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ON CHARACTERIZATION OF v-OPEN SETS IN A TOPOLOGICAL SPACES

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ABSTRACT

A new class of generalized open sets in a topological space, called v-open sets, is introduced and studied. This class contains all semi-open sets and all pre*open sets. Also a new class of sets, namely $*\,v$-open sets and $v^{\#}$-open sets are introduced in topological spaces. Also we find some basic properties and characterizations of v-open, *v-open sets and $v^{\#}$-open sets.*

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1. INTRODUCTION

Generalized closed sets in a topological space is introduced by Levine[4] in 1970. In 1963 Levine [3] introduced semi-open sets in topological spaces. After Levine's work, many mathematicians turned their attention to generalizing various concepts in topology by considering semi-open sets instead of open sets. Dunham [1] introduced the concept of generalized closure using Levine's generalized closed sets and defined a new topology τ^* and studied some of their properties. Robert. A[6] *et al.*, and selvi. T [7] introduced semi*-open sets and pre*-open sets respectively, using the generalized closure operator Cl^* due to Dunham. In this paper we define v-open sets, $*\,v$-open sets and $v^{\#}$-open sets and investigate fundamental properties of these sets.

2. PRELIMINARIES

Throughout this paper, spaces (X,τ) (or simply X) always mean non empty topological spaces on which no separation axioms are assumed unless explicity stated. For a subset A of a space (X,τ), $cl(A)$, $int(A)$ and X/A denote the closure of A, the interior of A and the complement of A respectively. Also $sint(A)$, $scl(A)$, $pint(A)$ and $pcl(A)$ denote the semi interior of A, semi closure of A, pre interior of A and pre closure of A respectively. The following definitions and results are very useful in the subsequent sections.

Definition 2.1: A subset A of a topological space(X,τ) is called
- (i) Pre-open set if $A \subseteq int(cl(A))$
- (ii) Semi*open set if $A \subseteq cl^*(int(A))$
- (iii) Pre* open set if $A \subseteq int^*(cl(A))$
- (iv) α^*-open set if $A \subseteq int^*(cl(int^*(A)))$
- (v) regular open set if $A = int(cl(A))$.

3. v-open sets

Now we consider a new class of generalized open sets.

Definition 3.1: A subset A of a topological space (X,τ) is said to be a v-open set if $A \subseteq int^*(cl(A)) \cup cl^*(int(A))$.

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Theorem 3.2: Every open set is v-open.

Proof: Let A be an open set of a topological space (X,τ). Then $A = int(A) \subseteq cl*(int(A))$. Also $int*(A) \subseteq int*(cl(A))$. Therefore $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and hence A is v-open.

Remark 3.3: The converse of the above theorem need not be true which is shown in the following example.

Example 3.4: Let X= {a, b, c} with τ={ϕ, {b}, {a, b}, X}. The sets {a} and {b, c} are v-open but not an open sets.

Theorem 3.5: Every pre-open set is v-open.

Proof: Let A be a pre-open set of a topological space (X,τ). Then $A \subseteq int(cl(A)) \subseteq int*(cl(A))$. Also $int(A) \subseteq cl*(int(A))$.Therefore $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and hence A is v-open.

Remark 3.6: The converse of the above theorem need not be true which is shown in the following example.

Example 3.7: Let X= {a, b, c} with τ = {ϕ, {b}, {a, b}, X}. The sets {a} is v-open but not a pre-open set.

Theorem 3.8: Every semi*-open set is v-open.

Proof: Let A be a semi*-open set of a topological space (X,τ). Then $A \subseteq cl*(int(A))$. Also $int*(A) \subseteq int*(cl(A))$. Therefore $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and hence A is v-open.

Remark 3.9: The converse of the above theorem need not be true which is shown in the following example.

Example 3.10: Let X= {a, b, c, d} with τ={ϕ, {a}, {a, b}, {c, d}, {a, c, d}, X}. The sets {c} and {a, d} are v-open but not a semi*-open sets.

Theorem 3.11: Every pre*-open set is v-open.

