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MORE TRIGONOMETRIC IDENTITIES
MAIN SUBJECT CLASSIFICATION: MATHEMATICS EDUCATION 97 G60 PLANE AND SPHERICAL TRIGONOMETRY

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ABSTRACT

There are certain trigonometric identities which look formidable and intimidating from the point of view of their solutions and appear tedious and cumbersome also. In ordinary ways, these seem practically unsolvable until we know the exact method to solve these. Most of the identities being considered here, according to me, are not known earlier and do not find mention in textbooks but attempt has been made to present and solve these by applying unique and different methods. Trigonometric ratios of sine and cosine are expandable in series expansion and also their multiple angles are expressible in single angle but their finite product is hardly found in textbooks, however, infinite product is commonly given. Attempt has been made to factorise these ratios and express these as finite product. In doing so, it turns out that such multiplication leads to Chebyshev polynomials. Hence roots of Chebyshev polynomials can be found out which are otherwise difficult to determine considering the degree n of the polynomials. To start with, a few such identities are given below, thereafter, their General formulae are derived and then sine and cosine of multiple angles are factorised and used in solving Chebyshev polynomials.

1. *(Cos^2 pi/3- cos^2 pi/250).(cos^2 pi/3- cos^2 3.pi/250).(cos^2 pi/3- cos^2 7pi/250).(cos^2 pi/3- cos^2 9pi/250) .(cos^2 pi/3- cos^2 11.pi/250).(cos^2 pi/3- cos^2 13.pi/250).(cos^2 pi/3- cos^2 17pi/250).(cos^2 pi/3-cos^2 19pi/250). (cos^2 pi/3- cos^2 21.pi/250).(cos^2 pi/3- cos^2 23pi/250).(cos^2 pi/3- cos^2 27pi/250).(cos^2 pi/3- cos^2 29.pi/250).(cos^2 pi/3- cos^2 31.pi/250).(cos^2 pi/3- cos^2 33.pi/250).(cos^2 pi/3- cos^2 37pi/250). (cos^2 pi/3- cos^2 39pi/250).(cos^2 pi/3- cos^2 41pi/250).(cos^2 pi/3- cos^2 43pi/250).(cos^2 pi/3- cos^2 47pi/250).(cos^2 pi/3- cos^2.49pi/250).(cos^2 pi/3- cos^2 51pi/250).(cos^2 pi/3- cos^2 53/250). (cos^2 pi/3- cos^2 57.pi/250).(cos^2 pi/3- cos^2 59.pi/250). (cos^2 pi/3- cos^2 61.pi/250). (cos^2 pi/3- cos^2 63.pi/250). (cos^2 pi/3- cos^2 67.pi/250).(cos^2 pi/3- cos^2 69pi/250).(cos^2 pi/3- cos^2 71.pi/250).(cos^2 pi/3- cos^2 73.pi/250).(cos^2 pi/3- Cos^2 77pi/250).(cos^2 pi/3- cos^2 79pi/250).(cos^2 pi/3- cos^2 81.pi/250). (cos^2 pi/3- cos^2 83.pi/250).(cos^2 pi/3- cos^2 87.pi/250).(cos^2 pi/3- cos^2 89pi/250).(cos^2 pi/3-cos^2 91pi/250).(cos^2 pi/3-Cos^2 93.pi/250).(cos^2 pi/3- cos^2 97pi/250).(cos^2 pi/3-cos^2 99.pi/250). (cos^2 pi/3- cos^2 101/250). (cos^2 pi/3- cos^2 103pi/250) (cos^2 pi/3- cos^2 107pi/250).(cos^2 pi/3- cos^2 109.pi/250).(cos^2 pi/3- cos^2 111pi/250).(cos^2 pi/3- cos^2 113.pi/250). (cos^2 pi/3- cos^2 117pi/250).(cos^2pi/3-cos^2 119pi/250). (cos^2 pi/3- cos^2 121pi/250).(cos^2 pi/3-cos^2 123/250) = (½)^100.*

2. *(Cos pi/3- cos pi/131). (cos pi/3+ cos 2pi/131) (cos pi/3- cos 3pi/131) (cos pi/3+ cos 4pi/131) (cos pi/3- cos 5pi/131).(cos pi/3+ cos 6pi/131) (cos pi/3- cos 7pi/131).(cos pi/3+ cos 8pi/131).(cos pi/3-cos 9pi/131).(cos pi/3+ cos 10pi/131).(cos pi/3- cos 11pi/131) (cos pi/3+ cos 12pi/131). (cos pi/3- cos 13pi/131) (cos pi/3+ cos 14pi/131).(cos pi/3- cos 15pi/131). (cos pi/3+ cos 16pi/131). (cos pi/3-cos 17pi/131) (cos pi/3+ cos 18pi/131). (cos pi/3- cos 19pi/131). (cos pi/3+ cos 20pi/131).(cos pi/3- cos 21pi/131).(cos pi/3+ cos 22pi/131). (cos pi/3- cos 23pi/131)(cos pi/3+ cos 24pi/131).(cos pi/3- cos 25pi/131).(cos pi/3+ cos 26pi/131). (cos pi/3- cos 27pi/131).(cos pi/3+ cos 28pi/131) (cos pi/3- cos 29pi/131).(cos pi/3+ cos 30pi/131).(cos pi/3- cos 31pi/131). (cos pi/3+ cos 32pi/131).(cos pi/3-cos 33pi/131).(cos pi/3+ cos 34pi/131).(cos pi/3- cos 35pi/131). (cos pi/3+ cos 36pi/131) (cos pi/3-cos 37pi/131).(cos pi/3+ cos 38pi/131).(cos pi/3- cos 39pi/131).(cos pi/3+ cos 40pi/131).(cos pi/3- cos 41pi/131).(cos pi/3+ cos 42pi/131).(cos pi/3- cos 43pi/131).(cos pi/3+cos 44pi/131). (cos pi/3- cos 45pi/131).(cos pi/3+ cos 46pi/131).(cos pi/3- cos 47pi/131).(cos pi/3+ cos 48pi/131). (cos pi/3-cos 49pi/131).(cos pi/3+ cos 50pi/131).(cos pi/3- cos 51pi/131).(cos pi/3+ cos 52pi/131).(cos pi/3-cos 53pi/131). (cos pi/3+ cos 54pi/131).(cos pi/3- cos 55pi/131).(cos pi/3+ cos 56pi/131).(cos pi/3- cos 57pi/131). (cos pi/3+ cos 58pi/131).(cos pi/3- cos 59pi/131).(cos pi/3+ cos 60pi/131).(cos pi/3- cos 61pi/131).(cos pi/3+ cos 62pi/131).(cos pi/3- cos 63pi/131).(cos pi/3+ cos 66pi/131).(cos pi/3- cos 67pi/131) = (½)^65.*

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Formulae on which these are based are given below. In addition, other identities are also derived using simple but unconventional methods of factorisation of sine and cosine of multiple angle. Symbols Π denotes multiplication of terms and symbol ^ denotes power to which the base is raised. That is 2^3 = 8.

A. *(1//2+ cos 2pi/n).(1/2+ cos 3.2pi/n).(1/2+ cos 9.2pi/n).(1/2+ cos 27.2pi/n) (1/2+ cos 81.2pi/n)………………. [1/2+ cos (3^(k-1)).2pi/n] = (½)^k*

In other words, $$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2\pi/n\} = (1/2)^k,$$

also by putting x = pi/n, $$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2x\} = (1/2)^k,$$

where integers m and k are given by relation (3^k + 1)/n = m and m is odd. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

OR

where integers m and k are given by relation (3^k - 1)/n = m and m is even. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

B. *(1/2+ cos 2pi/n).(1/2+ cos 3.2pi/n).(1/2+ cos 9.2pi/n).(1/2+ cos 27.2pi/n) (1/2+ cos 81.2pi/n)…[1/2+ cos (3^(k-1)).2pi/n] = -(½)^k*

In other words, $$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2\pi/n\} = -(1/2)^k,$$

also by putting x = pi/n, $$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2x\} = -(1/2)^k,$$

where integers m and k are given by relation (3^k + 1)/n = m and m is even. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

OR

where integers m and k are given by relation (3^k- 1)/n = m and m is odd. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

C. *When (2^k + 1)/n = m and m is odd*

or

when (2^k - 1)/n = m and m is even.

Cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n.2^(k-1) = (½)^k

or

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.\pi/n = (1/2)^k,$$

when integers k and m are given by relation (2^k + 1)/n= m and m is odd or relation (2^k - 1)/n = m and m is even for all odd n.

And also by putting π/n = x,

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x = (1/2)^k,$$

when integers m and k are given by relation (2^k + 1).x= m.π and m is odd

or

when integers m an k are given by relation (2^k - 1)/x= m.π and m is even. And n is odd in all cases.

For cases where (2^k + 1)/n = m and m is even

or

where (2^k - 1)/n = m and m is odd.

$$\prod_{j=1}^{k} \cos 2^{(j-1)}. \pi/n = -(1/2)^k,$$

when integers k and m are given by relation (2^k + 1)/n= m and m is even or relation (2^k - 1)/n = m and m is odd and also n is odd.

It can be written i n angle x form by putting π/n = x as

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x = -(1/2)^k,$$

when integers m and k are given by relation $(2^k + 1).x= m.\pi$ and m is odd

or

when integers m an k are given by relation $(2^k - 1)/n = m.\pi$ and m is even. And n is odd in all cases.

***D.**1).* $\cos n.x = 2^{(n-1)}.\{\cos x- \cos \pi/2n\}.\{\cos x- \cos 3.\pi/2n. \{\cos x- \cos 5.\pi/2n\}.......\{\cos x- \cos (2n- 1).\pi/2n\}$

Or

$$\cos n.x = 2^{(n-1)}. \prod_{k=0}^{n-1} \{\cos x- \cos (2k+ 1).\pi/2n\}$$

(2).
$$\prod_{k=0}^{n-1} \{1- \cos (2k+ 1).\pi/2n\}= (1/2)^{(n-1)}$$

or

$$\prod_{k=0}^{n-1} \sin (2k+ 1).\pi/4n = (1/2)^{(n-1/2)}$$

(3). When n is even,

$$\prod_{k=0}^{n-1} \{\cos (2k+ 1).\pi/2n = (-1)^{n/2}. (1/2)^{(n-1)}$$

***E.** (1) When n is odd and more than 1.*

$$\sin n.x =\{(-1)^{(n-1)/2}.2^{(n-1)}\}(\sin x).(\sin^2 x - \sin^2 \pi/n).(\sin^2 x - \sin^2 2\pi/n).(\sin^2 x - \sin^2 3\pi/n)......................, .(\sin^2 x- \sin^2 (n-1)\pi/2n)$$

Or

$$\sin n.x =\{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}.(\sin x) \prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)$$

Or

$$\sin n.x=\{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}. \prod_{k= -(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)]$$

(2). When n is odd and more than1,

$$\prod_{k= -(n-1)/2}^{(n-1)/2} \{1- \sin k\pi/n \}= (1/2)^{(n-1)}$$

Or

$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (1/2)^{(n-1)/2}$$

***F.** When n is even and more than 2,*

1.
$$\sin n.x= (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \sin x. \prod_{k=1}^{n/2-1} (\sin^2 x- \sin^2 k\pi/n)$$

Also when n is even and more than 2,

$$\sin n.x= (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \prod_{k= -n/2+1}^{n/2-1} (\sin x- \sin k\pi/n)$$

2.
$$\prod_{k=1}^{p} \cos k\pi/(4p+1) = (1/2)^{4p}$$

or $\cos \pi/(4p+1). \cos 2\pi/(4p+1). \cos 3\pi/(4p+1)..................... \cos 4p\pi/(4p+1)= (1/2)^{4p}.$

3.
$$\prod_{k=1}^{4p-2} \cos k\pi/(4p-1)= -(1/2)^{(4p-2)}$$

$\cos \pi/(4p-1). \cos 2\pi/(4p-1). \cos 3\pi/(4p-1)..................... \cos (4p-2)\pi/(4p-1)= -(1/2)^{(4p-2)}$

***Keywords:** Trigonometry Identities, Multiple Angles, Finite Product, Recursive Relation, Factorisation, Multiplication. Chebyshev Polynomial.*

1. INTRODUCTION

Cosine and sine of an angle x are ratios of base to hypotenuse, perpendicular to hypotenuse respectively in a right angled triangle with angle x between base and hypotenuse and these can be written as

cos x = {(1+ cos 2x)/2}^1/2 (1)

sin x = (1- cos 2x)/2}^1/2 (2)

on the basis of formula, cos (x + y) = cos x.cos y – cos x.cos y

by putting x = y. Sign ^ means, it is raised to power, for example sin^2 x = (sin x). (sin x).

