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ON THE ANNIHILATOR GRAPH OF A COMMUTATIVE $\Gamma-$ NEAR – RING

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ABSTRACT

Let M be a Commutative $\Gamma-$ near – ring with non zero identity. Let Z(M) be the set of all zero – divisors of M. For $x \in Z(M)$, let $ann_M(x) = \{y \in M/y\gamma x = 0\}$. We define the annihilator graph of M ,denoted by AG(M) ,as the undirected graph whose set of vertices is $Z(M)^* = Z(M) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $ann_M(x\gamma y) \neq ann_M(x) \cup ann_M(y)$.In this paper we study the ring theoretic properties of M and graph theoretic properties of $AG(M)$.

Keywords: *Annihilator graph, diameter, girth, zero – divisor graph.*

1. INTRODUCTION

The concept of a zero – divisor of a commutative ring was first introduced by I.Beck [4], where all the elements of the ring were taken as the vertices of the graph. In a commutative ring, for $x \in Z(M)$, let $ann_M(x) = \{y \in M/yx = 0\}$. A.Badawi [3] defined and studied the annihilator graph AG(M) of a commutative ring. The concept of a annihilator graph of a near ring was introduced and studied by T.Tamizh Chelvam and R.Rammoorthy [5]. Let M be a commutative $\Gamma-$ near – ring with non zero identity and Z(M) be its set of all zero – divisors. In this paper, we introduce the annihilator graph $AG(M)$ for a $\Gamma-$ near – ring M and study the connectivity of the annihilator graph. For a reduced $\Gamma-$ near – ring, we show that $Ann_G(M)$is identical to $\Gamma(M)$ if and only if M has exactly two distinct minimal prime ideals (Theorem 2.11). Among other things, we determine when $AG(M)$is a complete graph K_n, a complete bipartite graph ($K_{m,n}$), or a star graph($K_{1,n}$). If AG (M) is identical to Γ(M), then we write AG (M) = Γ(M) otherwise we write $AG(M) \neq \Gamma$(M) and also show that AG (M) is connected with diameter at most two. If AG (M) $\neq$ Γ(M), we show that gr (AG (M))$\in$ {3,4}.

2. MAIN RESULTS

Definition 2.1: The annihilator graph AG (M) for a $\Gamma-$ near – ring M, let $a \in Z(M)$ and let $ann_M(a) = \{m \in M/m\gamma a = 0, \gamma \in \Gamma\}$ The annihilator graph of M is the (undirected) graph AG (M) with vertices $Z(M)^* = Z(M) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $ann_M(x\gamma y) \neq ann_M(x) \cup ann_M(y)$, it follows that each edge (path) of Γ(M) is an edge (path) of AG (M).

Lemma 2.2: Let M be a commutative $\Gamma-$ near – ring. Then the following are hold.

i) Let x, y be distinct elements of $Z(M)^*$. Then x – y is not an edge of AG (M)if and only if $ann_M(x\gamma y) = ann_M(x)$ or $ann_M(x\gamma y) = ann_M(y)$
ii) If x – y is an edge of Γ(M) for some distinct x, y $\in Z(M)^*$, then x – y is an edge of AG (M).In particular if P is a path in Γ(M), then P is a path in AG (M).
iii) If x – y is not an edge of AG (M) for some distinct x, y $\in Z(M)^*$ then $ann_M(x) \subseteq ann_M(y)$ or $ann_M(y) \subseteq ann_M(x)$
iv) If $ann_M(x) \nsubseteq ann_M(y)$ and $ann_M(y) \nsubseteq ann_M(x)$ for some distinct x, y $\in Z(M)^*$ then x – y is an edge of AG (M)
v) If $d_{\Gamma(M)}(x,y) = 3$ for some distinct x, y $\in Z(M)^*$ then x – y is an edge of AG (M)
vi) If x- y is not an edge of $AG(M)$for some distinct x, y $\in Z(M)^*$, then there is a $w \in Z(M)^* - \{x,y\}$ such that x – w – y is a path in Γ(M) and hence x – w – y ia also a path in AG (M).