Proof: Let A be a pre*-open set of a topological space (X,τ). Then $A \subseteq int*(cl(A))$. Also $int(A) \subseteq cl*(int(A))$. Therefore $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and hence A is v-open.

Remark 3.12: The converse of the above theorem need not be true which is shown in the following example.

Example 3.13: Let X= {a, b, c} with τ={ϕ, {a}, {c}, {a, c}, X}. The sets {a, b} and {b, c} are v-open but not a pre*-open sets.

Theorem 3.14: Every α*-open set is v-open.

Proof: Let A be a α*-open set of a topological space (X,τ). Then $A \subseteq int*(cl(int*(A)))$. Now, $int*(cl(int*(A))) \subseteq int*(cl(A))$. Therefore $A \subseteq int*(cl(A))$. Also $int(A) \subseteq cl*(int(A))$. Therefore $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and hence A is v-open.

Remark 3.15: The converse of the above theorem need not be true which is shown in the following example.

Example 3.16: Let X= {a, b, c, d} with τ={ϕ, {b}, {a, b}, {b, c, d}, X}. The sets {c} and {b, c} are v-open but not a α*-open sets.

Theorem 3.17: The union of two v-open sets is v-open.

Proof: Let A and B be two v-open sets in a topological spaces (X,τ). Then $A \subseteq int*(cl(A)) \cup cl*(int(A))$ and $B \subseteq int*(cl(B)) \cup cl*(int(B))$.

Now, $A \cup B \subseteq [int*(cl(A)) \cup cl*(int(A))] \cup [int*(cl(B)) \cup cl*(int(B))] = [int*(cl(A)) \cup int*(cl(B))] \cup [cl*(int(A)) \cup cl*(int(B))] \subseteq [int*(cl(A) \cup cl(B))] \cup [cl*(int(A) \cup int(B))] \subseteq int*(cl(A \cup B) \cup cl*(int(A \cup B)$. That is, $A \cup B \subseteq int*(cl(A \cup B) \cup cl*(int(A \cup B)$ and hence $A \cup B$ is v-open.

Remark 3.18: Let $\{A_\alpha\}$ be a collection of v-open sets. Then $\bigcup_{\alpha\in I} A_\alpha$ is also a v-open set.

Remark 3.19: The finite intersection of v-open sets need not be a v-open set.

Example 3.20: Let X= {a, b, c} with τ={ϕ, {b}, {c}, {b, c}, X}. Here v-O(X,τ)={ϕ, {b},{c},{a, b},{a, c}, {b, c}, X}. The sets {a, b} and {a, c} are v-open but their intersection need not be a v-open set.

Theorem 3.21: If a topological space (X,τ), let $\tau_v = \{U \in v\text{-O}(X,\tau)/ U \cap A \in v\text{-O}(X,\tau)$ for all $A \in v\text{-O}(X,\tau)\}$. Then τ_v is a topology on X.

Proof: Clearly $\phi, X \in \tau_v$. Let $U_\beta \in \tau_v$ and U=$\cup\, U_\beta$. Since each $U_\beta \in \tau_v$, then by Remark 3.18, $U \in v$-O(X, τ). Let $A \in v$-O(X, τ). Then $U_\beta \cap A \in v$-O(X,τ) for each β.
Hence $U \cap A = (\cup\, U_\beta) \cap A = \cup\,(U_\beta \cap A) \in v$-O$(X,\tau)$. Therefore $U \in \tau_v$. Let $U_1, U_2 \in \tau_v$. Then $U_1, U_2 \in v$-O(X, τ) and from definition of τ_v , $U_1 \cap U_2 \in v$-O(X,τ). If $A \in v$-O(X, τ), and from definition of τ_v, $U_1 \cap U_2 \cap A \in v$-O$(X,\tau)$. Hence $U_1 \cap U_2 \in \tau_v$. This shows that τ_v is closed under finite intersection. Hence τ_v is a topology on X.

4. v-Closed Sets

Definition 4.1: A subset A of a topological space (X,τ) is called a v-closed set if $X\backslash A$ is v-open. The collection of all v-closed sets in (X,τ) is denoted by v-C(X,τ).