Similarly, cos 3x and sin 3x can be written as

cos 3x = 4.cos^3 x – 3.cos x (3)

sin 3x = 3 .sin x -4.sin^3 x (4)

by expanding cos (x+ 2x) and sin (x+ 2x) so as to reduce the angle from 3x to x.

Before proceeding further, it would be appropriate to be familiar with recursive relation as it will be used repeatedly in this paper. A recursive relation is that where in an equation, quantity on left side also appears in right hand side. In this way, quantity on RHS can be successively substituted by the quantity in left hand side. Such a relation forms self generating and never ending loop. For example the recursive relation x = 1+ √x has x in the left hand side as well as in the right side. Therefore, x = 1+ √x can be substituted for x in right hand side and that makes x = 1+ (1+ √x)^1/2. On successive substitution we get,

x = x = 1+[1+ {1+………….(1+ √x)^1/2……}^1/2]^1/2

We will be using the recursive relation, angle doubling and tripling formula to achieve the desired result. With this introduction, we proceed to main derivations.

An angle can be mentioned in degrees or radians and can also be converted from degrees to radians and vice versa by the relation pi written as π = 180 degrees. Therefore one radian equals 180/ π, it is also defined as an angle subtended by an arc equal in length to radius at the centre of the circle. In this way, an angle x can be written in radians as π/n and also, it can be transformed to this form by rearranging it. n can be any integer except 0, n can be a fraction also of the form p/q. Using recursive relations, value of finite product of trigonometric identities will be determined. At the same time, cos n.x and sin n.x will also be expressed as finite product.

2. THEORY AND CONCEPT

2.1 Part 1.

First, those cases where n is any integer except multiple of 3 or is a fraction p/q where p is any integer except multiple of 3, will be taken up.

That means angle x= π/n must not have n multiple of 3. Angle x may be of the form for examples, π/62, π/8, π/7, π/31 π/(8/3), π/(56/3) or π divided by any integer except multiple of 3 or π/(p/q) where p is any integer not multiple of 3 and q is odd.

Equation (4) can be written as

Sin x.(3- 4.sin^2 x) = sin 3x or

sin x = sin 3x/(3- 4.sin^2 x) (5)

This relation is recursive in nature as sin x appears on left hand side and also as sin 3x in right hand side. Sin x in LHS and sin 3x in RHS in numerator are being considered for recurrence.

From recursive relation (5), sin 3 x can also be written as

sin 3x = sin 9x/(3- 4.sin^2 3x),

sin 9 x = sin 27 x/(3- 4.sin^2 9x)

…………………………so on.

Sin x.3^(k-1) = sin 3^k/[3- 4.sin^2 {x.3^(k-1)}]

On substituting value of sin 3x = sin 9x/(3- 4.sin^2 3x), relation (5) can be written as

sin x = sin 9x/(3- 4.sin^2 x).(3- 4.sin^2 3x).

On successively substituting values of sin 9x, sin 27x ……………… Sin x.3^(k-1), equation takes the form

Sin x = sin x.3^k/[(3- 4.sin^2 x).(3- 4.sin^2 3x).(3- 4.sin^2 9x).(3- 4.sin^2 27x)……… {3.sin^2 {x.3^(k-1)}].

Putting x = π/n, this equation is written as

Sin π/n = sin π/n.3^k/[(3- 4.sin^2 π/n).(3- 4.sin^2 3 π/n).(3- 4.sin^2 9 π/n).(3- 4.sin^2 27 π/n)…………..{3.sin^2 {π/n.3^(k-1)}] (6)

Next task is to find out k so that equation (6) simplifies. For that a proposition is given that follows.

2.2. Proposition

If n is any integer but not multiple of three or is of form p/q where q is not multiple of 3 then (3^k + 1) or (3^k - 1) will always be divisible by n for some integer value of k. It is obvious 3^k + 1 and 3^k-1 will always be even on account of the fact that odd integer plus 1 or – 1 is always even. Also division of even integer by some other even integer is possible.

Here k can assume any value from 0 and then proceeding to 1, then 2, then 3 so on till that value of k is reached where it gets divided by n without any remainder. There are thus infinite positive integers available as value of k. Since even can be divided by even, then there is complete possibility for division of (3^k + 1) or (3^k - 1) by n.

Let *m = (3^k + 1)/n or*
m= (3^k - 1)/n
where m is that minimum number where (3^k + 1) or (3^k -1) is just divisible by n.

2.3Application of Proposition

Applying this proposition that there exists a value of k that makes (3^k + 1) or (3^k -1) divisible by n, we can easily say that *denominator of equation (6) has factors which are k in number*. First factor pertains to angle π/n.3^0, second to π/n.3^1, third to π/n.3^2, fourth to π/n.3^3…………so on and kth to π/n.3^(k-1).

From equation (6), it is clear, it has numerator sin π/n.3^k and this pertains to angle π/n.3^k. It is already proved
(3^k + 1)/n = m, or
(3^k - 1)/n = m.

First taking the case where (3^k + 1)/n = m and on multiplying both sides by π, makes
π/n.3^k = (m π - π/n)
and numerator of RHS of equation (6) becomes
sin π/n.3^k = sin (m π - π/n) = sin π/n when m is odd or equals - sin π/n when m is even.

For the cases where (3^k - 1)/n = m, numerator of RHS of equation (6) becomes
sin π/n.3^k = sin (m π+ π/n) = sin π/n when m is even or it equals - sin π/n when m is odd.

When (3^k + 1)/n = m and m is odd but n is not multiple of 3
or
when when (3^k - 1)/n = m and also m is even but n is not multiple of 3, equation (6) takes the form,
sin π/n = sin π/n [(3- 4.sin^2 π/n).(3- 4.sin^2 3 π/n).(3- 4.sin^2 9 π/n).(3- 4.sin^2 27 π/n)……{3.sin^2 {π/n.3^(k-1)}].

On cancelling sin π/n on both sides, we get
[(3- 4.sin^2 π/n).(3- 4.sin^2 3 π/n).(3- 4.sin^2 9 π/n).(3- 4.sin^2 27 π/n)……… {3.sin^2 {π/n.3^(k-1)}] = 1 (7)

We know from equation (3) that
sin x= (1- cos 2x)/2}^1/2. This can also be written as
cos 2x = 1- 2 sin^2 x.

Using this identity in identity (7), we get
(1+ 2.cos 2π/n).(1+ 2.cos 6π/n).(1+ 2.cos 18π/n).(1+ 2.cos 54π/n)..................1+ 2.cos {2π/n.3^(k-1)}=1 (8)

This can also be written as
(1/2+ cos 2π/n).(1/2+ cos 6π/n).(1/2+ cos 18π/n).(1/2+ cos 54π/n)..........{1/2+ cos {2π/n.3^(k-1)}= (1/2)^k (9)

In other words, $\prod_{j=1}^{k}$ *{1/2+ cos 3^(j-1).2pi/n} = (½)^k,* (10)

also by putting x = pi/n, $\prod_{j=1}^{k}$ *{1/2+ cos 3^(j-1).2x} = (½)^k,* (11)

When (3^k + 1)/n = m and m is even but n is not multiple of 3,
or
when when (3^k - 1)/n = m and also m is odd but n is not multiple of 3, right hand side of identities (7), (8) and (9) will be negative.

Therefore, in such cases,

[(3- 4.sin^2 π/n).(3- 4.sin^2 3 π/n).(3- 4.sin^2 9 π/n).(3- 4.sin^2 27 π/n)………….{3.sin^2 {π/n.3^(k-1)}] = -1 (7/1)

(1+ 2.cos 2π/n).(1+ 2.cos 6π/n).(1+ 2.cos 18π/n).(1+ 2.cos 54π/n)...........1+ 2.cos {2π/n.3^(k-1)}= -1 (8/1)

(1/2+ cos 2π/n).(1/2+ cos 6π/n).(1/2+ cos 18π/n).(1/2+ cos 54π/n)..........{1/2+ cos {2π/n.3^(k-1)}=-(1/2)^k. (9/1)

In other words, $\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2pi/n\} = - (½)^k$.. (10/1)

also by putting x = pi/n, $\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2x\} = - (½)^k$, (11/1)

2.4 Examples 1:

Sin pi/5

Here n is 5 and let us find out m and k by relation

m = (3^k + 1)/n or

m= (3^k - 1)/n.

It satisfies the relation

m = (3^k + 1)/n when k = 2, that makes m equals 2. That means there are two factors as k =2 and their product is -1 from identities (9) and 11 since m is even. Therefore,

(1 + 2 cos 2.pi/5). (1 + 2 cos 6.pi/5)= -1,

Or (1 + 2 cos 2.pi/5). (1 - 2 cos pi/5)= -1

Or (1/2 + cos 2.pi/5). (1/2 - cos pi/5)= -(1/2)^2 = -1/4.

From this, value of cos pi/5 can also be found out.

(1 + 2 cos 2.pi/5). (1 - 2 cos pi/5)= -1

1 + 2 cos 2.pi/5- 2 cos pi/5 – 4.cos pi/5.cos 2.pi/5 = -1

Or 1+2 cos 2.pi/5- 2 cos pi/5 – 2.(cos 3pi/5 + cos pi/5)= -1

Or 1 + 2 cos 2.pi/5- 2 cos pi/5 + 2.cos 2pi/5 – 2.cos pi/5 = -1

Or 2 (cos pi/5 – cos 2pi/5) = 2

4 cos^2 pi/5 -2.cos pi/5 -1 = 0.

This is a quadratic in cos pi/5 and its roots roots are

(1+5^1/2)/4 and (1- 5^1/2)/4.

Ignoring negative value, we get

cos pi/5 = (1+5^1/2)/4 or *Golden ratio phi = 2.cos pi/5 = (1+5^1/2)/2.*

Thus the identity is useful in derivation of value of cos pi/5 or sin pi/5.

Example 2:

Sin pi/10

Here n is 10 and let us find out m and k by relation

m = (3^k + 1)/n or

m= (3^k - 1)/n.

It satisfies the relation

m = (3^k + 1)/n when k = 2, that makes m equals 1. That means there are two factors and their product is 1 from identity (9) since m is odd. Therefore,

(1 + 2 cos pi/5). (1 + 2 cos 3.pi/5)= 1.

Example 3:

Sin pi/80

Here n is 80 and let us find out m and k by relation

m = (3^k + 1)/n or

m= (3^k - 1)/n.

It satisfies the relation
m = (3^k - 1)/n when k = 4, that makes m equals 1. That means there are four factors and their product is -1 from identities (7/1) and (8/1) since m is odd and relation m = (3^k - 1)/n is satisfied. Therefore,
(1 + 2 cos pi/40). (1 + 2 cos 3.pi/40).(1 + 2 cos 9pi/40). (1 + 2 cos 27pi/40) = -1.
On simplification,
(1 + 2 cos pi/40). (1 + 2 cos 3.pi/40).(1 + 2 cos 9pi/40). (1 - 2 cos 13.pi/40) = -1.

Example 5:
Sin pi/100

Here n is 100 and let us find out m and k by relation
m = (3^k + 1)/n or
m= (3^k - 1)/n.