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Proof:

i) Suppose that x – y is not an edge of AG (M). Then $ann_M(x\gamma y) = ann_M(x) \cup ann_M(y)$ (by Definition 2.1).Since $ann_M(x\gamma y)$ is a union of two ideals, We have, $ann_M(x\gamma y) = ann_M(x)\ or\ ann_M(x\gamma y) = ann_M(y)$. Conversely, suppose that $ann_M(x\gamma y) = ann_M(x)\ or\ ann_M(x\gamma y) = ann_M(y)$. Then $ann_M(x\gamma y) = ann_M(x) \cup ann_M(y)$ and thus x – y is not an edge of AG (M)

ii) Suppose x – y is an edge of Γ(M) for some distinct x, y ∈ Z$(M)^*$.Then $x\gamma y = 0$ and hence $ann_M(x\gamma y) = M$. Since $x \neq 0$ and $y \neq 0, ann_M(x) \neq M\ and\ ann_M(y) \neq M$. Thus x- y is an edge of AG (M). The 'in particular' is now clear.

iii) Suppose x – y is not an edge of AG (M) for some distinct x, y ∈ Z$(M)^*$. Then $ann_M(x) \cup ann_M(y) = ann_M(x\gamma y)$. Since $ann_M(x\gamma y)$ is a union of two ideals, we have $ann_M(x) \subseteq ann_M(y)$ or $ann_M(y) \subseteq ann_M(x)$

iv) This statement is now clear by (iii)

v) Suppose that $d_{\Gamma(M)}(x, y) = 3$ for some distinct x, y ∈ Z$(M)^*$. Then $ann_M(x) \nsubseteq ann_M(y)$ and $ann_M(y) \nsubseteq ann_M(x)$. Hence x – y is an edge of AG (M)by (iv)

vi) Suppose that x- y is not an edge of $AG(M)$ for some distinct x, y ∈ Z$(M)^*$.Then there is a $w \in ann_M(x) \cup ann_M(y)$ such that w≠ 0 by (iii). Since $x\gamma y \neq 0$ we have $w \in$ Z$(M)^* - \{x, y\}$.Hence x – w – y is a path in Γ(M) and thus x – w – y ia also a path in AG (M). by (iii).

In view of lemma 2.2, we have the following result

Theorem 2.3: Let M be a commutative Γ – near – ring with$|Z(M)^*| \geq 2$. Then $AG(M)$ is connected and $diam$(AG (M))≤ 2

Proof: obvious.

Lemma 2.4: Let M be a commutative Γ – near – ring and let x, y be distinct non zero elements. Suppose that x - y is an edge of AG (M) that is not an edge of Γ(M) for some distinct x, y ∈ Z$(M)^*$. If there is a $w \in ann_M(x\gamma y) - \{x, y\}$ such that $w\gamma x \neq 0$ and $w\gamma y \neq 0$, then x – w – y is a path in AG (M) that is not a path in Γ(M) and hence C : x – w – y – x is a cycle in $AG(M)$of length three and each edge of C is not an edge of Γ(M)

Proof: Suppose that x – y is an edge in AG (M) that is not an edge in Γ(M). Then $x\gamma y \neq 0$. Assume there is a $w \in ann_M(x\gamma y) - \{x, y\}$ such that $w\gamma x \neq 0$ and $w\gamma y \neq 0$. Since $y \in ann_M(x\gamma w) - (ann_M(x) \cup ann_M(w))$. We concludes that x – w is an edge of AG (M). Since $x \in ann_M(y\gamma w) - (ann_M(y) \cup ann_M(w))$.We conclude that y – w is an edge of AG(M).Hence x – w – y is a path in AG (M).Since $x\gamma w \neq 0$ and $y\gamma w \neq 0$, we have x – w – y is not a path in Γ(M).It is clear that x – w – y – x is a cycle in $AG(M)$ of length three and each edge of C is not an edge of Γ(M).

Theorem 2.5: Let M be a commutative Γ – near – ring. Suppose that x – y is an edge of AG (M) that is not an edge of Γ(M) for some distinct x, y ∈ Z$(M)^*$. If $x\gamma y^2 \neq 0$ and $x^2\gamma y \neq 0$, then there is a w ∈ Z$(M)^*$ such that x – w – y is a path in AG (M)that is not a path in Γ(M) and hence C : x – w – y – x is a cycle in AG (M)of length three and each edge of C is not an edge of Γ(M).