Theorem 4.2: A subset A of (X,τ) is v-closed if and only if $int*(cl(A)) \cap cl*(int(A)) \subseteq A$.

Proof: Let A be a v-closed set. Then X\A is v-open set. By definition, $X\backslash A \subseteq int*(cl(X\backslash A)) \cup cl*(int(X\backslash A)) = (X\backslash cl*(int(A))) \cup (X\backslash int*(cl(A))) = X\backslash (cl*(int(A)) \cap int*(cl(A)))$. Therefore $int*(cl(A)) \cap cl*(int(A)) \subseteq A$. Conversely, assume that $int*(cl(A)) \cap cl*(int(A)) \subseteq A$. Then $X\backslash A \subseteq X\backslash (cl*(int(A)) \cap int*(cl(A))) = (X\backslash cl*(int(A))) \cup (X\backslash int*(cl(A))) = int*(cl(X\backslash A)) \cup cl*(int(X\backslash A))$. Hence $X\backslash A$ is v-open and so A is v-closed.

Theorem 4.3: Arbitrary intersection of v-closed sets is v-closed.

Proof: Let $\{A_\alpha\}$ be a family of v-closed sets in a topological space (X,τ). Then $int*(cl(A_\alpha)) \cap cl*(int(A_\alpha)) \subseteq A_\alpha$, for every α. Since $\{{A_\alpha}^c\}$ is an arbitrary collection of v-open sets, hence $\bigcup A_\alpha^c$ is a v-open set. But $\bigcup A_\alpha^c = (\bigcap A_\alpha)^c$, is v-open set and hence $\bigcap A_\alpha$ is v-closed.

Theorem 4.4: For a topological space (X,τ),
- (i) Every closed set is v-closed.
- (ii) Every pre-closed set is v-closed.
- (iii) Every semi*-closed set is v-closed.
- (iv) Every pre*-closed set is v-closed.
- (v) Every α*-closed set is v-closed.

Theorem 4.5: Let A be v-closed in X. Then
- (i) $sint(A)$ is v-closed.
- (ii) If A is regular open, then $pint(A)$ and $scl(A)$ are also v-closed.
- (iii) If A is regular closed, then $pcl(A)$ is also v-closed.

Proof: Let A be a v-closed set of X.
- (i) Since $cl(int(A))$ is closed, then by theorem 4.4(i), $cl(int(A))$ is v-closed. By theorem 4.3, $sint(A)$ is v-closed.
- (ii) Suppose A is regular open, then $int(cl(A)) = A$. Implies that, $scl(A) = A$. Since A is v-closed, then $scl(A)$ is v-closed. Similarly $pint(A)$ is v-closed.
- (iii) Suppose A is regular closed, $cl(int(A)) = A$. Then by $pcl(A) = A$, and hence v-closed.

Theorem 4.6: A subset A of a topological space (X,τ) is v-open if and only if every closed set F containing A, there exists the union of maximal g-open set M contained in $cl(A)$ and the minimal g-closed set N containing $int(A)$ such that $A \subseteq M \cup N$.

Proof: Let A be a v-open set in a topological space (X,τ). Then $A \subseteq int*(cl(A)) \cup cl*(int(A))$. Let $A \subseteq F$ and F is closed so that $cl(A) \subseteq F$. Let $M = int*(cl(A))$, then M is the maximal g-open set contained in $cl(A)$. Let $N = cl*(int(A))$, then N is the minimal g-closed set containing $int(A)$. Now, $int*(cl(A)) \subseteq cl(A) \subseteq F$ and $cl*(int(A)) \subseteq cl(int(A)) \subseteq cl(A) \subseteq F$. Therefore $cl*(int(A)) \cup int*(cl(A)) \subseteq F$. Hence, $A \subseteq cl*(int(A)) \cup int*(cl(A)) \subseteq F$. That is, $A \subseteq M \cup N \subseteq F$. Conversely, assume that $A \subseteq M \cup N$, where A is a subset of a topological space, F is closed set containing A, M is the maximal g-open set contained in $cl(A)$ and N is the minimal g-closed set containing $int(A)$. Therefore, $M = int*(cl(A))$ and $N = cl*(int(A))$. Thus, $A \subseteq cl*(int(A)) \cup int*(cl(A)) \subseteq F$ and hence A is v-open.

