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COMMON COUPLED FIXED POINT THEOREMS FOR FOUR MAPPINGS IN DISLOCATED QUASI b-METRIC SPACES

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ABSTRACT

In this paper, we prove two common coupled fixed point theorems for four mappings in dislocated quasi b-metric spaces and provide two examples to support our theorems. Our results generalize some existing results in the literature.

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Keywords: *Dislocated quasi b-metric, coupled fixed points, w-compatible pair of maps, Cauchy sequence.*

1. INTRODUCTION

Hitzler [7] and Hitzler and Seda [6] introduced the notion of dislocated metric spaces and generalized the celebrated Banach contraction principle in such spaces.

Zeyada *et.al* [15] initiated the concept of dislocated quasi metric spaces and generalized the results of Hitzler and Seda [6] in dislocated quasi metric spaces.

The notion of b-metric space was introduced by Czerwic [3] in connection with some problems concerning with the convergence of non measurable functions with respect to measure.

Recently Klin-eam and Suanoom [8] introduced the concept of dislocated quasi b-metric spaces and which generalize b-metric spaces [3] and quasi b-metric spaces [13] and proved some fixed point theorems in it by using cyclic contractions.

The authors [1,5,8,10,11,12,14] etc. obtained fixed, common fixed points and common coupled fixed point theorems in dislocated quasi b-metric spaces using various contraction conditions for single and two maps.

In this note, we prove two common coupled fixed point theorems for four maps in dislocated quasi b-metric spaces and we also give examples to support our theorems.

Bhaskar and Lakshmi kantham [4] developed some coupled fixed point theorems in partially ordered metric spaces. Lakshmi kantham and Ciric [9] defined common coupled fixed points for a pair of mappings. Abbas *et al.* [2] introduced w-compatible mappings and proved some common coupled fixed point theorems in cone metric spaces.

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First we recall some known definitions and lemmas.

Definition 1.1: let X be a non-empty set, $s \geq 1$ (a fixed real number) and $d: X \times X \to [0, \infty)$ be a function. Consider the following condition on d.
(1.1.1) $d(x,x) = 0, \forall x \epsilon X$
(1.1.2) $d(x,y) = d(y,x) = 0 \Rightarrow x = y, \forall x, y \,\epsilon X$
(1.1.3) $d(x,y) = d(y,x), \forall x, y \epsilon X$
(1.1.4) $d(x,y) \leq d(x,z) + d(z,y), \forall x, y, z \,\epsilon X$
(1.1.5) $d(x,y) \leq s[d(x,z) + d(z,y)], \forall x, y, z \,\epsilon X$

(i) If d satisfies (1.1.2), (1.1.3) and (1.1.4) then d is called a dislocated metric and (X, d) is called a dislocated metric space.
(ii) If d satisfies (1.1.1), (1.1.2) and (1.1.4) then d is called a quasi metric and (X, d) is called a quasi metric space.
(iii) If d satisfies (1.1.2) and (1.1.4) then d is called a dislocated quasi metric or dq-metric and (X, d) is called a dislocated quasi metric space.
(iv) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.4) then d is called a metric and (X, d) is called a metric space.
(v) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.5) then d is called a b-metric and (X, d) is called a b-metric space.
(vi) If d satisfies (1.1.2) and (1.1.5) then d is called a dislocated quasi b-metric and (X, d) is called a dislocated quasi b-metric space or dq b-metric space.

Definition 1.2: Let (X, d) be a dq b-metric space. A sequence $\{x_n\}$ in (X, d) is said to be
(i) dq b-convergent if there exists some point $x \epsilon X$ such that $\lim_{n\to\infty.} d(x_n, x) = 0 = \lim_{n\to\infty.} d(x, x_n)$. In this case x is called a dq b-limit of $\{x_n\}$ and we write $x_n \to x \text{ as } n \to \infty$.
(ii) Cauchy sequence if $\lim_{n,m\to\infty.} d(x_n, x_m) = 0 = \lim_{n,m\to\infty.} d(x_m, x_n)$.

The space (X, d) is called complete if every Cauchy sequence in X is dq b-convergent.

One can prove easily the following Lemma.

Lemma 1.3: Let (X, d) be a dq b-metric space and $\{x_n\}$ be dq b-convergent to x in X and $y \epsilon X$ be arbitrary.Then

$$\frac{1}{s} d(x,y) \leq \lim_{n\to\infty} inf\, d(x_n, y) \leq \lim_{n\to\infty.} \sup d(x_n, y) \leq s d(x,y)$$

$$\frac{1}{s} d(y,x) \leq \lim_{n\to\infty} \inf d(y, x_n) \leq lim_{n\to\infty.} \sup d(y, x_n) \leq s d(y,x).$$

Definition 1.4([4]): Let X be a non-empty set .An element $(x,y) \epsilon X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X \text{ if } x = F(x,y) \text{ and } y = F(y,x)$.

Definition 1.5: Let X be anon-empty set and $F : X \times X \to X, f: X \to X$ be mappings .
(i) ([9]). An element $(x,y) \epsilon X \times X$ is called a coupled coincidence point of F and f if fx = F(x, y) and fy = F(y,x).
(ii) ([9]). An element $(x,y) \epsilon X \times X$ is called a common coupled fixed point of F and f if x = fx = F(x, y) and y = fy = F(y,x).
(iii) ([2]). The pair (F, f) is called w-compatible if $f\big(F(x,y)\big) = F(fx, fy) \text{ and } f\big(F(y,x)\big) = F(fy, fx)$ whenever there exist $x, y \,\epsilon X \text{ with } fx = F(x,y) \text{ and } fy = F(y,x)$.

