete metric space, there exists a point u_i (say) in X_i such that $\{y_i^m\} \rightarrow u_i$ as $m \rightarrow \infty$. Moreover $y_i^{2m+1} = P_i x^{2m} = T_i x^{2m+1} \rightarrow u_i$ and $y_i^{2m+2} = Q_i x^{2m+1} = S_i x^{2m+2} \rightarrow u_i$ as $m \rightarrow \infty$.

If systems of maps (P_1, \dots, P_n) and (S_1, \dots, S_n) are coordinatewise reciprocal continuous then

$$\lim_{m \rightarrow \infty} P_i(S_1 x^{2m}, \dots, S_n x^{2m}) = P_i u \text{ and } \lim_{m \rightarrow \infty} S_i(P_1 x^{2m}, \dots, P_n x^{2m}) = S_i u, \quad i = 1, \dots, n. \quad (2.7)$$

Now, coordinatewise R- weak commutativity of systems of maps (P_1, \dots, P_n) and (S_1, \dots, S_n) yields

$$d_i(P_i(S_1 x^{2m}, \dots, S_n x^{2m}), S_i(P_1 x^{2m}, \dots, P_n x^{2m})) \leq R d_i(P_i x^{2m}, S_i x^{2m}).$$

Taking $\lim m \rightarrow \infty$ and using condition (2.7), we have

$$d_i(P_i u, S_i u) = 0, \quad i = 1, \dots, n$$

and therefore

$$P_i u = S_i u, \quad i = 1, \dots, n. \quad (2.8)$$

Since $P_i(X) \subset T_i(X)$, so there exists a point $w = (w_1, \dots, w_n) \in X$ such that $P_i u = T_i w$ for $i = 1, \dots, n$.

Now from condition (2.5),

$$d_i(P_i u, Q_i w) \leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k u, T_k w), b \max \left\{ d_i(S_i u, P_i u), d_i(T_i w, Q_i w), \frac{d_i(S_i u, Q_i w) + d_i(T_i w, P_i u)}{2} \right\} \right\}.$$

Using condition (2.8), we get

$$\begin{aligned} d_i(P_i u, Q_i w) &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(P_k u, T_k w), b \max \left\{ d_i(P_i u, P_i u), d_i(P_i u, Q_i w), \frac{d_i(P_i u, Q_i w) + d_i(T_i w, P_i u)}{2} \right\} \right\} \\ &\leq b d_i(P_i u, Q_i w). \end{aligned}$$

So,

$$P_i u = Q_i w, \quad i = 1, \dots, n.$$

By the above we then have

$$P_i u = S_i u = T_i w = Q_i w, \quad \text{for } i = 1, \dots, n. \quad (2.9)$$

Coordinatewise R-weak commutativity of systems of maps (P_1, \dots, P_n) and (S_1, \dots, S_n) implies that there exists $R > 0$ such that

$$d_i(P_i(S_1 u, \dots, S_n u), S_i(P_1 u, \dots, P_n u)) \leq R d_i(P_i u, S_i u).$$

Using condition (2.9), we get

$$d_i(P_i(S_1 u, \dots, S_n u), S_i(P_1 u, \dots, P_n u)) = 0$$

and therefore

$$P_i(S_1 u, \dots, S_n u) = S_i(P_1 u, \dots, P_n u), \quad i = 1, \dots, n.$$

Again using condition (2.9), we have

$$P_i(S_1 u, \dots, S_n u) = P_i(P_1 u, \dots, P_n u) = S_i(P_1 u, \dots, P_n u) = S_i(S_1 u, \dots, S_n u), \quad i = 1, \dots, n.$$

Similarly, R-weak commutativity of systems of maps (Q_1, \dots, Q_n) and (T_1, \dots, T_n) implies

$$Q_i(T_1 w, \dots, T_n w) = Q_i(Q_1 w, \dots, Q_n w) = T_i(Q_1 w, \dots, Q_n w) = T_i(T_1 w, \dots, T_n w), \quad i = 1, \dots, n.$$