It satisfies the relation
m = (3^k - 1)/n when k = 20, that makes m equals 34867844 which is even. That means there are 20 factors and their product is 1 from identities (8) and (10) since m is even and relation m = (3^k - 1)/n is satisfied. Therefore,
(1 + 2 cos pi/50). (1 + 2 cos 3.pi/50).(1 + 2 cos 9pi/50). (1 + 2 cos 27pi/50).(1 + 2 cos 3^4.pi/50). (1 + 2 cos 3^5.pi/50). (1 + 2 cos 3^6.pi/50). (1 + 2 cos 3^7.pi/50).(1 + 2 cos 3^8.pi/50). (1 + 2 cos 3^9.pi/50).(1 + 2 cos 3^10.pi/50). (1 + 2 cos 3^11.pi/50).(1 + 2 cos 3^12.pi/50). (1 + 2 cos 3^13.pi/50).(1 +2 cos 3^14.pi/50). (1+2 cos 3^15.pi/50). (1 + 2 cos 3^16.pi/50). (1 + 2 cos 3^17.pi/50).(1 + 2 cos 3^18.pi/50). (1 + 2 cos 3^19.pi/50) = 1
Or (1 + 2 cos pi/50). (1 + 2 cos 3.pi/50).(1 + 2 cos 9pi/50). (1 - 2 cos 23pi/50).(1 + 2 cos 19.pi/50). (1 - 2 cos 7pi/50). (1 - 2 cos 21.pi/50). (1 + 2 cos 13.pi/50).(1 - 2 cos 11.pi/50). (1 + 2 cos 17.pi/50).(1 - 2 cos.pi/50). (1-2 cos 3pi/50). (1 - 2 cos 9.pi/50). (1 + 2 cos 23.pi/50).(1- 2 cos 19.pi/50). (1 + 2 cos 7.pi/100). (1+2 cos 21.pi/50). (1-2 cos 13pi/50). (1 + 2 cos 11.pi/50). (1- 2 cos 17.pi/50) = 1

Or (1- 4 cos^2 pi/50). (1- 4.cos^2 3pi/50).(1–4.cos^27pi/50).(1 – 4 cos^2 9pi/50).(1-4cos^2 11.pi/50) (1–4cos^2 13.pi/50). (1 – 4 cos^2 17.pi/50). (1 – 4 cos^2 19.pi/50)(1 – 4 cos^2 21.pi/50). (1 – 4 cos^2 23pi/50) = 1
Or *(1/4 - cos^2 pi/50). (1/4 - cos^2 3pi/50).(1/4 – cos^2 7pi/50).(1/4 – cos^2 9pi/50).(1/4 -cos^2 11.pi/50).* (1/4 – cos^2 13.pi/50).(1/4 – cos^2 17.pi/50).(1/4 – cos^2 19.pi/50).(1/4 –cos^2 21.pi/50).(1/4– cos^2 23pi/50) =(½)^20.

2.3b How to avoid excessive and difficult to calculate 3^k while calculating m in relations
m = (3^k + 1)/n or
m= (3^k - 1)/n.

A practical difficulty is observed while finding m particularly when power k reaches higher values. To avoid such difficulty, basic trigonometric identities may be applied to contain value of k. Basic identities which can be utilised are
Sin (pi- x) = sin x, cos (pi- x) = - cos x
Sin (pi+ x)= - sin x. cos (pi+ x) = - cos x
Sin (2pi – x) = - sin x cos (2pi - x) = cos x
Sin (2pi + x) = sin x cos (2pi + x) = sin x.

These identities can be extended as
Sin {(2k+ 1)pi- x} = sin x, cos {(2k+ 1)pi- x} = -cos x,
Sin {(2k+ 1)pi+ x} = - sin x cos {(2k+ 1)pi+ x} = -cos x,
Sin (2.k.pi+ x)= sin x cos (2.k.pi+ x)= cos x.
Sin (2.k.pi – x) = - sin x cos (2.k.pi - x)= cos x
where k is any number 0, 1, 2, 3…..

As and when value of angle (3^k)/n increases above ½, it can be brought down by the above identities, k is any number for calculating m. However, signs positive and negative must be kept in mind. The process should be continued till value of angle reaches pi.(n+ 1)/n or pi.(n -1)/n. Sign (positive or negative) of sin pi.(n+ 1)/n or sin pi.(n- 1)/n as the case may be, found out. This sign of sin pi.(n+ 1)/n or sin pi.(n- 1)/n then will be the sign of right hand side of the identity. For illustration, I take up the following example.

Example 4:
Sin pi/500.
This can be expanded as
Sin pi/500 = sin pi/500/ [(3- 4.sin^2 pi/500). (3- 4.sin^2 3.pi/500).(3- 4.sin^2 9pi/500).(3- 4.sin^2 27pi/500). (3- 4.sin^2 81pi/500).(3- 4.sin^2 243pi/500). (3+ 4.sin^2 229pi/500) (3- 4.sin^2 187pi/500). (3+ 4.sin^2 61pi/500). (3+ 4.sin^2 183pi/500). (3- 4.sin^2 49pi/500). (3- 4.sin^2 147pi/500). (3- 4.sin^2 59pi/500). (3- 4.sin^2 177pi/500). (3+ 4.sin^2 31pi/500). (3+ 4.sin^2 93pi/500). (3+ 4.sin^2 221pi/500). (3- 4.sin^2 163pi/500). (3- 4.sin^2 11pi/500).

(3- 4.sin^2 33pi/500) (3- 4.sin^2 99pi/500). (3- 4.sin^2 203pi/500). (3+ 4.sin^2 109pi/500). (3+ 4.sin^2 173pi/500). (3- 4.sin^2 19pi/500). (3- 4.sin^2 57pi/500). (3- 4.sin^2 171pi/500). (3+ 4.sin^2 13pi/500). (3+ 4.sin^2 39pi/500). (3+4.sin^2 117pi/500). (3+ 4.sin^2 149pi/500). (3+ 4.sin^2 53pi/500). (3+ 4.sin^2 159pi/500). (3+ 4.sin^2 23pi/500). (3+ 4.sin^2 69pi/500). (3+ 4.sin^2 207pi/500). (3- 4.sin^2 121pi/500). (3- 4.sin^2 137pi/500). (3- 4.sin^2 89pi/500). (3- 4.sin^2 233pi/500). (3+ 4.sin^2 199pi/500). (3- 4.sin^2 97pi/500). (3- 4.sin^2 209pi/500). (3+ 4.sin^2 127pi/500). (3+ 4.sin^2 119pi/500). (3+ 4.sin^2 143pi/500). (3+ 4.sin^2 71pi/500). (3+ 4.sin^2 213pi/500). (3- 4.sin^2 139pi/500). (3- 4.sin^2 83pi/500). (3- 4.sin^2 249pi/500). (3+ 4.sin^2 247pi/500). (3- 4.sin^2 241pi/500). (3+ 4.sin^2 223pi/500). (3- 4.sin^2 169pi/500). (3+ 4.sin^2 7pi/500). (3+ 4.sin^2 21pi/500). (3+ 4.sin^2 63pi/500). (3+ 4.sin^2 189pi/500). (3- 4.sin^2 67pi/500). (3- 4.sin^2 201pi/500). (3+ 4.sin^2 103pi/500). (3+ 4.sin^2 191pi/500). (3- 4.sin^2 73pi/500). (3- 4.sin^2 219pi/500). (3+ 4.sin^2 157pi/500). (3+ 4.sin^2 29pi/500). (3+ 4.sin^2 87pi/500). (3+ 4.sin^2 239pi/500). (3- 4.sin^2 217pi/500). (3+ 4.sin^2 151pi/500). (3+ 4.sin^2 47pi/500). (3+ 4.sin^2 141pi/500). (3+ 4.sin^2 77pi/500). (3+ 4.sin^2 231pi/500). (3- 4.sin^2 193pi/500). (3+ 4.sin^2 79pi/500). (3+ 4.sin^2 237pi/500). (3- 4.sin^2 211pi/500). (3+ 4.sin^2 133pi/500). (3+ 4.sin^2 101pi/500). (3+ 4.sin^2 197pi/500). (3- 4.sin^2 91pi/500). (3- 4.sin^2 227pi/500). (3+ 4.sin^2 181pi/500). (3- 4.sin^2 43pi/500). (3- 4.sin^2 129pi/500). (3- 4.sin^2 113pi/500). (3- 4.sin^2 161pi/500). (3- 4.sin^2 17pi/500). (3- 4.sin^2 51pi/500). (3- 4.sin^2 153pi/500). (3- 4.sin^2 41pi/500). (3- 4.sin^2 123pi/500). (3- 4.sin^2 131pi/500). (3- 4.sin^2 107pi/500). (3- 4.sin^2 179pi/500). (3+ 4.sin^2 37pi/500). (3+ 4.sin^2 111pi/500). (3+ 4.sin^2 167pi/500)]

Here we have obtained lastly, - sin 501pi/500 and that is - sin (pi+pi/500) = sin pi/500 which has positive sign. In this way, there does not appear necessity of finding ‘m’ to decide sign (positive or negative) on right hand side. Here sign obtained is positive sin pi/500 without finding value of ‘m’. In this way we obtain identity as

[(3- 4.sin^2 pi/500). (3- 4.sin^2 3.pi/500).(3- 4.sin^2 9pi/500).(3- 4.sin^2 27pi/500). (3- 4.sin^2 81pi/500). (3- 4.sin^2 243pi/500). (3+ 4.sin^2 229pi/500) (3- 4.sin^2 187pi/500). (3+ 4.sin^2 61pi/500). (3+ 4.sin^2 183pi/500). (3- 4.sin^2 49pi/500). (3- 4.sin^2 147pi/500). (3- 4.sin^2 59pi/500). (3- 4.sin^2 177pi/500). (3+ 4.sin^2 31pi/500). (3+ 4.sin^2 93pi/500). (3+ 4.sin^2 221pi/500). (3- 4.sin^2 163pi/500). (3- 4.sin^2 11pi/500). (3- 4.sin^2 33pi/500) (3- 4.sin^2 99pi/500). (3- 4.sin^2 203pi/500). (3+ 4.sin^2 109pi/500). (3+ 4.sin^2 173pi/500). (3- 4.sin^2 19pi/500). (3- 4.sin^2 57pi/500). (3- 4.sin^2 171pi/500). (3+ 4.sin^2 13pi/500). (3+ 4.sin^2 39pi/500). (3+4.sin^2 117pi/500). (3+ 4.sin^2 149pi/500). (3+ 4.sin^2 53pi/500). (3+ 4.sin^2 159pi/500). (3+ 4.sin^2 23pi/500). (3+ 4.sin^2 69pi/500). (3+ 4.sin^2 207pi/500). (3- 4.sin^2 121pi/500). (3- 4.sin^2 137pi/500). (3- 4.sin^2 89pi/500). (3- 4.sin^2 233pi/500). (3+ 4.sin^2 199pi/500). (3- 4.sin^2 97pi/500). (3- 4.sin^2 209pi/500). (3+ 4.sin^2 127pi/500). (3+ 4.sin^2 119pi/500). (3+ 4.sin^2 143pi/500). (3+ 4.sin^2 71pi/500). (3+ 4.sin^2 213pi/500). (3- 4.sin^2 139pi/500). (3- 4.sin^2 83pi/500). (3- 4.sin^2 249pi/500). (3+ 4.sin^2 247pi/500). (3- 4.sin^2 241pi/500). (3+ 4.sin^2 223pi/500). (3- 4.sin^2 169pi/500). (3+ 4.sin^2 7pi/500). (3+ 4.sin^2 21pi/500). (3+ 4.sin^2 63pi/500). (3+ 4.sin^2 189pi/500). (3- 4.sin^2 67pi/500). (3- 4.sin^2 201pi/500). (3+ 4.sin^2 103pi/500). (3+ 4.sin^2 191pi/500). (3- 4.sin^2 73pi/500). (3- 4.sin^2 219pi/500). (3+ 4.sin^2 157pi/500). (3+ 4.sin^2 29pi/500). (3+ 4.sin^2 87pi/500). (3+ 4.sin^2 239pi/500). (3- 4.sin^2 217pi/500). (3+ 4.sin^2 151pi/500). (3+ 4.sin^2 47pi/500). (3+ 4.sin^2 141pi/500). (3+ 4.sin^2 77pi/500). (3+ 4.sin^2 231pi/500). (3- 4.sin^2 193pi/500). (3+ 4.sin^2 79pi/500).(3+ 4.sin^2 237pi/500). (3- 4.sin^2 211pi/500). (3+ 4.sin^2 133pi/500).(3+ 4.sin^2 101pi/500). (3+ 4.sin^2 197pi/500). (3- 4.sin^2 91pi/500). (3- 4.sin^2 227pi/500). (3+ 4.sin^2 181pi/500). (3- 4.sin^2 43pi/500). (3- 4.sin^2 129pi/500). (3- 4.sin^2 113pi/500). (3- 4.sin^2 161pi/500). (3- 4.sin^2 17pi/500). (3- 4.sin^2 51pi/500). (3- 4.sin^2 153pi/500). (3- 4.sin^2 41pi/500). (3- 4.sin^2 123pi/500). (3- 4.sin^2 131pi/500). (3- 4.sin^2 107pi/500). (3- 4.sin^2 179pi/500). (3+ 4.sin^2 37pi/500). (3+ 4.sin^2 111pi/500). (3+ 4.sin^2 167pi/500) = 1.