2. MAIN RESULT

Before proving our main theorems, we state the following

Definition 2.1: For the integer $s \geq 1$, let $\mathbf{\Phi}_s$ denote the set of all functions $\varphi: [0, \infty) \to [0, \infty)$ satisfying the following
(i) φ is monotonically non-decreasing,
(ii) $\sum_{n=1}^{\infty} s^n \varphi^n(t) < \infty$ for all $t > 0$, (iii) $\varphi(t) < t$ for t > 0.

From (i) and (iii), it is clear that $\varphi(0) = 0$.

Theorem 2.2: Let (X, d) be a complete dislocated quasi b- metric space with fixed integer $s \geq 1$ and F, G : $X \times X \to X$ and $S, T: X \to X$ be continuous mappings satisfying
(2.2.1) $d\big(F(x,y), G(u,v)\big) \leq \varphi(max\{\, d(Sx, Tu), d(Sy, Tv)\})$ for all $x, y, u, v \in X, where\ \varphi \in \mathbf{\Phi}_s$,
(2.2.2) $d\big(G(x,y), F(u,v)\big) \leq \varphi(max\{d(Tx, Su), d(Ty, Sv)\})$ for all $x, y, u, v \in X, where\ \varphi \in \mathbf{\Phi}_s$,
(2.2.3) $F(X \times X) \subseteq T(X), G(X \times X) \subseteq S(X)$,
(2.2.4) $FS = SF \text{ and } GT = TG$.
Then F, G, S and T have a unique common coupled fixed point in $X \times X$.

Proof: Let $(x_0, y_0) \epsilon X \times X$.

From (2.2.3), there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that

$$\mathrm{F}(x_{2n}, y_{2n}) = Tx_{2n+1} = z_{2n},$$
$$\mathrm{F}(y_{2n}, x_{2n}) = Ty_{2n+1} = w_{2n},$$
$$\mathrm{G}(x_{2n+1}, y_{2n+1}) = Sx_{2n+2} = z_{2n+1},$$
$$\mathrm{G}(y_{2n+1}, x_{2n+1}) = Sy_{2n+2} = w_{2n+1}\ , n = 0,1,2, \dots$$

Case-(i): Suppose max $\{d(z_{2n}, z_{2n-1}), d(z_{2n-1}, z_{2n}), d(w_{2n}, w_{2n-1}), d(w_{2n-1}, w_{2n})\} = 0$ for some n. Then $z_{2n-1} = z_{2n}$ and $w_{2n-1} = w_{2n}$ from (1.1.2). Now from (2.2.1),

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= d\big(\mathrm{F}(x_{2n}, y_{2n}), \mathrm{G}(x_{2n+1}, y_{2n+1})\big) \\ &\le \varphi(max\{d(z_{2n-1}, z_{2n}), d(w_{2n-1}, w_{2n})\}). \end{aligned} \quad (1)$$

From (2.2.2) we have

$$\begin{aligned} d(z_{2n+1}, z_{2n}) &= d\big(\mathrm{G}(x_{2n+1}, y_{2n+1}), \mathrm{F}(x_{2n}, y_{2n})\big) \\ &\le \varphi(max\{d(z_{2n}, z_{2n-1}), d(w_{2n}, w_{2n-1})\}). \end{aligned} \quad (2)$$

From (2.2.1) and (2.2.2), we have

$$\begin{aligned} d(w_{2n}, w_{2n+1}) &= d\big(\mathrm{F}(y_{2n}, x_{2n}), \mathrm{G}(y_{2n+1}, x_{2n+1})\big) \\ &\le \varphi(max\{d(w_{2n-1}, w_{2n}), d(z_{2n-1}, z_{2n})\}). \end{aligned} \quad (3)$$

and

$$\begin{aligned} d(w_{2n+1}, w_{2n}) &= d\big(\mathrm{G}(y_{2n+1}, x_{2n+1}), \mathrm{F}(y_{2n}, x_{2n})\big) \\ &\le \varphi(max\{d(w_{2n}, w_{2n-1}), d(z_{2n}, z_{2n-1})\}). \end{aligned} \quad (4)$$

Since φ is monotonically non-decreasing, we have

$$\begin{aligned} max\left\{\begin{matrix} d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n}), \\ d(w_{2n}, w_{2n+1}), d(w_{2n+1}, w_{2n}) \end{matrix}\right\} &\le \varphi\left(max\left\{\begin{matrix} d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n-1}), \\ d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n-1}) \end{matrix}\right\}\right) \\ &= \varphi(0) = 0. \end{aligned} \quad (5)$$

Thus $z_{2n} = z_{2n+1} and\ w_{2n} = w_{2n+1}$ from (1.1.2).

Continuing in this way, we have $z_{2n-1} = z_{2n} = z_{2n+1} = \cdots$ and $w_{2n-1} = w_{2n} = w_{2n+1} = \cdots$

Hence $\{z_n\}$ and $\{w_n\}$ are constant Cauchy sequences in X.