5. v-NEIGHBOURHOOD

Definition 5.1: Let X be a topological space and let $x \in X$. A subset N of X is said to be a v-neighbourhood (shortly, v-nbhd) of x if there exsits a v-open set U such that $x \in U \subseteq N$.

Definition 5.2: A subset N of a space X, is called a v-nbhd of $A \subseteq X$ if there exists a v-open set U such that $A \subseteq U \subseteq N$.

Theorem 5.3: Every nbhd N of $x \in X$ is a v-nbhd of x.

Proof: Let N be a nbhd of point $x \in X$. Then there exists an open set U such that $x \in U \subseteq N$. Since every open set is v-open, U is a v-open set such that $x \in U \subseteq N$. This implies, N is a v-nbhd of x.

Remark 5.4: The converse of the above theorem need not be true which is shown in the following example.

Example 5.5: Let X = {a, b, c, d} with topology τ = {ϕ, {a}, {a, b}, {c, d}, {a, c, d}, X}. In this topological space (X, τ), v-O(X) = {X, ϕ, {a}, {c}, {d}, {a, b}, {a, c}, {a, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}}. The set {c, d} is the v-nbhd of d, since {c, d} is v-open set such that d ∈ {c,d }⊆ {c, d}. However, the set {c, d} is not a nbhd of the point d.

Remark 5.6: Every v-open set is a v-nbhd of each of its points.

Remark 5.7: The converse of the above theorem need not be true in general as seen from the following example.

Example 5.8: Let X = {a, b, c, d} with the topology τ ={ϕ, {a}, {a, b}, {c, d}, {a, c, d}, X}. In this topological spaces v-O(X) ={X, ϕ, {a}, {c}, {d}, {a, b}, {a, c}, {a, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}}. The set {b, c} is a v-nbhd of a point c, since c ∈{c}⊆{b, c} . However {b, c} is not a v-open.

Theorem 5.9: If F is a v-closed subset of X and $x \in X\backslash F$, then there exists a v-nbhd N of x such that $N \cap F = \phi$.

Proof: Let F be v-closed subset of X and $x \in X\backslash F$. Then $X\backslash F$ is a v-open set of X. By Theorem 4.6, $X\backslash F$ contains a v-nbhd of each of its points. Hence there exists a v-nbhd N of x such that $N \subseteq X\backslash F$. Hence $N \cap F = \phi$.

Definition 5.10: The collection of all v-neighborhoods of $x \in X$ is called the v-neighborhood system of x and is denoted by v-$N(x)$.

Theorem 5.11: Let (X, τ) be a topological space and $x \in X$. Then
(i) v-$N(x) \neq \phi$ and x ∈each member of v-$N(x)$
(ii) If $N \in v$-$N(x)$ and $N \subseteq M$, then $M \in v$-$N(x)$.
(iii) Each member $N \in v$-$N(x)$ is a superset of a member $G \in v$-$N(x)$ where G is a v-open set.

Proof:
(i) Since X is a v-open set containing x, it is a v-nbhd of every $x \in X$. Thus for each $x \in X$, there exists atleast one v-nbhd, namely X. Therefore, v-$N(x) \neq \phi$. Let $N \in v$-$N(x)$. Then N is a v-nbhd of x. Hence there exists a v-open set G such that $x \in G \subseteq N$, so $x \in N$. Therefore x ∈ every member N of v-$N(x)$.
(ii) If $N \in v$-$N(x)$, then there is a v-open set G such that $x \in G \subseteq N$. Since $N \subseteq M$, M is v-nbhd of x. Hence $M \in v$-$N(x)$.
(iii) Let $N \in v$-$N(x)$. Then there is a v-open set G, such that $x \in G \subseteq N$. Since G is v-open and $x \in G$, G is a v-nbhd of x. Therefore $G \in v$-$N(x)$ and also $G \subseteq N$.