Proof: Suppose that x – y is an edge of AG (M) that is not an edge of Γ(M).Then $x\gamma y \neq 0$ and there is a $w \in ann_M(x\gamma y) - (ann_M(x) \cup ann_M(y))$. We show $w \notin \{x, y\}$.Assume $w \in \{x, y\}$.Then either $x^2\gamma y = 0$ or $y^2\gamma x = 0$, which is a contradiction. Thus $w \notin \{x, y\}$.Hence x – w – y is the desired path in $AG(M)$ by Lemma 2.4

Corollary 2.6: Let M be a reduced commutative Γ – near – ring. Suppose that x – y is an edge of AG (M) that is not an edge of Γ(M) for some distinct,$y \in Z(M)^*$. Then there is a $w \in ann_M(x\gamma y) - \{x, y\}$ such that x – w – y is a path in AG (M) that is not a path in Γ(M) and AG (M)contains a cycle C of length 3 such that at least two edges C are not the edges of Γ(M).

Proof: Suppose that x – y is an edge of AG (M) that is not an edge of Γ(M) for some distinct x,$y \in Z(M)^*$. Since M is reduced, we have$(x\gamma y)^2 \neq 0, \gamma \in \Gamma$. This implies $x^2\gamma y \neq 0\ and\ x\gamma y^2 \neq 0$. Thus the claim is now clear by Theorem 2.5.

Corollary 2.7: Let M be a reduced commutative Γ – near – ring and suppose that AG (M) $\neq \Gamma(M)$. Then gr($AG(M)$) = 3. Moreover, there is a cycle C of length 3 in $AG(M)$ such that at least two edges of C are not the edges of $\Gamma(M)$.

Proof: Since AG (M) $\neq \Gamma(M)$, there are some distinct $x, y \in Z(M)^*$such that x – y is an edge of AG (M) that is not an edge of $\Gamma(M)$. Since M is reduced, we have $(x\gamma y)^2 \neq 0, \gamma \in \Gamma$. This implies $x^2\gamma y \neq 0\ and\ x\gamma y^2 \neq 0$. Thus the claim is now clear by Theorem 2.5.

Theorem 2.8: Let M be a commutative $\Gamma -$ near – ring and suppose that $AG(M) \neq \Gamma(M)$ with gr(AG (M)) $\neq 3$. Then there are some distinct $x, y \in Z(M)^*$ such that x – y is an edge of AG (M) that is not an edge of $\Gamma(M)$ and there is no path of length 2 from x to y in $\Gamma(M)$.

Proof: Since AG (M) $\neq \Gamma(M)$, there are some distinct $x, y \in Z(M)^*$such that x – y is an edge of AG (M) that is not an edge of $\Gamma(M)$. If possible suppose that x – w – y is a path of length 2 in $\Gamma(M)$. Then x – w – y is a path of length 2 in AG (M) by lemma 2.2(i). Therefore x – w – y – x is a cycle of length 3 in AG (M) and hence gr (AG (M)) = 3, a contradiction. Thus there is no path of length from x to y in $\Gamma(M)$.

Lemma 2.9: Let M be a reduced $\Gamma -$ near – ring that is not an gamma near- integral domain and let $z \in Z(M)^*$. Then

i) $ann_M(x) = ann_M(z^n)$ for each positive integer $n \geq 2$

ii) If $c + z \in Z(M) for\ some\ c \in ann_R(z) \setminus \{0\}\ then\ ann_R(z + c) \subset ann_R(z)$ ((ie) $ann_M(c + z) \subset ann_M(z)$) In particular if Z(M) is an ideal of M and $c \in ann_M(x) - \{0\}$, then $ann_M(z + c)$ is properly contained in $ann_M(z)$.

Proof:

i) Let $n \geq 2$. It is clear that $ann_M(z) \subseteq ann_M(z^n)$ let $f \in ann_M(z^n)$. Since $f\gamma z^n = 0\ and$ M is reduced, we have $f\gamma z = 0.$ Thus $ann_M(z^n) = ann_M(z)$.

ii) Let $c \in ann_M(z) \setminus \{0\}$ and suppose that $c + z \in Z(M)$.Since $z^2 \neq 0$, we have $c + z \neq 0$ and hence $c + z \in Z(M)^*$.Since $c \in ann_M(z)$ and M is reduced, we have $c \notin ann_M(c + z)$. Hence $ann_M(c + z) \neq ann_M(z)$. Since $ann_M(c + z) \subset ann_M\big(z\gamma(c + z)\big) = ann_M(z^2)$ and $ann_M(z^2) = ann_M(z)\ by (i)$ It follows that $ann_M(c + z) \subset ann_M(z)$.

Lemma 2.10: Let M be a commutative $\Gamma -$ near – ring. Then diam $(\Gamma(M)) = 2$ if and only if either of the following is true:

i) M is reduced with exactly two minimal primes and at least three non – zero zero – divisors or

ii) Z(M) is an ideal whose square is not $\{0\}$ and each pair of distinct zero – divisors has a non – zero annihilator.