Case-(ii): Suppose max $\{d(z_{n-1}, z_n), d(z_n, z_{n-1}) d(w_{n-1}, w_n), d(w_n, w_{n-1})\} \neq 0$ for n=1, 2, 3,…..As in

Case(i), we have from (5) that

$$max\left\{\begin{matrix} d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n}), \\ d(w_{2n}, w_{2n+1}), d(w_{2n+1}, w_{2n}) \end{matrix}\right\} \le \varphi\left(max\left\{\begin{matrix} d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n-1}), \\ d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n-1}) \end{matrix}\right\}\right)$$

This is true for n=1, 2, 3,…

Hence using the monotonically non-decreasing property of φ, we get

$$\begin{aligned} max\left\{\begin{matrix} d(z_n, z_{n+1}), d(z_{n+1}, z_n), \\ d(w_n, w_{n+1}), d(w_{n+1}, w_n) \end{matrix}\right\} &\le \varphi\left(max\left\{\begin{matrix} d(z_{n-1}, z_n), d(z_n, z_{n-1}), \\ d(w_{n-1}, w_n), d(w_n, w_{n-1}) \end{matrix}\right\}\right) \\ &\le \varphi^2\left(max\left\{\begin{matrix} d(z_{n-2}, z_{n-1}), d(z_{n-1}, z_{n-2}), \\ d(w_{n-2}, w_{n-1}), d(w_{n-1}, w_{n-2}) \end{matrix}\right\}\right) \\ &\quad \dots\dots\dots \\ &\le \varphi^n\left(max\left\{\begin{matrix} d(z_0, z_1), d(z_1, z_0), \\ d(w_0, w_1), d(w_1, w_0) \end{matrix}\right\}\right) \end{aligned} \quad (6)$$

Now for all positive integers n and p, consider, using (6),

$$\begin{aligned} d(z_n, z_{n+p}) &\le s\ d(z_n, z_{n+1}) + s^2 d(z_{n+1}, z_{n+2}) + \cdots + s^p d(z_{n+p-1}, z_{n+p}) \\ &\le s\ \varphi^n(t) + s^2\varphi^{n+1}(t) + \cdots + s^p\varphi^{n+p-1}(t), \text{ where } t = max\left\{\begin{matrix} d(z_0, z_1), d(z_1, z_0), \\ d(w_0, w_1), d(w_1, w_0) \end{matrix}\right\} \\ &\le s^n\ \varphi^n(t) + s^{n+1}\varphi^{n+1}(t) + \cdots + s^{n+p-1}\varphi^{n+p-1}(t), \text{ since } s \ge 1 \\ &= \textstyle\sum_{i=n}^{n+p-1} s^i\ \varphi^i(t) \le \sum_{i=n}^{\infty} s^i\ \varphi^i(t). \end{aligned}$$

Since $\sum_{i=1}^{\infty} s^i\ \varphi^i(t)$ converges for all $t > 0$, its remainder after n terms tends to zero as n $\to\infty$.

Hence, we have $\lim_{n\to\infty} d(z_n, z_{n+p}) = 0$. Also using (6), we have

$$\begin{aligned} d(z_{n+p}, z_n) &\le s\, d(z_{n+p}, z_{n+1}) + s\, d(z_{n+1}, z_n) \\ &\le s^2\, d(z_{n+p}, z_{n+2}) + s^2\, d(z_{n+2}, z_{n+1}) + s\, d(z_{n+1}, z_n) \\ &\le s^3\, d(z_{n+p}, z_{n+3}) + s^3\, d(z_{n+3}, z_{n+2}) + s^2\, d(z_{n+2}, z_{n+1}) + s\, d(z_{n+1}, z_n) \\ &\ldots\ldots\ldots \\ &\le s^{p-1}\, d(z_{n+p}, z_{n+p-1}) + s^{p-1}\, d(z_{n+p-1}, z_{n+p-2}) + \cdots . + s^2\, d(z_{n+2}, z_{n+1}) + s\, d(z_{n+1}, z_n) \\ &\le s^{p-1}\, \varphi^{n+p-1}(t) + s^{p-1}\varphi^{n+p-2}(t) + \cdots . + s^2\, \varphi^{n+1}(t) + s\, \varphi^n(t), \text{ where t is as in above} \\ &\le s^{n+p-1}\, \varphi^{n+p-1}(t) + s^{n+p-2}\varphi^{n+p-2}(t) + \cdots . + s^{n+1}\, \varphi^{n+1}(t) + s^n\, \varphi^n(t), \text{ since } s \ge 1 \\ &= \textstyle\sum_{i=n}^{n+p-1} s^i\, \varphi^i(t) \le \sum_{i=n}^{\infty} s^i\, \varphi^i(t). \end{aligned}$$

As in above, we have $\lim_{n\to\infty} d(z_{n+p}, z_n) = 0$.

Similarly we can show that $\lim_{n\to\infty} d(w_n, w_{n+p}) = 0$ and $\lim_{n\to\infty} d(w_{n+p}, w_n) = 0$.

Thus $\{z_n\}$ *and* $\{w_n\}$ are Cauchy sequences in X.

Since X is a complete dislocated quasi b- metric space, there exist $z, w \in X$ such that $\{z_n\}$ converges to z and $\{w_n\}$ converges to w.