This can also be written as

(1 + 2 cos pi/250). (1 + 2 cos 3.pi/250).(1+2 cos 9pi/250). (1 +2 cos 27pi/250).(1+2 cos 81.pi/250).(1-2 cos 7pi/250). (1 - 2 cos 21.pi/250). (1 - 2 cos 63.pi/250).(1 + 2 cos 61.pi/250).(1 - 2 cos 67.pi/250) (1+ 2 cos.49pi/250). (1 -2 cos 103pi/250).(1 + 2 cos 59.pi/250). (1 - 2 cos 73.pi/250).(1+ 2 cos 31.pi/250).(1 + 2 cos 93.pi/259). (1- 2 cos 29.pi/250). (1 -2 cos 87.pi/250). (1 + 2 cos 11.pi/250). (1+ 2 cos 33.pi/250).(1 + 2 cos 99.pi/250). (1 - 2 cos 47pi/250). (1 + 2 cos 109.pi/250).(1 - 2 cos 77pi/250). (1+ 2 cos 19pi/250). (1+2 cos.57.pi/250). (1 - 2 cos 79pi/250).(1 + 2 cos 13.pi/250). (1+ 2 cos 39pi/250). (1+ 2 cos 117pi/250).(1- 2 cos 101/250). (1+2 cos 53/250).(1-2 cos 91pi/250).(1+ 2 cos 23pi/250).(1+2 cos 69pi/250). (1- 2 cos 43pi/250). (1+2 cos 121pi/250). (1- 2 cos 113.pi/250).(1+ 2 cos.89pi/250). (1- 2 cos 17pi/250).(1 - 2 cos 51pi/250). (1 + 2 cos 97pi/250). (1- 2 cos 41pi/250). (1 -2 cos 123/259). (1+ 2 cos 119pi/250). (1- 2 cos 107pi/250). (1+2 cos 71.pi/250).(1- 2 cos 37pi/250). (1- 2 cos 111pi/250).(1+ 2 cos 83.pi/250).(1-2 cos.pi/250).(1-2 cos 3.pi/250). (1 - 2 cos9pi/250). (1- 2 cos 27pi/250). (1-2 cos 81.pi/250). (1 + 2 cos 7pi/250).(1+ 2 cos 21.pi/250). (1+ 2 cos 63.pi/250).(1- 2 cos 61.pi/250). (1+ 2 cos 67.pi/250).(1-2 cos.49pi/250). (1+ 2 cos 103pi/250). (1- 2 cos 59.pi/250).(1 + 2 cos 73.pi/250).(1- 2 cos 31.pi/250). (1- 2 cos 93.pi/259). (1+ 2 cos 29.pi/250). (1+ 2 cos 87.pi/250).(1 - 2 cos 11.pi/250). (1- 2 cos 33.pi/250).(1 - 2 cos 99.pi/250).(1+ 2 cos 47pi/250). (1- 2 cos 109.pi/250).(1 + 2 cos 77pi/250). (1- 2 cos 19pi/250).(1 -2 cos.57.pi/250). (1+ 2 cos 79pi/250). (1 - 2 cos 13.pi/250).(1- 2 cos 39pi/250). (1- 2 cos 117pi/250).(1+ 2 cos 101/250). (1- 2 cos 53/250).(1+ 2 cos 91pi/250). (1- 2 cos 23pi/250).(1- 2 cos 69pi/250). (1+ 2 cos 43pi/250).(1- 2 cos 121pi/250). (1+ 2 cos 113.pi/250). (1- 2 cos.89pi/250). (1+ 2 cos 17pi/250).(1 + 2 cos 51pi/250). (1- 2 cos 97pi/250).(1+ 2 cos 41pi/250).

(1+ 2 cos 123/250). (1- 2 cos 119pi/250).(1+ 2 cos 107pi/250). (1 - 2 cos 71.pi/250). (1+ 2 cos 37pi/250). (1+ 2 cos 111pi/250).(1- 2 cos 83.pi/250) = 1

Or (1- 4.cos^2 pi/250). (1 – 4.cos^2 3.pi/250).(1 -4 cos^29pi/250). (1- 4 cos^2 27pi/250).(1- 4.cos^2 81.pi/250). (1 –4.cos^2 7pi/250).(1 - 4 cos^2 21.pi/250). (1 - 4 cos^263.pi/250).(1- 4 cos^2 61.pi/250). (1 – 4.cos^2 67.pi/250). (1- 4.cos^2.49pi/250). (1 – 4.cos^2 103pi/250).(1 -4. cos^259.pi/250). (1 – 4.cos^2 73.pi/250).(1- 4.cos^2 31.pi/250). (1 -4. Cos^2 93.pi/259). (1- 4.cos^2 29.pi/250). (1 – 4.cos^287.pi/250). (1 -4.cos^2 11.pi/250). (1- 4.cos^2 33.pi/250). (1-4.cos^2 99.pi/250).(1-4.cos^2 47pi/250). (1 – 4.cos^2109.pi/250).(1 -4. Cos^2 77pi/250). (1- 4.cos^2 19pi/250). (1- 4.cos^2 57.pi/250). (1- 4.cos^2 79pi/250).(1-4.cos^2 13.pi/250). (1-4.cos^2 39pi/250). (1-4.cos^2 117pi/250). (1- 4.cos^2 101/250). (1-4.cos^2 53/250).(1- 4.cos^2 91pi/250).(1-4.cos^2 23pi/250).(1-4.cos^2 69pi/250). (1- 4.cos^243pi/250). (1- 4.cos^2 121pi/250). (1- 4.cos^2 113.pi/250).(1-4.cos^2 89pi/250). (1- 4.cos^2 17pi/250). (1-4.cos^251pi/250). (1 – 4.cos^2 97pi/250).(1- 4.cos^2 41pi/250). (1 – 4.cos^2 123/259). (1-4.cos^2 119pi/250). (1-4.cos^2107pi/250). (1-4.cos^2 71.pi/250). (1- 4.cos^2 37pi/250).(1 – 4.cos2 111pi/250).(1- 4.cos^2 83.pi/250) = 1.

Or (1/4- cos^2 pi/250). (1/4 –cos^2 3.pi/250).(1/4- cos^2 9pi/250). (1/4- cos^2 27pi/250).(1/4- cos^2 81.pi/250). (1/4- cos^2 7pi/250).(1/4- cos^2 21.pi/250). (1/4- cos^2 63.pi/250).(1/4- cos^2 61.pi/250). (1/4- cos^2 67.pi/250). (1/4- cos^2.49pi/250). (1/4- cos^2 103pi/250).(1/4- cos^2 59.pi/250). (1/4- cos^2 73.pi/250).(1/4- cos^2 31.pi/250). (1/4- Cos^2 93.pi/259). (1/4- .cos^2 29.pi/250). (1/4- cos^2 87.pi/250). (1/4- cos^2 11.pi/250). (1/4- cos^2 33.pi/250). (1/4- cos^2 99.pi/250).(1/4- cos^2 47pi/250). (1/4- cos^2 109.pi/250).(1/4- Cos^2 77pi/250). (1/4- cos^2 19pi/250). (1/4- cos^2 57.pi/250). (1/4- cos^2 79pi/250).(1/4- cos^2 13.pi/250). (1/4- cos^2 39pi/250). (1/4- cos^2 117pi/250). (1/4- cos^2 101/250). (1/4- cos^2 53/250).(1/4- cos^2 91pi/250). (1/4- cos^2 23pi/250).(1/4- cos^2 69pi/250). (1/4- cos^2 43pi/250).(1/4- cos^2 121pi/250). (1/4- cos^2 113.pi/250).(1/4- cos^289pi/250). (1/4- cos^2 17pi/250). (1/4- cos^2 51pi/250). (1/4- cos^2 97pi/250).(1/4- cos^2 41pi/250).(1/4- cos^2 123/250). (1/4- cos^2 119pi/250). (1/4- cos^2 107pi/250). (1/4- cos^2 71.pi/250). (1/4 - cos^2 37pi/250).(1/4- cos2 111pi/250).(1/4 - cos^2 83.pi/250) = (½)^100.

On rearranging,
(1/4-cos^2 pi/250).(1/4–cos^2 3.pi/250).(1/4-cos^2 7pi/250).(1/4-cos^2 9pi/250).(1/4-cos^2 11.pi/250). (1/4- cos^2 13.pi/250). (1/4- cos^2 17pi/250).(1/4- cos^2 19pi/250).(1/4- cos^2 21.pi/250).(1/4- cos^2 23pi/250). (1/4- cos^2 27pi/250). (1/4- .cos^2 29.pi/250).(1/4- cos^2 31.pi/250).(1/4- cos^2 33.pi/250).(1/4 - cos^2 37pi/250). (1/4- cos^2 39pi/250). (1/4- cos^2 41pi/250).(1/4- cos^2 43pi/250).(1/4- cos^2 47pi/250).(1/4- cos^2.49pi/250). (1/4- cos^2 51pi/250). (1/4- cos^2 53/250).(1/4- cos^2 57.pi/250).(1/4- cos^2 59.pi/250).(1/4- cos^2 61.pi/250). (1/4- cos^2 63.pi/250). (1/4- cos^2 67.pi/250).(1/4- cos^2 69pi/250).(1/4- cos^2 71.pi/250).(1/4- cos^2 73.pi/250). (1/4- Cos^2 77pi/250). (1/4- cos^2 79pi/250).(1/4- cos^2 81.pi/250). (1/4 - cos^2 83.pi/250).(1/4- cos^2 87.pi/250). (1/4- cos^2 89pi/250). (1/4- cos^2 91pi/250).(1/4- Cos^2 93.pi/259).(1/4- cos^2 97pi/250).(1/4- cos^2 99.pi/250). (1/4- cos^2 101/250). (1/4- cos^2 103pi/250) (1/4- cos^2 107pi/250). (1/4- cos^2 109.pi/250).(1/4- cos2 111pi/250). (1/4- cos^2 113.pi/250). (1/4- cos^2 117pi/250).(1/4- cos^2 119pi/250).(1/4- cos^2 121pi/250).(1/4- cos^2 123/250)= (½)^100.