Theorem 2.11: Let M be a reduced $\Gamma -$ near – ring with$|Min(M)| \geq 3$. (Possibly Min (M) is infinite) Then AG (M) $\neq \Gamma(M)$ and gr $(AG(M))$ = 3

Proof: If Z(M) is an ideal of M then AG (M) $\neq \Gamma(M)$ by Theorem 2.3 Hence assume that Z (M) is not an ideal of M. Since $|Min(M)| \geq 3$,we have diam $(\Gamma(M)) = 3$ by lemma 2.10(ii) and thus AG (M) $\neq \Gamma(M)$ by Theorem 2.3.Since M is reduced and AG (M) $\neq \Gamma(M)$, we have gr $(AG(M))$ = 3.

Theorem 2.12: Let M be a reduced $\Gamma -$ near – ring that is not an gamma near- integral domain. Then AG (M) $= \Gamma(M)$ if and only if $|Min(M)| = 2$

Proof: Suppose that AG (M) $= \Gamma(M)$.Since M is a reduced $\Gamma -$ near – ring that is not an gamma near- integral domain $|Min(M)| = 2$ by Theorem 2.5.Conversely, suppose that $|Min(M)| = 2$.Let P_1, P_2 be the minimal prime ideals of M. Since M is reduced, we have Z (M) = $P_1 \cup P_2$ and $P_1 \cap P_2 = \{0\}$. Let a, b $\in Z(M)^*$.Assume that a, b $\in P_1$.Since $P_1 \cap P_2 = \{0\}$ neither a $\in P_2$ nor b $\in P_2$ and thus $a\gamma b \neq 0$.Since $P_1 \Gamma P_2 \subseteq P_1 \cap P_2 = \{0\}$, it follows that $ann_M(a\gamma b) = ann_M(a) = ann_M(b) = P_2$. Thus a – b is not an edge of $AG(M)$. Similarly, if a, b $\in P_2$ then a – b is not an edge of $AG(M)$ If a $\in P_1$ b $\in P_2$then $a\gamma b = 0$ and thus a – b is an edge of AG (M).Hence each edge of $AG(M)$is an edge of $\Gamma(M)$ and therefore AG (M) $= \Gamma(M)$

For the remainder of this section, we study the case when M is non reduced

Theorem 2.13: Let M be a non reduced $\Gamma -$ near – ring with $|Nil(M)^*| \geq 2$ and let $AG_N(M)$ be the (induced) sub graph of AG (M) with vertices $Nil(M)^*$. Then $AG_N(M)$ is complete.

Proof: Suppose there are non – zero distinct elements $a, b \in Nil(M)$ such that $a\gamma b \neq 0, \gamma \in \Gamma$.Assume that $ann_M(a\gamma b) = ann_M(a) \cup ann_M(b)$. Hence $ann_M(a\gamma b) = ann_M(a) or\ ann_M(a\gamma b) = ann_M(b)$.Without loss of generality, we may assume that $ann_M(a\gamma b) = ann_M(a)$.Let n be the least positive integer such that $b^n = 0$.Suppose that $a\gamma b^k \neq 0$ for each k, $1 \leq k \leq n$. Then $b^{n-1} \in ann_M(a\gamma b) \setminus ann_M(a)$, a contradiction. Hence assume that k, $1 \leq k \leq n$ is the least positive integer such that $a\gamma b^k = 0$. Since $a\gamma b \neq 0, 1 < k < n$. Hence $b^{k-1} \in ann_M(a\gamma b) - ann_M(a)$, a contradiction. Thus a – b is an edge of $AG_N(M)$.

Theorem 2.14: Let M be a non reduced $\Gamma -$ near – ring with $|Nil(M)^*| \geq 2$ and let $\Gamma_N(M)$ be the induced sub graph of Γ(M) with vertices Nil (M)*. Then Γ_N(M) is complete if and only if Nil $(M)^2$={0}.

Proof: If Nil $(M)^2$={0}, then it is clear that Γ_N(M) is complete. Hence assume that Γ_N(M) is complete. We need only show that $w^2 = 0$ for each w ϵ Nil (M)*. Let w ϵ Nil (M)* and assume that $w^2 \neq 0$. Let n be the least positive integer such that $w^n = 0$. Then n$\geq$ 3. Thus $w\gamma(w^{n-1} + w)$=0 and $w^n = 0$.We have $w^2 = 0 \Rightarrow\Leftarrow$.Thus $w^2 = 0$ for each w $\in$ Nil (M).