Now from condition (2.5) and assuming

$$d_i(P_i(P_1 u, \dots, P_n u), P_i u) \leq r_i, \quad i = 1, \dots, n.$$

We have

$$\begin{aligned} d_i(P_i(P_1 u, \dots, P_n u), P_i u) &= d_i(P_i(P_1 u, \dots, P_n u), Q_i w) \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k(P_1 u, \dots, P_n u), T_k w), \right. \\ &\quad \left. b \max \left\{ d_i(S_i(P_1 u, \dots, P_n u), P_i(P_1 u, \dots, P_n u)), d_i(T_i w, Q_i w), \right. \right. \\ &\quad \left. \left. \frac{d_i(S_i(P_1 u, \dots, P_n u), Q_i w) + d_i(T_i w, P_i(P_1 u, \dots, P_n u))}{2} \right\} \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(P_k(P_1 u, \dots, P_n u), P_k u), \right. \\ &\quad \left. b \max \left\{ d_i(P_i(P_1 u, \dots, P_n u), P_i(P_1 u, \dots, P_n u)), d_i(T_i w, Q_i w), \right. \right. \\ &\quad \left. \left. \frac{d_i(P_i(P_1 u, \dots, P_n u), P_i u) + d_i(P_i u, P_i(P_1 u, \dots, P_n u))}{2} \right\} \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} r_k, b d_i(P_i(P_1 u, \dots, P_n u), P_i u) \right\} \\ &\leq \max \{ h r_i, b r_i \} \leq c r_i, \quad \text{where } c = \max\{h, b\}. \end{aligned}$$

Inductively, we can prove that

$$d_i(P_i(P_1 u, \dots, P_n u), P_i u) \leq c^m r_i, \quad m = 0, 1, 2, \dots$$

and consequently

$$P_i(P_1 u, \dots, P_n u) = P_i u, \quad i = 1, \dots, n.$$

By the above we then have

$$P_i(P_1 u, \dots, P_n u) = S_i(P_1 u, \dots, P_n u) = P_i u. \quad (2.10)$$

Similarly, by using the condition (2.5), we can obtain

$$Q_i(Q_1 w, \dots, Q_n w) = T_i(Q_1 w, \dots, Q_n w) = Q_i w. \quad (2.11)$$

Since

$$P_i u = Q_i w = v_i \text{ (say) for } i = 1, \dots, n, \quad (2.12)$$

then condition (2.10), (2.11) and (2.12) together implies that

$$P_i v = Q_i v = v_i = S_i v = T_i v, \quad i = 1, \dots, n.$$

This proves that the system of equation (2.6) has a common solution when systems of maps (P_1, \dots, P_n) and (S_1, \dots, S_n) are coordinatewise reciprocal continuous. In case, when systems of maps (Q_1, \dots, Q_n) and (T_1, \dots, T_n) are coordinatewise reciprocal continuous, the proof may be accomplished in an analogous manner.

Now, to prove the uniqueness of the solution, we assume that $v_i, \bar{v}_i \in X_i$ such that $v_i \neq \bar{v}_i$ and $P_i \bar{v} = Q_i \bar{v} = S_i \bar{v} = T_i \bar{v} = \bar{v}$. We can assume that

$$d_i(v_i, \bar{v}_i) \leq r_i, \quad i = 1, \dots, n.$$

From condition (2.5), we get

$$\begin{aligned} d_i(v_i, \bar{v}_i) &= d_i(P_i v, Q_i \bar{v}) \leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(S_k v, T_k \bar{v}), b \max \left\{ d_i(S_i v, P_i v), d_i(T_i \bar{v}, Q_i \bar{v}), \frac{d_i(S_i v, Q_i \bar{v}) + d_i(T_i \bar{v}, P_i v)}{2} \right\} \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(v_i, \bar{v}_i), b \max \left\{ d_i(v_i, \bar{v}_i), \frac{d_i(v_i, \bar{v}_i) + d_i(\bar{v}_i, v_i)}{2} \right\} \right\} \\ &\leq \max \left\{ \sum_{k=1}^n a_{ik} r_k, b d_i(v_i, \bar{v}_i) \right\} = \max \{ h r_i, b r_i \} = c r_i, \text{ where } c = \max \{ h, b \}. \end{aligned}$$

Inductively

$$d_i(v_i, \bar{v}_i) \leq c^m r_i, \quad m = 0, 1, 2, \dots$$

and consequently

$$v_i = \bar{v}_i, \quad i = 1, \dots, n.$$

This completes the proof.