On further rearranging,
(cos^2 pi/3- cos^2 pi/250).(cos^2 pi/3- cos^2 3.pi/250).(cos^2 pi/3- cos^2 7pi/250).(cos^2 pi/3- cos^2 9pi/250).(cos^2 pi/3- cos^2 11.pi/250).(cos^2 pi/3- cos^2 13.pi/250).(cos^2 pi/3- cos^2 17pi/250).(cos^2 pi/3- cos^2 19pi/250).(cos^2 pi/3- cos^2 21.pi/250).(cos^2 pi/3- cos^2 23pi/250).(cos^2 pi/3-- cos^2 27pi/250).(cos^2 pi/3-cos^2 29.pi/250).(cos^2 pi/3- cos^2 31.pi/250).(cos^2 pi/3- cos^2 33.pi/250).(cos^2 pi/3- cos^2 37pi/250).(cos^2 pi/3- cos^2 39pi/250).(cos^2 pi/3- cos^2 41pi/250).(cos^2 pi/3- cos^2 43pi/250).(cos^2 pi/3- cos^2 47pi/250).(cos^2 pi/3- cos^2.49pi/250).(cos^2 pi/3- cos^2 51pi/250).(cos^2 pi/3- cos^2 53/250).(cos^2 pi/3- cos^2 57.pi/250).(cos^2 pi/3- cos^2 59.pi/250). (cos^2 pi/3- cos^2 61.pi/250). (cos^2 pi/3- cos^2 63.pi/250).(cos^2 pi/3- cos^2 67.pi/250).(cos^2 pi/3- cos^2 69pi/250).(cos^2 pi/3- cos^2 71.pi/250).(cos^2 pi/3- cos^2 73.pi/250).(cos^2 pi/3-Cos^2 77pi/250).(cos^2 pi/3- cos^2 79pi/250).(cos^2 pi/3- cos^2 81.pi/250). (cos^2 pi/3- cos^2 83.pi/250).(cos^2 pi/3- cos^2 87.pi/250).(cos^2 pi/3- cos^2 89pi/250).(cos^2 pi/3- cos^2 91pi/250).(cos^2 pi/3- Cos^2 93.pi/250).(cos^2 pi/3- cos^2 97pi/250).(cos^2 pi/3- cos^2 99.pi/250).(cos^2 pi/3- cos^2 101/250).(cos^2 pi/3- cos^2 103pi/250) (cos^2 pi/3- cos^2 107pi/250). (cos^2 pi/3- cos^2 109.pi/250).(cos^2 pi/3- cos^2 111pi/250).(cos^2 pi/3- cos^2 113.pi/250). (cos^2 pi/3- cos^2 117pi/250).(cos^2 pi/3- cos^2 119pi/250).(cos^2 pi/3- cos^2 121pi/250).(cos^2 pi/3- cos^2 123/250) = (½)^100.

Thus the identity given in the abstract is solved.

Example 5:
Sin 7pi/131
Here n is of the form p/q= 131/7.
Sin 7pi/131 = - sin 138.pi/131/[(3- 4.sin^2 7pi/131).(3- 4.sin^2 21.pi/131) (3- 4.sin^2 63pi/131) (3+ 4.sin^2 58pi/131) (3- 4.sin^2 43pi/131) (3- 4.sin^2 2pi/131) (3- 4.sin^2 6pi/131) (3- 4.sin^2 18.pi/131).(3- 4.sin^2 54pi/131).(3+ 4.sin^2 31pi/131) (3+ 4.sin^2 38pi/131) (3+ 4.sin^2 17pi/131) (3+ 4.sin^2 51pi/131) (3- 4.sin^2

22pi/131) (3- 4.sin^2 65pi/131) (3+ 4.sin^2 64pi/131) (3- 4.sin^2 61pi/131) .(3+ 4.sin^2 52pi/131) (3- 4.sin^2 25pi/131) (3- 4.sin^2 56pi/131) (3+ 4.sin^2 37pi/131) (3+ 4.sin^2 20pi/131) (3+ 4.sin^2 60pi/131) (3- 4.sin^2 49pi/131).(3+4.sin^2 16pi/131) (3+ 4.sin^2 48pi/131) (3- 4.sin^2 13pi/131) (3- 4.sin^2 39pi/131) (3- 4.sin^2 14pi/131) (3- 4.sin^2 42pi/131) (3- 4.sin^2 5pi/131) (3- 4.sin^2 15pi/131) (3- 4.sin^2 45pi/131) .(3+ 4.sin^2 4pi/131) (3+ 4.sin^2 12pi/131) (3+ 4.sin^2 36pi/131) (3+ 4.sin^2 23pi/131) (3+4.sin^2 62pi/131) (3- 4.sin^2 55pi/131) (3+ 4.sin^2 34pi/131).(3+ 4.sin^2 29pi/131) (3+ 4.sin^2 44pi/131) (3- 4.sin^2 pi/131) (3- 4.sin^2 3pi/131) (3- 4.sin^2 9pi/131) (3- 4.sin^2 27pi/131) (3- 4.sin^2 50pi/131) (3+ 4.sin^2 19pi/131) (3+ 4.sin^2 57pi/131) .(3- 4.sin^2 40pi/131) (3- 4.sin^2 11pi/131) (3- 4.sin^2 33pi/131) (3- 4.sin^2 32pi/131)(3- 4.sin^2 35pi/131) (3- 4.sin^2 26pi/131) (3- 4.sin^2 53pi/131).(3+ 4.sin^2 28pi/131) (3+ 4.sin^2 47pi/131) (3- 4.sin^2 10pi/131) (3- 4.sin^2 30pi/131) (3- 4.sin^2 41pi/131)(3- 4.sin^2 8pi/131) (3- 4.sin^2 24pi/131) (3- 4.sin^2 59pi/131) (3+ 4.sin^2 46pi/131).

In this way numerator on right hand side comes as -sin 138pi/131= - sin (pi+ 7pi/131) = sin 7.pi/131 which is positive. Therefore, on cancelling sin 7.pi/131 from LHS and RHS, we get
(3- 4.sin^2 7pi/131).(3- 4.sin^2 21.pi/131) (3- 4.sin^2 63pi/131) (3+ 4.sin^2 58pi/131) (3- 4.sin^2 43pi/131) (3- 4.sin^2 2pi/131) (3- 4.sin^2 6pi/131) (3- 4.sin^2 18.pi/131).(3- 4.sin^2 54pi/131).(3+ 4.sin^2 31pi/131) (3+ 4.sin^2 38pi/131) (3+ 4.sin^2 17pi/131) (3+ 4.sin^2 51pi/131) (3- 4.sin^2 22pi/131) (3- 4.sin^2 65pi/131) (3+ 4.sin^2 64pi/131) (3- 4.sin^2 61pi/131) .(3+ 4.sin^2 52pi/131) (3- 4.sin^2 25pi/131) (3- 4.sin^2 56pi/131) (3+ 4.sin^2 37pi/131) (3+ 4.sin^2 20pi/131) (3+ 4.sin^2 60pi/131) (3- 4.sin^2 49pi/131).(3+ 4.sin^2 16pi/131) (3+ 4.sin^2 48pi/131) (3- 4.sin^2 13pi/131) (3- 4.sin^2 39pi/131) (3- 4.sin^2 14pi/131) (3- 4.sin^2 42pi/131) (3- 4.sin^2 5pi/131) (3- 4.sin^2 15pi/131) (3- 4.sin^2 45pi/131) .(3+ 4.sin^2 4pi/131) (3+ 4.sin^2 12pi/131) (3+ 4.sin^2 36pi/131) (3+ 4.sin^2 23pi/131) (3+ 4.sin^2 62pi/131) (3- 4.sin^2 55pi/131) (3+ 4.sin^2 34pi/131).(3+ 4.sin^2 29pi/131) (3+ 4.sin^2 44pi/131) (3- 4.sin^2 pi/131) (3- 4.sin^2 3pi/131) (3- 4.sin^2 9pi/131) (3- 4.sin^2 27pi/131) (3- 4.sin^2 50pi/131) (3+ 4.sin^2 19pi/131) (3+ 4.sin^2 57pi/131) .(3- 4.sin^2 40pi/131) (3- 4.sin^2 11pi/131) (3- 4.sin^2 33pi/131)(3- 4.sin^2 32pi/131) (3- 4.sin^2 35pi/131) (3- 4.sin^2 26pi/131) (3- 4.sin^2 53pi/131).(3+ 4.sin^2 28pi/131) (3+ 4.sin^2 47pi/131) (3- 4.sin^2 10pi/131) (3- 4.sin^2 30pi/131)(3- 4.sin^2 41pi/131) (3- 4.sin^2 8pi/131) (3- 4.sin^2 24pi/131) (3- 4.sin^2 59pi/131) (3+ 4.sin^2 46pi/131) = 1.

On applying identity cos 2x = 1- 2.sin^2 x, it can be rewritten as
(1+ 2.cos 14pi/131).(1+ 2.cos 42pi/131).(1- 2.cos 5pi/131). (1- 2.cos 15pi/131). (1-2.cos 45pi/131). (1+ 2.cos 4pi/131).(1+ 2.cos 12pi/131). (1+ 2.cos 36pi/131).(1-2.cos 23pi/131) (1+ 2.cos 62pi/131). (1- 2.cos 55pi/131). (1+ 2.cos 34pi/131). (1- 2.cos 29pi/131) (1+ 2.cos 44pi/131). (1- 2.cos pi/131).(1-2 cos 3pi/131) . (1- 2.cos 9pi/131). (1- 2.cos 27pi/131). (1+ 2.cos 50pi/131). (1-2.cos 19pi/131).(1- 2.cos 57pi/131). (1+ 2.cos 40pi/131). (1- 2.cos 11pi/131).(1- 2.cos 33pi/131).(1+ 2.cos 32pi/131). (1- 2.cos 35pi/131).(1+ 2.cos 26pi/131). (1- 2.cos 53pi/131). (1+ 2.cos 28pi//131). (1- 2.cos 47pi/131). (1+ 2.cos 10pi/131). (1+ 2.cos 30pi/131) (1- 2.cos 41pi/131). (1+ 2.cos 8pi/131). (1+ 2.cos 24pi/131). (1- 2.cos 59pi/131). (1+ 2.cos 46pi/131). (1- 2.cos 7pi/131). (1- 2.cos 21pi/131). (1- 2.cos 63pi/131). (1+ 2.cos 58pi/131). (1- 2.cos 43pi/131). (1+ 2.cos 2pi/131). (1+ 2.cos 6pi/131). (1+ 2.cos 18pi/131). (1+ 2.cos 54pi/131). (1- 2.cos 31pi/131). (1+ 2.cos 38pi/131). (1- 2.cos 17pi/131). (1- 2.cos 51pi/131). (1+ 2.cos 22pi/131). (1+ 2.cos 66pi/131). (1- 2.cos 67pi/131).(1- 2.cos 61pi/131). (1+ 2.cos 52pi/131).(1- 2.cos 25pi/131). (1+ 2.cos 56pi/131). (1- 2.cos 37pi/131). (1+ 2.cos 20pi/131). (1+ 2.cos 60pi/131). (1- 2.cos 49pi/131). (1+ 2.cos 16pi/131). (1+ 2.cos 48pi/131). (1- 2.cos 13pi/131). (1- 2.cos 39pi/131) = 1.