Since SF = FS and S and F are continuous, we have

$$\begin{aligned} Sz &= \lim_{n\to\infty} S(z_{2n}) = \lim_{n\to\infty} S(F(x_{2n}, y_{2n})) = \lim_{n\to\infty} F(Sx_{2n}, Sy_{2n}) = \lim_{n\to\infty} F(z_{2n-1}, w_{2n-1}) \\ &= F(\lim_{n\to\infty} z_{2n-1}, \lim_{n\to\infty} w_{2n-1}) = F(z, w). \end{aligned}$$

Similarly we have $Sw = F(w, z)$.

Since TG= GT and T and G are continuous, we have

$$\begin{aligned} Tz &= \lim_{n\to\infty} T(G(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} G(Tx_{2n+1}, Ty_{2n+1})) = \lim_{n\to\infty} G(z_{2n}, w_{2n}) \\ &= G(\lim_{n\to\infty} z_{2n}, \lim_{n\to\infty} w_{2n}) = G(z, w). \end{aligned}$$

Similarly we have $Tw = G(w, z)$.

$$d(Sz, Tz) = d(F(z, w), G(z, w)) \le \varphi(max\{d(Sz, Tz), d(Sw, Tw)\}) \text{ from (2.2.1)}$$
$$d(Sw, Tw) = d(F(w, z), G(w, z)) \le \varphi(max\{d(Sz, Tz), d(Sw, Tw)\}) \text{ from (2.2.1)}$$

Thus we have $max\{d(Sz, Tz), d(Sw, Tw)\} \le \varphi(max\{d(Sz, Tz), d(Sw, Tw)\})$,

which in turn yields that $d(Sz, Tz) = 0 = d(Sw, Tw)$, since $\varphi(t) < t \; for\; all\; t > 0$.

Similarly using (2.2.2), we can show that

$$d(Tz, Sz) = 0 = d(Tw, Sw).$$

Hence by (1.1.2), we have $Sz = Tz \; and \; Sw = Tw$.

Let $\propto = Sz = Tz \; and \; \beta = Sw = Tw$.

Then $S\alpha = S^2 z = S(F(z, w)) = F(Sz, Sw) = F(\alpha, \beta)$,

$$\begin{aligned} S\beta &= S^2 w = S(F(w, z)) = F(Sw, Sz) = F(\beta, \alpha), \\ T\alpha &= T^2 z = T(G(z, w)) = G(Tz, Tw) = G(\alpha, \beta), \\ T\beta &= T^2 w = T(G(w, z)) = G(Tw, Tz) = G(\beta, \alpha). \end{aligned}$$

Now using (2.2.1) and (2.2.2), we have

$$\begin{aligned} d(S\alpha, \alpha) &= d(F(\alpha, \beta), Tz) = d(F(\alpha, \beta), G(z, w)) \le \varphi(max\{d(S\alpha, \alpha), d(S\beta, \beta)\}), \\ d(\alpha, S\alpha) &= d(Tz, F(\alpha, \beta)) = d(G(z, w), F(\alpha, \beta)) \le \varphi(max\{d(\alpha, S\alpha), d(\beta, S\beta)\}), \\ d(S\beta, \beta) &= d(F(\beta, \alpha), Tw) = d(F(\beta, \alpha), G(w, z)) \le \varphi(max\{d(S\beta, \beta), d(S\alpha, \alpha)\}), \\ d(\beta, S\beta) &= d(Tw, F(\beta, \alpha)) = d(G(w, z), F(\beta, \alpha)) \le \varphi(max\{d(\alpha, S\alpha), d(\beta, S\beta)\}). \end{aligned}$$

Since φ is monotonically non-decreasing, we have

$$max\{d(S\alpha, \alpha), d(\alpha, S\alpha), d(S\beta, \beta), d(\beta, S\beta)\} \le \varphi(max\{d(S\alpha, \alpha), d(\alpha, S\alpha), d(S\beta, \beta), d(\beta, S\beta)\})$$

which in turn yields that $S\alpha = \alpha \; and \; S\beta = \beta$, since $\varphi(t) < t \; for \; t > 0$ and from (1.1.2).

Similarly we can show that $T\alpha = \alpha$ and $T\beta = \beta$.

Thus $F(\alpha,\beta) = S\alpha = \alpha = T\alpha = G(\alpha,\beta)$ and $F(\beta,\alpha) = S\beta = \beta = T\beta = G(\beta,\alpha)$.

Hence (α,β) is a common coupled fixed point of F, G, S and T.

UNIQUENESS:

Let (p, q) be another common coupled fixed point of F, G, S and T. Then $F(p,q) = Sp = p = Tp = G(p,q)$ and $F(q,p) = Sq = q = Tq = G(q,p)$.

Consider $d(\alpha,p) = d\big(F(\alpha,\beta), G(p,q)\big) \le \varphi(max\{d(\alpha,p), d(\beta,q)\})$ from (2.2.1),
$d(p,\alpha) = d\big(G(p,q), F(\alpha,\beta)\big) \le \varphi(max\{d(p,\alpha), d(q,\beta)\})$ from (2.2.2),
$d(\beta,q) = d\big(F(\beta,\alpha), G(q,p)\big) \le \varphi(max\{d(\alpha,p), d(\beta,q)\})$ from (2.2.1),
$d(q,\beta) = d\big(G(q,p), F(\beta,\alpha)\big) \le \varphi(max\{d(p,\alpha), d(q,\beta)\})$ from (2.2.2).