Corollary 2.1: Let $(X_i, d_i), i = 1, 2, \dots, n$, be complete metric spaces and $P_i, T_i : X \rightarrow X_i, i = 1, \dots, n$, be such that

$$P_i(X) \subset T_i(X), \quad i = 1, \dots, n; \quad (2.13)$$

Systems of maps (P_1, \dots, P_n) and (T_1, \dots, T_n) are coordinatewise R- weakly commuting; (2.14)

Systems of maps (P_1, \dots, P_n) and (T_1, \dots, T_n) are coordinatewise reciprocal continuous; (2.15)

If there exist non- negative numbers b and $a_{ik}, i, k = 1, \dots, n$ such that (1.1), (1.2), (1.4) and the following hold:

$$0 \leq b < 1 - h, \text{ where } h \text{ is defined in (1.6) and} \quad (2.16)$$

$$d_i(P_i x, P_i y) \leq \max \left\{ \sum_{k=1}^n a_{ik} d_k(T_k x, T_k y), b \max \left\{ d_i(P_i x, T_i x), d_i(P_i y, T_i y), \frac{d_i(P_i x, T_i y) + d_i(P_i y, T_i x)}{2} \right\} \right\} \quad (2.17)$$

for all $x, y \in X$, then the system of equations

$$P_i x = T_i x = x_i, \quad (2.18)$$

has a unique common solution (x_1, \dots, x_n) such that $x_i \in X_i, i = 1, \dots, n$.

Proof: Proof may be completed by putting $P_i = Q_i$ and $S_i = T_i, i = 1, \dots, n$, in the proof of Theorem 2.1.

Example 2.1: Let $X_1 = X_2 = [2, 20]$ be usual metric spaces and $P_i, T_i : X_1 \times X_2 \rightarrow X_i, i = 1, 2$, such that

$$\begin{aligned} P_1(x_1, x_2) &= \begin{cases} 2 & x_1 = 2 \\ 6 & 2 < x_1 \leq 6, \\ 2 & x_1 > 6 \end{cases} & P_2(x_1, x_2) &= \begin{cases} 2 & x_2 = 2 \\ 6 & 2 < x_2 \leq 6, \\ 2 & x_2 > 6 \end{cases} \\ T_1(x_1, x_2) &= \begin{cases} 2 & x_1 = 2 \\ 12 & 2 < x_1 \leq 6, \\ x_1 - 3 & x_1 > 6 \end{cases} & T_2(x_1, x_2) &= \begin{cases} 2 & x_2 = 2 \\ 12 & 2 < x_2 \leq 6, \\ x_2 - 3 & x_2 > 6 \end{cases} \end{aligned}$$

for all $(x_1, x_2) \in X_1 \times X_2$. This example satisfy all the conditions of our Corollary 2.1 and have a unique common solution at $x = (2, 2)$. It is notable that two systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are coordinatewise reciprocal continuous but none of P_1, P_2, T_1, T_2 is continuous, not even at the point $x = (2, 2)$.

Remark 2.1: If we assume $(M, d) = (X_i, d_i)$, $P = P_i$, $Q = Q_i$, $S = S_i$, $T = T_i$, $i = 1, \dots, n$, and $n = 1$ then (2.5) with $a_{11} = k$ may be written as:

$$P, Q, S, T : M \rightarrow M, 0 < k < 1,$$

$$(*) \quad d(Px, Qy) \leq \beta \max \left\{ d(Sx, Ty), d(Sx, Px), d(Ty, Qy), \frac{d(Sx, Qy) + d(Ty, Px)}{2} \right\}, x, y \in X,$$

where $0 < \beta = \max \{k, b\} < 1$.

Remark 2.2: A multitude of fixed point theorems generalizing the Jungck contraction principle have been establish under the condition (*) and its particular cases (see for [2], [3], [7], [12], [15], [17], [27], [29]-[30], [34]-[38]). It may also be mentioned that Kubiak main result proved exactly under the condition (*) uses the commutativity of P with S and that of Q with T .

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