Rearranging
(1- 2.cos pi/131).(1+ 2.cos 2pi/131) (1-2 cos 3pi/131) (1+ 2.cos 4pi/131) (1- 2.cos 5pi/131).(1+ 2.cos 6pi/131) (1- 2.cos 7pi/131).(1+ 2.cos 8pi/131).(1- 2.cos 9pi/131).(1+ 2.cos 10pi/131).(1- 2.cos 11pi/131) (1+ 2.cos 12pi/131). (1- 2.cos 13pi/131) (1+ 2.cos 14pi/131).(1- 2.cos 15pi/131). (1+ 2.cos 16pi/131). (1- 2.cos 17pi/131) (1+ 2.cos 18pi/131).(1- 2.cos 19pi/131). (1+ 2.cos 20pi/131).(1- 2.cos 21pi/131).(1+ 2.cos 22pi/131).(1- 2.cos 23pi/131)(1+ 2.cos 24pi/131).(1- 2.cos 25pi/131).(1+ 2.cos 26pi/131).(1- 2.cos 27pi/131).(1+ 2.cos 28pi/131) (1- 2.cos 29pi/131) (1+ 2.cos 30pi/131) (1- 2.cos 31pi/131). (1+ 2.cos 32pi/131). (1- 2.cos 33pi/131).(1+ 2.cos 34pi/131). (1- 2.cos 35pi/131). (1+ 2.cos 36pi/131) (1- 2.cos 37pi/131).(1+ 2.cos 38pi/131).(1- 2.cos 39pi/131).(1+ 2.cos 40pi/131).(1- 2.cos 41pi/131).(1+ 2.cos 42pi/131).(1- 2.cos 43pi/131).(1+ 2.cos 44pi/131).(1- 2.cos 45pi/131).(1+ 2.cos 46pi/131).(1- 2.cos 47pi/131).(1+ 2.cos 48pi/131).(1- 2.cos 49pi/131). (1+ 2.cos 50pi/131).(1- 2.cos 51pi/131). (1+ 2.cos 52pi/131).(1- 2.cos 53pi/131). (1+ 2.cos 54pi/131).(1- 2.cos 55pi/131).(1+ 2.cos 56pi/131).(1- 2.cos 57pi/131).(1+ 2.cos 58pi/131).(1- 2.cos 59pi/131).(1+ 2.cos 60pi/131).(1- 2.cos 61pi/131).(1+ 2.cos 62pi/131).(1- 2.cos 63pi/131).(1+ 2.cos 66pi/131).(1- 2.cos 67pi/131) = 1.

It can also be written as
(1/2- cos pi/131).(1/2+ cos 2pi/131) (1/2- cos 3pi/131) (1/2+ cos 4pi/131) (1/2- cos 5pi/131).(1/2+ cos 6pi/131) (1/2-cos 7pi/131).(1/2+ cos 8pi/131).(1/2- cos 9pi/131).(1/2+ cos 10pi/131).(1/2- cos 11pi/131) (1/2+ cos 12pi/131). (1/2- cos 13pi/131) (1/2+ cos 14pi/131).(1/2- cos 15pi/131). (1/2+ cos 16pi/131). (1/2- cos 17pi/131) (1/2+ cos 18pi/131).(1/2- cos 19pi/131). (1/2+ cos 20pi/131).(1/2- cos 21pi/131).(1/2+ cos 22pi/131).(1/2- cos 23pi/131)(1/2+ cos 24pi/131).(1/2- cos 25pi/131).(1/2+ cos 26pi/131).(1/2- cos 27pi/131).(1/2+ cos 28pi/131) (1/2- cos 29pi/131).(1/2+ cos 30pi/131).(1/2- cos 31pi/131). (1/2+ cos 32pi/131).(1/2- cos 33pi/131).(1/2+ cos 34pi/131). (1/2- cos 35pi/131).(1/2- cos

36pi/131) (1/2-cos 37pi/131).(1/2+ cos 38pi/131).(1/2- cos 39pi/131).(1/2+ cos 40pi/131).(1/2- cos 41pi/131).(1/2+ cos 42pi/131).(1/2- cos 43pi/131).(1/2+ cos 44pi/131).(1/2- cos 45pi/131).(1/2+ cos 46pi/131).(1/2- cos 47pi/131).(1/2+ cos 48pi/131).(1/2- cos 49pi/131). (1/2+ cos 50pi/131).(1/2- cos 51pi/131). (1/2+ cos 52pi/131).(1/2- cos 53pi/131).(1/2+ cos 54pi/131).(1/2- cos 55pi/131).(1/2+ cos 56pi/131).(1/2- cos 57pi/131).(1/2+ cos 58pi/131).(1/2- cos 59pi/131).(1/2+ cos 60pi/131).(1/2- cos 61pi/131).(1/2+ cos 62pi/131).(1/2- cos 63pi/131).(1/2+ cos 66pi/131).(1/2- cos 67pi/131) = (½)^65.

Also
(cos pi/3- cos pi/131).(cos pi/3+ cos 2pi/131) (cos pi/3- cos 3pi/131) (cos pi/3+ cos 4pi/131) (cos pi/3- cos 5pi/131). (cos pi/3+ cos 6pi/131) (cos pi/3- cos 7pi/131).(cos pi/3+ cos 8pi/131).(cos pi/3- cos 9pi/131).(cos pi/3+cos 10pi/131). (cos pi/3- cos 11pi/131) (cos pi/3+ cos 12pi/131). (cos pi/3- cos 13pi/131) (cos pi/3+ cos 14pi/131).(cos pi/3- cos 15pi/131). (cos pi/3+ cos 16pi/131). (cos pi/3- cos 17pi/131) (cos pi/3+ cos 18pi/131).(cos pi/3- cos 19pi/131). (cos pi/3+ cos 20pi/131).(cos pi/3- cos 21pi/131).(cos pi/3+ cos 22pi/131).(cos pi/3- cos 23pi/131)(cos pi/3+ cos 24pi/131).(cos pi/3- cos 25pi/131).(cos pi/3+ cos 26pi/131).(cos pi/3- cos 27pi/131).(cos pi/3+ cos 28pi/131) (cos pi/3- cos 29pi/131).(cos pi/3+ cos 30pi/131).(cos pi/3- cos 31pi/131). (cos pi/3+ cos 32pi/131).(cos pi/3- cos 33pi/131). (cos pi/3+ cos 34pi/131).(cos pi/3- cos 35pi/131).(cos pi/3+ cos 36pi/131) (cos pi/3- cos 37pi/131). (cos pi/3+ cos 38pi/131).(cos pi/3- cos 39pi/131).(cos pi/3+ cos 40pi/131).(cos pi/3- cos 41pi/131).(cos pi/3+ cos 42pi/131).(cos pi/3-cos 43pi/131).(cos pi/3+ cos 44pi/131).(cos pi/3- cos 45pi/131).(cos pi/3+ cos 46pi/131). (cos pi/3- cos 47pi/131).(cos pi/3+ cos 48pi/131).(cos pi/3- cos 49pi/131).(cos pi/3+ cos 50pi/131).(cos pi/3- cos 51pi/131).(cos pi/3+ cos 52pi/131).(cos pi/3- cos 53pi/131).(cos pi/3+ cos 54pi/131).(cos pi/3- cos 55pi/131). (cos pi/3+ cos 56pi/131).(cos pi/3- cos 57pi/131).(cos pi/3+ cos 58pi/131).(cos pi/3- cos 59pi/131).(cos pi/3+ cos 60pi/131).(cos pi/3- cos 61pi/131).(cos pi/3+ cos 62pi/131).(cos pi/3- cos 63pi/131).(cos pi/3+ cos 66pi/131). (cos pi/3- cos 67pi/131) = (½)^65.

Thus the identity mentioned in abstract is solved.

After solving these example and taking the theory in view, it is concluded that

A. *(1/2+ cos 2pi/n).(1/2+ cos 3.2pi/n).(1/2+ cos 9.2pi/n).(1/2+ cos 27.2pi/n) (1/2+ cos 81.2pi/n)… [1/2+ cos (3^k-1). 2pi/n] = (½)^k*

In other words

$$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2\pi/n\} = (½)^k,$$

where integers m and k are given by relation (3^k + 1)/n = m and m is odd. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

OR

where integers m and k are given by relation (3^k - 1)/n = m and m is even. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

B. *(1/2+ cos 2pi/n).(1/2+ cos 3.2pi/n).(1/2+ cos 9.2pi/n).(1/2+ cos 27.2pi/n) (1/2+ cos 81.2pi/n)………………………….. [1/2+ cos (3^k-1).2pi/n] = -(½)^k*

In other words,

$$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2\pi/n\} = - (½)^k$$

where integers m and k are given by relation (3^k + 1)/n = m and m is even. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

OR

where integers m and k are given by relation (3^k- 1)/n = m and m is odd. And n may be integer but not multiple of 3 and if fraction p/q, p is not multiple of 3.

On putting pi/n = x in radians, the above identities transform to
A.(1/2+ cos 2x).(1/2+ cos 3.2x).(1/2+ cos 9.2x).(1/2+ cos 27.2x) (1/2+ cos 81.2x)……. [1/2+ cos 3^(k-1).2x] = (½)^k,
this can also be written as

$$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2x\} = (½)^k,$$

where integers m and k are given by relation (3^k + 1)/x = m.pi and m is odd. And 1/x is not multiple of 3.

OR

where integers m and k are given by relation (3^k - 1)/x = m.pi and m is even. And 1/x is not multiple of 3.

B. *(1/2+ cos 2x).(1/2+ cos 3.2x).(1/2+ cos 9.2x).(1/2+ cos 27.2x) (1/2+ cos 81.2x)…………………………. [1/2+ cos 3^(k-1).2x] = -(½)^k*

$$\prod_{j=1}^{k} \{1/2+ \cos 3^{(j-1)}.2x\} = - (½)^k,$$

where integers m and k are given by relation (3^k + 1)/x = m.pi and m is even. And 1/x is not multiple of 3.

OR

where integers m and k are given by relation (3^k- 1)/n = m.pi and m is odd and 1/x is not multiple of 3.

2.4Cases where n is not multiple of 2 or is a fraction p/q where integer p is not multiple of 2 are taken up here.

In such cases, angle x is 'π divided by odd integer' or π is divided by fraction p/q where p is odd integer. That is angle x may be of the form, for example, π/9 or π/13 or π/31 π/(21/8) or π/(57/7) or π divided any integer except multiplier of 2 or π/(p/q) where p is any integer not multiplier of 2.

Elementary identity
2.sin x.cos x = sin 2x or
sin x = sin 2x/2.cos x (12)
is recursive in nature as sine appears in left hand side as well as in right hand side. Sin x in LHS and sin 2x in RHS in numerator are being considered for recurrence.

From this recursive relation, sin 2x can also be written as
sin 2x = sin 4x/(2.cos 2x)
sin 4x= sin 8x/(2.cos 4x)
……………………………………………..
Sin x.2^(k-1) = sin 2^k/[2.cos 2^(k-1)]

On substituting value of sin 2x = sin 4x/(2.sin 2x), equation 12 can be written as
sin x = sin 4x/(2.cos x.2.cos 2x) = sin 4x/(4.cos x.cos 2x)

On successively substituting value of sin 4x, sin 8x …………… sin x.2^(k-1), we get
sin x = (2^k.sin x)/[2^k.cos x.cos 2x.cos 4x.cos 8x……………….cos x,2^(k-1)]

Putting x = π/n, this equation is written as
sin π/n = sin π/n.2^k/[2^k.cos π/n.cos 2π/n cos 4π/n. cos 8 π/n………cos π/n,2^(k-1)] (13)

If n is any integer except even integer or is form p/q where p is not even then (2^k + 1) or (2^k - 1) will always be divisible by n for some integer value of k. It is obvious 2^k + 1 and 2^k-1 will always be odd as even integer plus or minus 1 always makes odd integer. Also division of odd integer by some other odd integer is possible.

Here k can assume any value starting from 0, and then proceeding to 1, then 2, then 3, so on till that value of k is reached where it gets divided by n without any remainder. There are thus many positive k integers available and since odd can be divided by odd then there is complete possibility for division of (3^k + 1) or (3^k - 1) by n.

Let m = (2^k + 1)/n or
m= (2^k - 1)/n
where m is that minimum number where (3^k + 1) or (3^k -1) is just divisible by n.

Applying this concept that there exists a value of k that makes (2^k + 1) or (2^k -1) divisible by n, we can easily say that *denominator of equation (13) has factors which are k in number*. First factor pertains to angle π/n.2^0, second to π/n.2^1, third to π/n.2^2, fourth to π/n.2^3…………so on and kth to π/n.2^(k-1).