Since φ is monotonically non-decreasing, we have
$$max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\} \le \varphi\,(max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\})$$

which in turn yields that $\alpha = p\ and\ \beta = q$, since $\varphi(t) < t\ for\ t > 0$ and from (1.1.2).

Thus (α,β) is the unique common coupled fixed point of F, G, S and T.

Example 2.3: Let $X = [0,1]\ and\ d(x,y) = |x-y|^2 + |x|$. Let F, G : $X \times X \to X$ and $S,T: X \to X$ be defined by $F(x,y) = \frac{x+y}{64}, Sx = \frac{x}{2}, G(x,y) = \frac{x+y}{96}, Tx = \frac{x}{3}$. Let $\varphi: [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \frac{t}{4}$.

(i) Clearly $d(x,y) = d(y,x) = 0 \Rightarrow x = y$

(ii)
$$\begin{aligned} d(x,y) = |x-y|^2 + |x| &= |x-z+z-y|^2 + |x| \\ &\le 2[|x-z|^2 + |z-y|^2] + |x| \\ &\le 2[|x-z|^2 + |x| + |z-y|^2 + |z|] \\ &= s[d(x,z) + d(z,y)] \end{aligned}$$
, where s = 2.

$$\begin{aligned} d\big(F(x,y), G(u,v)\big) &= d\left(\frac{x+y}{64}, \frac{u+v}{96}\right) = \left|\frac{x+y}{64} - \frac{u+v}{96}\right|^2 + \left|\frac{x+y}{64}\right| \\ &= \left|\frac{3x-2u+3y-2v}{6\times 32}\right|^2 + \frac{x}{64} + \frac{y}{64} \\ &\le \frac{1}{36\times 32\times 32}\ 2[|3x-2u|^2 + |3y-2v|^2] + \frac{x}{64} + \frac{y}{64} \\ &= \frac{1}{16\times 32}\left[\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \left|\frac{y}{2} - \frac{v}{3}\right|^2\right] + \frac{x}{64} + \frac{y}{64} \\ &= \frac{1}{32}\left[\frac{1}{16}\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \frac{1}{16}\left|\frac{y}{2} - \frac{v}{3}\right|^2 + \frac{x}{2} + \frac{y}{2}\right] \\ &\le \frac{1}{32}\left[\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \left|\frac{y}{2} - \frac{v}{3}\right|^2 + \frac{x}{2} + \frac{y}{2}\right] \\ &= \frac{1}{32}[d(Sx,Tu) + d(Sy,Tv)] \\ &\le \frac{1}{16}\max\{d(Sx,Tu), d(Sy,Tv)\} \\ &\le \frac{1}{4}\max\{d(Sx,Tu), d(Sy,Tv)\} \\ &= \varphi(max\{d(Sx,Tu), d(Sy,Tv)\}), since\ \varphi(t) = \frac{t}{4}. \end{aligned}$$

Similarly we can show that $d\big(G(x,y), F(u,v)\big) \le \varphi(max\{d(Tx,Su), d(Ty,Sv)\})$.

Also it is clear that F, G, S and T are continuous, FS = SF, GT = TG and $F(X \times X) \subseteq T(X), G(X \times X) \subseteq S(X)$. Thus all conditions of Theorem 2.2 are satisfied. Clearly (0, 0) is the unique common coupled fixed point of F, G, S and T in $X \times X$.

Now replacing the completeness of X, continuities of F, G, S and T and commutatively of pairs (F, S) and (G, T) by w-compatible pairs (F, S) and (G, T) and completeness of one of S(X) and T(X), we prove a unique common coupled fixed point theorem. In fact, we prove the following theorem.

Theorem 2.4: Let (X, d) be a dislocated quasi b- metric space with fixed integer $s \geq 1$ and $F, G: X \times X \to X$ and $S, T: X \to X$ be mappings satisfying

(2.4.1) $d\big(F(x,y), G(u,v)\big) \leq \varphi\left(\frac{1}{2s^2} max\{\, d(Sx,Tu), d(Sy,Tv) \,\}\right)$ for all $x, y, u, v \in X,$ where φ ϵ $\mathbf{\Phi}_s$ and φ is continuous,

(2.4.2) $d\big(G(x,y), F(u,v)\big) \leq \varphi\left(\frac{1}{2s^2} max\{d(Tx,Su), d(Ty,Sv)\}\right)$ for all $x, y, u, v \in X, where$ φ ϵ $\mathbf{\Phi}_s$ and φ is continuous ,

(2.4.3) $F(X \times X) \subseteq T(X), G(X \times X) \subseteq S(X),$

(2.4.4) one of $S(X)$ *and* $T(X)$ is a complete sub space of X,

(2.4.5) the pairs (F,S) and (G,T) are w-compatible.

Then F, G, S and T have a unique common coupled fixed point in $X \times X$.

Proof: As in proof of Theorem (2.2), for $x_0, y_0 \in X,$ *the sequences* $\{z_n\}$ *and* $\{w_n\}$ are Cauchy in X.

Suppose S(X) is a complete sub space of X.