6. *v-OPEN SETS AND $v^{\#}$-OPEN SETS

Definition 6.1: A subset A of a topological space (X, τ) is said to be a *v-open set if $A \subseteq int^*(cl(A)) \cap cl^*(int(A))$.

Definition 6.2: A subset A of a topological space (X, τ) is said to be a $v^{\#}$-open set if $A = int^*(cl(A)) \cup cl^*(int(A))$.
It is note worthy to see that every $v^{\#}$-open set is a v-open set. However the converse is not true as shown in the following example.

Example 6.3: Let X= {a, b, c} with τ={ϕ, {a}, {c}, {a, c}, X}. The sets {a}, {c} and {a, c} are v-open but not a $v^{\#}$-open sets.

Theorem 6.4: Every $*\upsilon$-open set is υ-open.

Proof: Let A be a $*\upsilon$-open. Then $A \subseteq int*(cl(A)) \cap cl*(int(A))$. This implies, $A \subseteq int*(cl(A)) \cup cl*(int(A))$. Hence A is υ-open.

Theorem 6.5: Every $\upsilon^{\#}$-open set is $*\upsilon$-closed.

Proof: Let A be a $\upsilon^{\#}$-open. Then $A = int*(cl(A)) \cup cl*(int(A))$. This implies, $int*(cl(A)) \cup cl*(int(A)) \subseteq A$. Hence A is υ-closed.

Theorem 6.6:

(i) Every open set is $*\upsilon$-open.
(ii) Every $*\upsilon$-open set is pre*open.
(iii) Every $*\upsilon$-open set is semi*open.
(iv) Every $*\upsilon$-open set is semi-open.
(v) Every $\upsilon^{\#}$-open set is semi*-closed.
(vi) Every $\upsilon^{\#}$-open set is pre*-closed

Proof: Straight Forward.

Remark 6.7: The converse of the above theorem need not be true in general as seen from the following examples.

Example 6.8: Let X= {a, b, c} with τ={ϕ, {a}, {c}, {a, c}, X}.

(i) The sets {a, b} and {a, c} are υ-open but not a $*\upsilon$-open sets.
(ii) The set {b} is υ-closed but not a $\upsilon^{\#}$-open set.
(iii) The sets {a, b} and {b, c} are semi*-open but not a $\upsilon*$-open sets.
(iv) The sets {a, b} and {b, c} are semi-open but not a $\upsilon*$-open sets.

Example 6.9: Let X= {a, b, c} with τ={ϕ, {b}, {a, b}, X}.

(i) The set {b, c} are $*\upsilon$-closed but not open.
(ii) The set {a} is pre*-open but not $*\upsilon$-open.
(iii) The sets {c}, {a, c} are semi*-closed but not $\upsilon^{\#}$-open.
(iv) The sets {c}, {a, c}, {b, c} are pre*-closed but not $\upsilon^{\#}$-open.

REFERENCES

1. Dunham, W., A New Closure Operator for Non-T1 Topologies, *Kyungpook Math. J*. 22 (1982), 55-60.
2. Khalimsky, E.D, Applications of Connected Ordered Topological spaces in Topology, *Conference of Math*. Department of Povolsia, 1970.
3. Levine, N., Semi-Open Sets and Semi-Continuity in Topological Space, *Amer. Math. Monthly*. 70 (1963), 36-41.
4. Levine, N., Generalized Closed Sets in Topology, *Rend. Circ. Mat. Palermo*.19 (2) (1970), 89-96.
5. Mashhour.A.S,Abd.M.E,E1-Monsef and S.N.E1-Deep, On pre-continuous and weak-pre continuous mapping, Proc. Math.and Phys.Soc.Egypt, 53(1982), 47-53.
6. Robert, A., and Pious Missier, S. A New Class of Nearly Open Sets, International Journal of mathametical archive.
7. Selvi. T and A. Punitha Dharani, Some new class of nearly closed and open sets, Asian Journal of Current Engineering and Maths 1:5 Sep – Oct (2012) 305-307.
8. Stone M. Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc 1937; 41:374-481.

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