From equation (13), it is clear, RHS has numerator sin π/n.2^k and this pertains to angle π/n.2^k. It is already proved (2^k + 1)/n = m, that makes
π/n.2^k = (m π - π/n)

Similarly, for the case where (2^k - 1)/n = m,
π/n.2^k = (m π + π/n).

Therefore *for the cases where (2^k + 1)/n = m*, numerator of RHS of equation (13) becomes
sin π/n.2^k = sin (m π - π/n) = sin π/n when m is odd and = - sin π/n when m is even.

For the cases where (2^k - 1)/n = m, numerator of RHS of equation (13) becomes
sin π/n.2^k = sin (m π+ π/n) = sin π/n when m is even and it equals - sin π/n when m is odd.

When (2^k + 1)/n = m and m is odd
or
when (2^k - 1)/n = m and m is even.
Equation (13) becomes
sin π/n = sin π/n/[2^k.cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1)]

On cancelling sin π/n on both sides, we get
[2^k.cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1)]= 1
Or *cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1) = (½)^k* (14)

It can also be written as

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.\pi/n = (1/2)^k,$$

when integers k and m are given by relation (2^k + 1)/n= m and m is odd or relation (2^k - 1)/n = m and m is even for all odd n.

It can be be written in angle x form by putting π/n = x as

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x = (1/2)^k,$$

when integers m and k are given by relation (2^k + 1).x= m.π and m is odd
or
when integers m an k are given by relation (2^k - 1)/x= m.π and m is even. And n is odd in all cases. .

Similarly for cases
where (2^k + 1)/n = m and m is even
or
where (2^k - 1)/n = m and m is odd.
cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1) = - (1/2)^k (14/1)

It can also be written as

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.\pi/n = -(1/2)^k,$$

When integers k and m are given by relation (2^k + 1)/n= m and m is even or relation (2^k - 1)/n = m and m is odd for all odd n.

It can be be written in angle x form by putting π/n = x as

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x = -(1/2)^k,$$

when integers m and k are given by relation (2^k + 1).x= m.π and m is odd
or
when integers m an k are given by relation (2^k - 1)/n = m.π and m is even. And n is odd in all cases

2.5Example

Example 1
Sin π/63
Here n = 63 and it satisfies relation
(2^k - 1)/n = m as (2^6 - 1)/63
where k = 6, m = 1.

Hence equation (16) is satisfied, therefore,
cos π/63. cos 2π/63. cos 4π/63. cos 8π/63. cos 16π/63.cos 32π/63= - (1/2)^6.

Example 2

Sin π/81

Cos π/81.Cos 2π/81. Cos 4π/81. Cos 8π/81. Cos 16π/81. Cos 32π/81. Cos 17π/81 Cos 34π/81.Cos 13π/81. Cos 26π/81. Cos 29π/81. Cos 23π/81. Cos 35π/81. Cos 11π/81 Cos 22π/81.Cos 37π/81. Cos 7π/81. Cos 14π/81. Cos 28π/81. Cos 25π/81. Cos 31π/81. Cos 19π/81.Cos 38π/81.Cos 5π/81.Cos 10π/81. Cos 20π/81. Cos 40π/81= (½)^127.

Example 3

Sin π/243

Cos π/243. Cos 2π/243.Cos 4π/243.Cos 8π/243. Cos 16π/243. Cos 32π/243 Cos 64π/243 Cos 115π/243 Cos 13π/243 Cos 26π/243Cos 52π/243.Cos 104π/243 Cos 35π/243 Cos 70π/243 Cos 103π/243 Cos 37π/243 Cos 74π/243 Cos 95π/243 Cos 53π/243Cos 106π/243 Cos 31π/243.Cos 62π/243 Cos 119π/243Cos 5π/243 Cos 10π/243 Cos 20π/243 Cos 40π/243 Cos 80π/243 Cos 83π/243 Cos 77π/243 Cos 89π/243.Cos 65π/243 Cos 113π/243Cos 17π/243 Cos 34π/243 Cos 68π/243 Cos 107π/243 Cos 29π/243 Cos 58π/243 Cos 116π/243 Cos 11π/243.Cos 22π/243Cos 44π/243 Cos 88π/243 Cos 67π/243 Cos 109π/243 Cos 25π/243 Cos 50π/243 Cos 100π/243 Cos 43π/243 Cos 86π/243.Cos 71π/243 Cos 101π/243 Cos 41π/243 Cos 82π/243 Cos 79π/243 Cos 85π/243 Cos 73π/243 Cos97π/243 Cos 49π/243 Cos 98π/243.Cos 47π/243 Cos 94π/243 Cos 55π/243 Cos 110π/243Cos 23π/243 Cos 46π/243 Cos 92π/243 Cos 59π/243 Cos 118π/243 Cos 7π/243.Cos 14π/243 Cos 28π/243 Cos 56π/243Cos 112π/243 Cos 19π/243 Cos 38π/243 Cos 76π/243 Cos 91π/243 Cos 61π/243. Cos121π/243 = (½)^81.

From these we can say that

A. *Cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n.2^(k-1) = (½)^k,*
in other words product cos 2^j.π/n=(½)^k and j varies from 0 to k-1
where integers m and k are given by relation (2^k + 1)/n = m.And m and n are odd

or

where integers m and k are given by relation (2^k - 1)/n = m. And m is even and n is odd.

B. *Cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1) = - (1/2)^k*

In other words, $\prod_{j=1}^{k} \cos 2^{(j-1)}.\pi/n = -(½)^k,$

where integers m and k are given by relation (2^k + 1)/n = m. And m is even and n is odd.

or

where integers m and k are given by relation (2^k - 1)/n = m. And m and n are odd.

A. If we put *π/n* = x, we get

Cos x.cos 2x. cos 4x. cos 8x………cos 2^(k-1).x = (½)^k,

In other words,

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x= (½)^k,$$

where integers m and k are given by relation (2^k + 1).x = m.π and m is odd and 1/x is not multiple of 2.

or

where integers m and k are given by relation (2^k - 1)/n = m and m is even and 1/x is not multiple of 2.

B. Cos π/n.cos 2π/n cos 4π/n. cos 8 π/n ………cos π/n,2^(k-1) = - (1/2)^k in other words,

$$\prod_{j=1}^{k} \cos 2^{(j-1)}.x= - (½)^k,$$

where integers m and k are given by relation (2^k + 1)/n = m and m is even and 1/x is not multiple of 2

or

where integers m and k are given by relation (2^k - 1)/n = m and m is odd and 1/x is not multiple of 2.

2.6 Part 2

Factorisation Of Cosine Of An Angle

Now we come to factorisation of cosine of multiple angle. Let it be cos 3x.

By putting cos 3x = 0, its roots correspond to angles that can be found as
3x = (2k + 1).*π/2,* where k is 0, 1, 2, ………

Therefore, x is *π/6*, *π/2, 5π/6.* On putting k =0, 1, 2 and its roots are found as cos *π/6,* cos *π/2,* cos 5*π*/6. Now when roots are known, its equation can be written as
(cos x – cos *π/6).* (cos x – cos *π/2).*(cos x – cos 5*π/6)= 0* and it has three factors.

If we multiply this equation by 2^(j-1), where j equals number of factors, it does not make any difference and equation becomes
$2^2.(\cos x - \cos \pi/6).(\cos x - \cos \pi/2).(\cos x - \cos 5\pi/6) = 0$ and these factors correspond to cos 3x.

$$\cos 3x = \text{its factors} = 2^2.(\cos x - \cos \pi/6).(\cos x - \cos \pi/2).(\cos x - \cos 5\pi/6)$$
$$= 2^2.\cos x.(\cos x - \cos \pi/6)..(\cos x + \cos \pi/6)$$
$$= 4.\cos x.(\cos^2 x - 3/4)$$

Similarly, it can be found that cos 4x has four roots, $\cos\pi/8$, $\cos 3\pi/8$, $\cos 5\pi/8$ $\cos 7\pi/8$ and
$$\cos 4x = 2^3.(\cos x - \cos \pi/8).(\cos x - \cos 3\pi/8).(\cos x - \cos 5\pi/8).(\cos x - \cos 7\pi/8)$$
$$= 2^3.(\cos x - \cos \pi/8).(\cos x - \cos 3\pi/8).(\cos x + \cos 3\pi//8).(\cos x + \cos \pi/8)$$
$$= 2^3.(\cos^2 x - \cos^2 \pi/8).(\cos^2 x + \cos^3\pi/8).$$

Similarly,
$\cos 9x = 2^8.(\cos x - \cos \pi/18).(\cos x - \cos 3\pi/18)(\cos x - \cos 5\pi/18)(\cos x - \cos 7\pi/18)(\cos x - \cos 9\pi/18).(\cos x - \cos 11\pi/18).(\cos x - \cos 13\pi/18)(\cos x - \cos 15\pi/18)(\cos x - \cos 17\pi/18) = 2^8.\cos x.(\cos^2 x - \cos^2 \pi/18).(\cos^2 x - \cos^2 3\pi/18).(\cos^2 x - \cos^2 5\pi/18).(\cos^2 x - \cos^2 7\pi/18).$

By mathematical induction, it can be written as
$\cos n.x = 2^{(n-1)}.\{\cos x - \cos \pi/2n\}.\{\cos x - \cos 3.\pi/2n.\ \{\cos x - \cos 5.\pi/2n\}\ldots\ldots.\{\cos x - \cos (2n-1).\pi/2n\}$ (15)

When n is even, it can also be written as
$\cos n.x = 2^{(n-1)}.\{\cos^2 x - \cos^2 \pi/2n\}.\{\cos^2 x - \cos^2 3\pi/2n.\ \{\cos^2 x - \cos^2 5.\pi/2n\}\ldots.\{\cos^2 x - \cos^2 (n-1).\pi/2n\}$ (15/4)

$$\cos n.x = 2^{(n-1)}.\prod_{k=0}^{n-1} \{\cos x - \cos (2k+1).\pi/2n\} \quad (15/1)$$

Since cos n.x = Tn (cos x) where Tn (cos x) is Chebyshev polynomial of first kind in cos x, therefore,
$T_n(\cos x) = 2^{(n-1)}.\{\cos x - \cos \pi/2n\}.\{\cos x - \cos 3.\pi/2n.\ \{\cos x - \cos 5.\pi/2n\}\ldots\ldots\{\cos x - \cos (2n-1).\pi/2n\}.$

That means Chebyshev polynomial of first kind in cos x has roots
$\cos \pi/2n, \cos 3\pi/2n, \cos 5\pi/2n\ldots\ldots \cos (2n-1)\pi/2n.$

Also $\cos n.x = (-1)^{n/2}.T_n(\sin x)$ when n is even and identity (15/4) can be written in sin x as
$\cos n.x = (-1)^{n/2}.2^{(n-1)}.\{\sin^2 x - \sin^2 \pi/2n\}.\{\sin^2 x - \sin^2 3\pi/2n.\{\sin^2 x - \sin^2 5.\pi/2n\}\ldots\{\sin^2 x - \sin^2 (n-1).\pi/2n\},$

Therefore,
$T_n(\sin x) = 2^{(n-1)}.\{\sin^2 x - \sin^2 \pi/2n\}.\{\sin^2 x - \sin^2 3\pi/2n.\ \{\sin^2 x - \sin^2 5.\pi/2n\}\ldots.\{\sin^2 x - \sin^2 (n-1).\pi/2n\}$ when n is even.