Since $z_{2n+1} = Sx_{2n+2} \subseteq S(X)$, there exist z, u $\in X$ such that $z_{2n+1} \to z = Su$ and since $w_{2n+1} = Sy_{2n+2} \subseteq S(X)$, there exist $w, v \in X$ such that $w_{2n+1} \to w = Sv$. Hence clearly $z_{2n} \to z$ and $w_{2n} \to w$.

By Lemma 1.3, we have

$$\begin{aligned}\frac{1}{s} d(F(u,v),z) &\leq \lim_{n\to\infty} \inf d\big(F(u,v), G(x_{2n+1}, y_{2n+1})\big)\\ &\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\{d(Su, Tx_{2n+1}), d(Sv, Ty_{2n+1})\}\right), \text{ from (2.4.1)}\\ &= \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\{d(z, z_{2n}), d(w, w_{2n})\}\right)\\ &= \varphi(0), \text{ since } \varphi \text{ is continuous, } z_{2n} \to z \text{ and } w_{2n} \to w.\\ &= 0\end{aligned}$$

Thus $d(F(u,v),z) = 0$.

Also by Lemma 1.3 and (2.4.2), we can prove that $d\big(z, F(u,v)\big) = 0$.

Hence Su = z = $F(u,v)$.Similarly we can show that Sv = w = F(v, u).

Thus (u, v) is a coupled coincidence point of S and F.

Since the pair (F, S) is w-compatible, we have

$$Sz = S(Su) = S\big(F(u,v)\big) = F(Su,Sv) = F(z,w) \text{ and}$$
$$Sw = S(Sv) = S\big(F(v,u)\big) = F(Sv,Su) = F(w,z).$$

Now using Lemma 1.3, (2.4.1) and monotonically non-decreasing property of φ, we have

$$\begin{aligned}\frac{1}{s} d(Sz,z) = \frac{1}{s} d(F(z,w),z) &\leq \lim_{n\to\infty} \inf d\big(F(z,w), G(x_{2n+1}, y_{2n+1})\big)\\ &\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\{d(Sz, z_{2n}), d(Sw, w_{2n})\}\right)\\ &\leq \varphi\left(\frac{1}{2s^2} max\{s\, d(Sz,z), s\, d(Sw,w)\}\right)\\ &\leq \varphi\left(\frac{1}{s} max\{d(Sz,z), d(Sw,w)\}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{1}{s} d(z,Sz) = \frac{1}{s} d\big(z, F(z,w)\big) &\leq \lim_{n\to\infty} \inf d\big(G(x_{2n+1}, y_{2n+1}), F(z,w)\big)\\ &\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\{d(z_{2n}, Sz), d(w_{2n}, Sw)\}\right)\\ &\leq \varphi\left(\frac{1}{2s^2} max\{s\, d(z,Sz), s\, d(w,Sw)\}\right)\\ &\leq \varphi\left(\frac{1}{s} max\{d(z,Sz), d(w,Sw)\}\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{s} d(w,Sw) = \frac{1}{s} d\big(w, F(w,z)\big) &\leq \lim_{n\to\infty} \inf d\big(G(y_{2n+1}, x_{2n+1}), F(w,z)\big)\\ &\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\{d(w_{2n}, Sw), d(z_{2n}, Sz)\}\right)\\ &\leq \varphi\left(\frac{1}{2s^2} max\{s\, d(w,Sw), s\, d(z,Sz)\}\right)\\ &\leq \varphi\left(\frac{1}{s} max\{d(z,Sz), d(w,Sw)\}\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{s}d(Sw,w)=\frac{1}{s}d(F(w,z),w) &\le \lim_{n\to\infty}\inf d\big(F(w,z),G(y_{2n+1},x_{2n+1})\big)\\ &\le \lim_{n\to\infty}\inf \varphi\left(\frac{1}{2s^2}max\{d(Sw,w_{2n}),d(Sz,z_{2n})\}\right)\\ &\le \varphi\left(\frac{1}{2s^2}max\{s\,d(Sw,w),s\,d(Sz,z)\}\right)\\ &\le \varphi\left(\frac{1}{s}max\{d(Sw,w),d(Sz,z)\}\right).\end{aligned}$$

Since φ is monotonically non decreasing, we have

$$\frac{1}{s}max\{d(Sz,z),d(z,Sz),d(Sw,w),d(w,Sw)\}\le\varphi\left(\frac{1}{s}max\{d(Sz,z),d(z,Sz),d(Sw,w),d(w,Sw)\}\right)$$

Since $\varphi(t)<t\ for\ all\ t>0$, we have

$max\{d(Sz,z),d(z,Sz),d(Sw,w),d(w,Sw)\}=0$ which in turn yields that $Sz=z,Sw=w$.

Thus $z=Sz=F(z,w)\ and\ w=Sw=F(w,z)$. (1)

Since $F(X\times X)\subseteq T(X)$, there exist $\propto,\beta\ in\ X$ such that

$T\propto=F(z,w)=Sz=z\ and\ T\beta=F(w,z)=Sw=w$.