Theorem 2.15: Let M be a $\Gamma -$ near – ring such that AG (M) $\neq$ Γ(M). Then the following statements are equivalent

i. Γ(M) is a star graph
ii. $\Gamma(M) = K_{1,2}$
iii. AG (M) = K_3

Proof:
(i)=>(ii): Since gr (Γ(M)) = ∞ and AG (M)$\neq$ Γ(M),We have M is non reduced by Theorem 2.11 and $|Z(M)^*| \geq 3$. Since Γ(M) is a star graph, there are two sets A, B such that $Z(R)^* = A \cup B$ with $|A| = 1, A \cap B = \emptyset, A\gamma B = \{0\}$ and $b_1\gamma b_2 \neq 0$ for every $b_1, b_2 \in B$. Since $|A| = 1$, we may assume that $A = \{w\}$ for some $w \in Z(M)^*$. Since each edge of Γ(M) is an edge of $AG(M)$ and AG (M) $\neq$ Γ(M), there are some x, y $\in B$ such that $x\gamma y$ is an edge of Γ(M), but not an edge of $Ann_G(M)$. Since $ann_M(c) = w\ for\ each\ c \in B\ and\ ann_M(x\gamma y) \neq ann_M(x) \cup ann_M(y)$

We have $ann_M(x\gamma y) \neq w$. Thus $ann_M(x\gamma y) = B\ and\ x\gamma y = w$. Since A = $\{x\gamma y\}\ and\ A\gamma B = \{0\}$. We have $(x\gamma y)\gamma x = x^2\gamma y = 0$ and $(x\gamma y)\gamma y = y^2\gamma x = 0$. We show that B ={x, y} and hence $|B| = 2$. Thus assume there is a $c \in B$ such that $c \neq x\ and\ c \neq y$.Then w$\gamma c = x\gamma y\gamma c = 0$.We show that$(x\gamma c + x\gamma y) \neq x\ and\ (x\gamma c + x\gamma y) \neq x\gamma y(note\ that\ x\gamma y = w)$.Suppose that $(x\gamma c + x\gamma y) = x$.Then $(x\gamma c + x\gamma y)\gamma y = x\gamma c\gamma y + x\gamma y^2 = 0\ and\ x\gamma y = 0, a\ contradiction.$ Hence $x \neq (x\gamma c + x\gamma y)$.Since $x, c \in B\ we\ have\ x\gamma c \neq 0\ and\ thus\ (x\gamma c + x\gamma y), x\gamma y\ are\ distinct\ elements\ of\ Z(M)^*$.

Since $x^2\gamma y = 0\ and\ y \in B\ either\ x^2 = 0\ or\ x^2 = x\gamma y\ or\ x^2 = y$. Suppose that $x^2 = y$. Since $x\gamma y = w \neq 0$. We have $x\gamma y = x\gamma(x^2) = x^3 = w \neq 0$. Since $x^2\gamma y = 0,\ we\ have\ x^4 = 0$. Since $x^4 = 0,\ and\ x^3 \neq 0, we\ have$ $x^2, x^3, x^2 + x^3$ are distinct elements of $Z(M)^*$, and thus $x^2 - x^3 - x^2 + x^3 - x^2$ is a cycle of length three in Γ(M)=><=. Hence we assume that either $x^2 = 0\ or\ x^2 = x\gamma y = w$. In both cases, we have $x^2\gamma c = 0$. Since $x, (x\gamma c + x\gamma y), x\gamma y$ are distinct elements of $Z(M)^*$ and $x\gamma y^2 = \gamma y x^2 = x^2\gamma c = 0$. We have $x - (x\gamma c + x\gamma y) - x\gamma y - x$ is a cycle of length three in Γ(M) =><=. Thus B = {x, y} and $|B| = 2$. Hence $\Gamma(M) = K_{1,2}$.

(ii)=>(iii): Since each edge of Γ(M) is an edge of AG (M) and AG (M) $\neq$ Γ(M), and $\Gamma(M)=K_{1,2}$. It is clear that AG (M) must be K_3.

(iii)=>(i): Since $|Z(M)^*| = 3$ and Γ(M) is connected and AG (M) $\neq$ Γ(M) exactly one edge of $AG(M)$ is not an edge of Γ(M).Thus Γ(M) is a star graph.

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