Since φ is monotonically non decreasing and s ≥ 1, we have

$$\begin{aligned}d\big(T\propto,G(\propto,\beta)\big)&=d\big(F(z,w),G(\propto,\beta)\big)\\ &\le\varphi\left(\frac{1}{2s^2}max\{d(Sz,T\propto),d(Sw,T\beta)\}\right)\\ &\le\varphi\left(\frac{1}{2s^2}max\left\{\begin{matrix}s\,d\big(T\propto,G(\propto,\beta)\big)+s\,d(G(\propto,\beta),T\propto),\\ s\,d\big(T\beta,G(\beta,\alpha)\big)+s\,d(G(\beta,\alpha),T\beta)\end{matrix}\right\}\right)\\ &\le\varphi\big(max\{d\big(T\propto,G(\propto,\beta)\big),d(G(\propto,\beta),T\propto),d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta)\}\big),\end{aligned}$$

$$\begin{aligned}d(G(\propto,\beta),T\propto)&=d\big(G(\propto,\beta),F(z,w)\big)\\ &\le\varphi\left(\frac{1}{2s^2}max\{d(T\propto,Sz),d(T\beta,Sw)\}\right)\\ &\le\varphi\left(\frac{1}{2s^2}max\left\{\begin{matrix}s\,d\big(T\propto,G(\propto,\beta)\big)+s\,d(G(\propto,\beta),T\propto),\\ s\,d\big(T\beta,G(\beta,\alpha)\big)+s\,d(G(\beta,\alpha),T\beta)\end{matrix}\right\}\right)\\ &\le\varphi\big(max\{d\big(T\propto,G(\propto,\beta)\big),d(G(\propto,\beta),T\propto),d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta)\}\big),\end{aligned}$$

$$\begin{aligned}d\big(T\beta,G(\beta,\alpha)\big)&=d\big(F(w,z),G(\beta,\alpha)\big)\\ &\le\varphi\left(\frac{1}{2s^2}max\{d(Sw,T\beta),d(Sx,T\alpha)\}\right)\\ &\le\varphi\left(\frac{1}{2s^2}max\left\{\begin{matrix}s\,d\big(T\beta,G(\beta,\alpha)\big)+s\,d(G(\beta,\alpha),T\beta),\\ s\,d\big(T\alpha,G(\alpha,\beta)\big)+s\,d(G(\alpha,\beta),T\alpha)\end{matrix}\right\}\right)\\ &\le\varphi\big(max\{d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta),d\big(T\alpha,G(\alpha,\beta)\big),d(G(\alpha,\beta),T\alpha)\}\big),\end{aligned}$$

$$\begin{aligned}d(G(\beta,\propto),T\beta)&=d\big(G(\beta,\alpha),F(w,z)\big)\\ &\le\varphi\left(\frac{1}{2s^2}max\{d(T\beta,Sw),d(T\alpha,Sz)\}\right)\\ &\le\varphi\left(\frac{1}{2s^2}max\left\{\begin{matrix}s\,d\big(T\beta,G(\beta,\alpha)\big)+s\,d(G(\beta,\alpha),T\beta),\\ s\,d\big(T\alpha,G(\alpha,\beta)\big)+s\,d(G(\alpha,\beta),T\alpha)\end{matrix}\right\}\right)\\ &\le\varphi\big(max\{d\big(T\propto,G(\propto,\beta)\big),d(G(\propto,\beta),T\propto),d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta)\}\big).\end{aligned}$$

Thus we have

$$max\left\{\begin{matrix}d\big(T\propto,G(\propto,\beta)\big),d(G(\propto,\beta),T\propto),\\ d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta)\end{matrix}\right\}\le\ \varphi\left(max\left\{\begin{matrix}d\big(T\propto,G(\propto,\beta)\big),d(G(\propto,\beta),T\propto),\\ d\big(T\beta,G(\beta,\alpha)\big),d(G(\beta,\alpha),T\beta)\end{matrix}\right\}\right)$$

which in turn yields that $T\propto=G(\propto,\beta)$ and $T\beta=G(\beta,\alpha)$. Since the pair (G, T) is w-compatible, we have

$Tz=T(T\propto)=T\big(G(\propto,\beta)\big)=G(T\propto,T\beta)=G(z,w)$ and

$Tw=T(T\beta)=T\big(G(\beta,\alpha)\big)=G(T\beta,T\alpha)=G(w,z)$.

Now we have

$$\begin{aligned}d\big(z,G(z,w)\big)&=d\big(F(z,w),G(z,w)\big)\\ &\le\varphi\left(\frac{1}{2s^2}max\{d(Sz,Tz),d(Sw,Tw)\}\right)\\ &=\varphi\left(\frac{1}{2s^2}max\{d\big(z,G(z,w)\big),d\big(w,G(w,z)\big)\}\right)\\ &\le\varphi\big(max\{d\big(z,G(z,w)\big),d\big(w,G(w,z)\big)\}\big),\end{aligned}$$

$$\begin{aligned} d(G(z,w),z) &= d\big(G(z,w),F(z,w)\big) \\ &\le \varphi\left(\tfrac{1}{2s^2} max\{d(Tz,Sz),d(Tw,Sw)\}\right) \\ &= \varphi\left(\tfrac{1}{2s^2} max\{d(G(z,w),z),d(G(w,z),w)\}\right) \\ &\le \varphi\ (max\{d(G(z,w),z),d(G(w,z),w)\}), \end{aligned}$$

$$\begin{aligned} d\big(w,G(w,z)\big) &= d\big(F(w,z),G(w,z)\big) \\ &\le \varphi\left(\tfrac{1}{2s^2} max\{d(Sw,Tw),d(Sz,Tz)\}\right) \\ &= \varphi\left(\tfrac{1}{2s^2} max\{d\big(w,G(w,z)\big),d\big(z,G(z,w)\big)\}\right) \\ &\le \varphi\ \big(max\{d\big(w,G(w,z)\big),d\big(z,G(z,w)\big)\}\big), \end{aligned}$$

$$\begin{aligned} d(G(w,z),w) &= d\big(G(w,z),F(w,z)\big) \\ &\le \varphi\left(\tfrac{1}{2s^2} max\{d(Tw,Sw),d(Tz,Sz)\}\right) \\ &= \varphi\left(\tfrac{1}{2s^2} max\{d(G(w,z),w),d(G(z,w),z)\}\right) \\ &\le \varphi\ (max\{d(G(w,z),w),d(G(z,w),z)\}). \end{aligned}$$

Thus we have

$$max\begin{Bmatrix} d\big(z,G(z,w)\big),d(G(z,w),z), \\ d\big(w,G(w,z)\big),d(G(w,z),w)\end{Bmatrix} \le \varphi\left(max\begin{Bmatrix} d\big(z,G(z,w)\big),d(G(z,w),z), \\ d\big(w,G(w,z)\big),d(G(w,z),w)\end{Bmatrix}\right)$$

which in turn yields that $z = G(z,w)\ and\ w = G(w,z)$.

Thus $z = G(z,w) = Tz, and\ w = G(w,z) = Tw.$ (2)

From (1) and (2), (z, w) is a common coupled fixed point of F,G,S and T. Uniqueness of common coupled fixed point of F, G, S and T follows as in Theorem 2.2.

Now we give an example to illustrate Theorem 2.4.

Example 2.6: Let $X = [0,1]\ and$ define $d(x,y) = |x-y|^2 + |x|$. Let F, G : $X \times X \to X$ and $S,T: X \to X$ be defined by $F(x,y) = \frac{x^2+y^2}{128}, G(x,y) = \frac{x^2+y^2}{256}, Sx = \frac{x^2}{2}, Tx = \frac{x^2}{4}$. Let $\varphi: [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \frac{t}{4}$. As in Example 2.3, d is a dislocated quasi b-metric with s = 2.

Consider

$$\begin{aligned} d\big(F(x,y),G(u,v)\big) &= d\left(\frac{x^2+y^2}{128},\frac{u^2+v^2}{256}\right) = \left|\frac{x^2+y^2}{128} - \frac{u^2+v^2}{256}\right|^2 + \frac{x^2+y^2}{128} \\ &= \frac{|2x^2+2y^2-u^2-v^2|^2}{256\times 256} + \frac{x^2}{128} + \frac{y^2}{128} \\ &\le \frac{2\left[|2x^2-u^2|^2 + |2y^2-v^2|^2\right]}{256\times256} + \frac{x^2}{128} + \frac{y^2}{128} \\ &= \left\{\frac{16}{128\times 256}\left[\left|\frac{x^2}{2} - \frac{u^2}{4}\right|^2 + \left|\frac{y^2}{2} - \frac{v^2}{4}\right|^2\right]\right\} + \frac{x^2}{128} + \frac{y^2}{128} \\ &= \left\{\frac{1}{128\times16}\left[\left|\frac{x^2}{2} - \frac{u^2}{4}\right|^2 + \left|\frac{y^2}{2} - \frac{v^2}{4}\right|^2\right]\right\} + \frac{x^2}{128} + \frac{y^2}{128} \\ &= \frac{1}{64}\left[\frac{1}{32}\left|\frac{x^2}{2} - \frac{u^2}{4}\right|^2 + \frac{1}{32}\left|\frac{y^2}{2} - \frac{v^2}{4}\right|^2 + \frac{x^2}{2} + \frac{y^2}{2}\right] \\ &\le \frac{1}{64}\left[\left|\frac{x^2}{2} - \frac{u^2}{4}\right|^2 + \frac{x^2}{2} + \left|\frac{y^2}{2} - \frac{v^2}{4}\right|^2 + \frac{y^2}{2}\right] \\ &= \frac{1}{64}[d(Sx,Tu) + d(Sy,Tv)] \\ &\le \frac{1}{32}\ \max\{d(Sx,Tu),d(Sy,Tv)\} \\ &= \frac{1}{4}.\frac{1}{2s^2}\max\{d(Sx,Tu),d(Sy,Tv)\}\text{ , since s = 2} \\ &= \varphi\left(\frac{1}{2s^2}\ max\{d(Sx,Tu),d(Sy,Tv)\}\right)\text{, since } \varphi(t) = \frac{t}{4}. \end{aligned}$$

Similarly we can show that $d\big(G(x,y),F(u,v)\big) \le \varphi\left(\frac{1}{2s^2} max\{d(Tx,Su),d(Ty,Sv)\}\right)$.

Also it is clear that S(X) and T (X) are complete subspaces of X, the pairs (F,S) and (G,T) are w-compatible and $F(X\times X) \subseteq T(X), G(X\times X) \subseteq S(X)$. Thus all conditions of Theorem 2.4 are satisfied.

Clearly (0, 0) is the unique common coupled fixed point of F, G, S and T in $X \times X$.

Remark: Theorem 2.4 is a generalization of Theorem 4.1 of [14], Theorem 3.2 of [11] and Theorem 2.1 of [1].

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