When n is odd, identity (15) can be written as
$\cos n.x = 2^{(n-1)}.\cos x.(\cos^2 x - \cos^2 \pi/2n\}.\{\cos^2 x - \cos^2 3\pi/2n.\ \{\cos^2 x - \cos^2 5.\pi/2n\}\ldots\ldots\{\cos^2 x - \cos^2 (n-2).\pi/2n\}.$
Or $\cos n.x = 2^{(n-1)}.(-1)^{(n-1)/2}.\ \cos x,(\sin^2 \pi/2n - \sin^2 x).(\sin^2 3\pi/2n - \sin^2 x).(\sin^2 - \sin 5\pi/2n)\ldots\ldots\ldots\{\sin^2 \pi/2n - \sin^2 (n-2).\pi/2n\}.$

Also $\cos n.x = .(-1)^{(n-1)/2}.\cos x.U_{n-1}(\sin x).$

Therefore $U_{n-1}(\sin x) = 2^{n-1}.(\sin^2 x - \sin^2 \pi/2n).(\sin^2 x - \sin^2 3\pi/2n).(\sin^2 x - \sin 5\pi/2n)\ldots\ldots\{\sin^2 x - \sin^2 (n-2).\pi/2n\}$
and its roots are $\sin \pi/2n, -\sin \pi/2n, \sin 3\pi/2n, -\sin 3\pi/2n, \sin 5\pi/2n, -\sin 5\pi/2n, \ldots\ldots\ldots \sin (n-1)\pi/2n, -\sin (n-1)\pi/2n$ *when is odd.* $U_{n-1}(\sin x)$ *is Chebyshev polynomial of second kind in sin x.*

Coming to equation (15), when x is zero, equation (15) reduces to
$1 = 2^{(n-1)}.(1 - \cos \pi/2n).(1 - \cos 3\pi/2n).(1 - \cos 5\pi/2n)\ldots\ldots\ldots\ldots.\{1 - \cos (2n-1).\pi/2n\},$
Or
$$\prod_{k=0}^{n-1} \{1 - \cos (2k+1).\pi/2n\} = (1/2)^{(n-1)} \quad (15/2)$$

Now since $1-\cos(2k+1).\pi/2n = 2.\sin^2(2k+1).\pi/4n$, therefore above identity takes the form

$$2^n.\prod_{k=0}^{n-1}\sin^2(2k+1).\pi/4n = (1/2)^{(n-1)}$$

$$\text{or}\quad \prod_{k=0}^{n-1}\sin^2(2k+1).\pi/4n = (1/2)^{(2n-1)}$$

$$\text{or}\quad \prod_{k=0}^{n-1}\sin(2k+1).\pi/4n = (1/2)^{(n-1/2)} \tag{15/3}$$

When $x=\pi/2$, $\cos\pi/2 = 0$.

Equation (15/1) reduces to

$$\text{Cos } n.\pi/2 = 2^{(n-1)}.(-1)^n.\prod_{k=0}^{n-1}\{\cos(2k+1).\pi/2n\}$$

Case-1:
Let n be odd and x is $\pi/2$, where p is any integer, then this identity takes the form

$$\text{Cos }(2p+1).\pi/2 = \{2^{2p}\}.\{(-1)^{(2p+1)}\}\prod_{k=0}^{2p}\{\cos(2k+1).\pi/4p+2\},$$

$$\text{or}\quad \prod_{k=0}^{2p}\{\cos(2k+1).\pi/(4p+2\} = 0.$$

This product will always be zero as it contains one factor that pertains to integer $2p+1$ when k will be equal to p. In that case $\cos(2p+1)/\pi/(4p+2) = \cos\pi/2 = 0$.

Case-2:
Let n is even of the form 4p, where p is any integer and $x = \pi/2$, then the identity 15/1 takes the form

$$\text{Cos } 4p.\pi/2. = \{2^{(4p-1)}\}\{(-1)^{4p}.\prod_{k=0}^{4p-1}\{\cos(2k+1).\pi/8p$$

$$\text{or}\quad \prod_{k=0}^{4p-1}\{\cos(2k+1).\pi/8p = (1/2)^{(4p-1)}$$

Case-3:
Let n be even and of the form $2(2p+1)$ where p is any integer and $x = \pi/2$, the identity takes the form,

$$\text{Cos } 2(2p+1)\pi/2. = 2^{(4p+1)}.(-1)^{(4p+2)}.\prod_{k=0}^{4p+1}\{\cos(2k+1).\pi/(8p+4)\}$$

$$\text{or}\quad \prod_{k=0}^{4p+1}\{\cos(2k+1).\pi/(8p+4)\} = -(1/2)^{(4p+1)}$$

From above, the identities can be generalised as

$$\prod_{k=0}^{n-1}\{\cos(2k+1).\pi/2n = (-1)^{n/2}.(1/2)^{(n-1)}$$

Factorisation Of Sine Of An Angle

Now factorisation of sine of multiple angle is taken up when angle is odd.That is it is of form $2k+1$ where k is 0, 1, 2, 3………….. Let it be sin 3x.

By putting $\sin 3x = 0$, its roots correspond to angles that can be found as
$3x = +k\pi$,
$3x = -k\pi$,

where k is 0, 1, 2, …therefore on putting k = 1 and also k = -1, x equals $\pi/3$ and $-\pi/3$. On putting k =0, x equals 0. Therefore roots of sin 3x are $\sin \pi/3$, $-\sin \pi/3$, sin 0. Now when roots are known, its equation can be written as
$(\sin x - \sin \pi/3).(\sin x + \sin \pi/3)(\sin x - \sin 0) = 0$.

If we multiply this equation by $(-1)^{(n-1)/2}. 2^{(j-1)}$, where j is number of factors and n is multiplier of x, here it is 2, it does not make any difference and equation becomes
$(-1)^{(n-1)/2}. 2^{(j-1)},(\sin x - \sin \pi/3).(\sin x + \sin \pi/3)(\sin x - \sin 0) = 0$
and these factors correspond to sin 3x.
Sin 3x = its factors = $(-1)^{(n-1)/2}. 2^{(j-1)}, (\sin x - \sin \pi/3).(\sin x + \sin \pi/3)(\sin x - \sin 0)$
$= 2^2.(\sin \pi/3 - \sin x).(\sin \pi/3 + \sin x).\sin x$
$= 4.(3/4 - \sin^2 x).\sin x$
$= \sin x.(3 - 4 \sin^2 x)$

Similarly, it can be found that sin 5x has five roots, $\sin \pi/5$, $-\sin \pi/5$, $\sin 3\pi/5$, $-\sin 3\pi/5$, 0 and it can be factorised as
$\sin 5x = (-1)^{(5-1)/2}.2^4.(\sin x - \sin \pi/5).(\sin x + \sin \pi/5).(\sin x - \sin 2\pi/5).(\sin x + \sin 2\pi//5),(\sin x)$
$= 2^4. \sin x.(\sin^2 - x \sin^2 \pi/5).(\sin^2 x - \sin^2 2\pi/5)$

Similarly $\sin 17x = \{-1)^{(17-1)/2}\}.2^{16}. \sin x.(\sin^2 x - \sin^2 \pi/17) (\sin^2 x - \sin^2 2\pi/17) (\sin^2 x\sin^2 3\pi/17)$
$(\sin^2 x - \sin^2 4 \pi/17).(\sin^2 x - \sin^2 5\pi/17) .(\sin^2 x - \sin^2 6\pi//17) .(\sin^2 x - \sin^2 7\pi/17) .(\sin^2 x - \sin^2 \pi/17)$

By mathematical induction, it can be written as
$\sin n = \{(-1)^{(n-1)/2}\}.2^{(n-1)}\}(\sin x).(\sin^2 x - \sin^2 \pi/n).(\sin^2 x - \sin^2 2\pi/n).(\sin^2 x - \sin^2 3\pi/n)$………,,…..,, $.(\sin^2 x - \sin^2 (n-1)\pi/2n)$, when n is odd.

In other words, $$\sin n.x = \{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}.(\sin x)\prod_{k=1}^{\mathbf{(n-1)/2}}(\sin^2 x - \sin^2 k\pi/n)$$
when n is odd.
This can also be written as
$$\sin n.x=\{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}. \prod_{k=-(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)] \qquad (16)$$
when n is odd.

Since $\sin n.x = (-1)^{(n-1)/2}.T_n (\sin x)$ where $T_n (\sin x)$ is Chebyshev polynomial of first kind in sin x, therefore,
$$T_n (\sin x) = 2(^{n-1}).(\sin x)\prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)$$
Coming to equation (16), its three cases will be considered.

Case-1:
When x = 0 and n is odd, the identity (16) transforms to
$$\prod_{k=-(n-1)/2}^{(n-1)/2} \sin k\pi/n = 0.$$

It is trivial due to the fact, above product contains term sin 0 and sin 0 being equal to 0, makes the product 0.

Case-2:
When $x = \pi/2$ and $n = 4p+ 1$ where p is 0, 1, 2, 3………….., the identity 16 transforms to
$$\sin (4p+1) \pi/2 =(-1)^{2p}. (2^{4p}). \prod_{k=-2p}^{2p} \{1- \sin k\pi/(4p+1)\}$$
$$\text{Or} \quad \prod_{k=-2p}^{2p} \{1- \sin k\pi/(4p+1)\}= (½)^{4p}.$$
$$\text{Also} \quad \prod_{k=1}^{2p} (1- \sin^2 k\pi/(4p+1) = (½)^{4p}$$

$$\text{Or} \quad \prod_{k=1}^{2p} \cos k\pi/(4p+1) = (½)^{2p}$$

Case-3:

When $x = \pi/2$ and $n = 4p- 1$ where p is 1, 2, 3………….., the identity 16 transforms to

$$\sin (4p-1)\,\pi/2 = (-1)^{(2p-1)}\,(2^{4p-2}) \prod_{k=-2p+1}^{2p-1} \{1- \sin k\pi/(4p-1)\}$$

$$\prod_{k=-2p+1}^{2p-1} \{1- \sin k\pi/(4p-1)\} = (½)^{(4p-2)}$$

Also $$\prod_{k=1}^{2p-1} \{1- \sin^2 k\pi/(4p-1) = (½)^{(4p-2)}$$

or $$\prod_{k=1}^{2p-1} \cos k\pi/(4p-1) = (½)^{(2p-1)}$$

From above, it is clear that it does not matter whether n is of the form $4p+ 1$ or of form $4p-1$, the above identities therefore can be generalised as follows.

When n is odd and more than 1.

$$\prod_{k= -(n-1)/2}^{(n-1)/2} \{1- \sin k\pi/n \}= (½)^{(n-1)} \tag{17}$$

and

$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (½)^{(n-1)/2} \tag{18}$$

Now factorisation of sine of multiple angle is taken up when angle is even.That is it is of form 2k where k is 1, 2, 3………….. Let it be sin 2x.

By putting $\sin 2x = 0$, its roots correspond to angles that can be found as

$2x= +k\pi$,

$2x = -k\,\pi$,

where k is 0, 1, -1, ……… therefore on putting $k = 1$ and also $k = -1$, x equals $\pi/2$ and $-\pi/2$. Therefore, sin 2x should have factors $(\sin x- \sin \pi/2).(\sin x + \sin \pi/2)$ but it is $1- \sin^2 x$ which equals $\cos^2 x$. However, sin 2x has factors $2.\sin x. \cos x$. If we include, $k = 0$ also, then sin x can be considered its factor. That means if we consider three roots by putting k= 1, 0, -1, then its factors are $\sin x. (\sin^2 x -1)$ or $- \sin x. \cos^2$. But sin 2x can have two factors, therefore whenever factor $(\cos^2 x-1)$ appears and if it is assumed to correspond to cos x, then the problem is solved and sin 2x will have two factors sin x and cos x and multiplying it with 2, it equates to sin 2x. It can therefore be written as

$\sin 2x = (-1)^{(2/2+1)} . 2^1. \cos x. \sin x.$

Likewise sin 4x will have factors corresponding to k = -1, 0, 1, -2 and 2 but as proposed above factor corresponding to k = 4/2, -4/2 will be considered pertaining to – cos x, therefore,

$\sin 4x = (-1)^{(4/2+1)}. 2^{(4-1)}. \cos x. \sin x. (\sin^2 x- \sin^2 \pi/4)$

$= - 2^3. \cos x. \sin x. (\sin^2 x- 1/4).$

Likewise $\sin 6x = (-1)^{(6/2+1)}. 2^{(6-1)}. \cos x. \sin x. (\sin^2 x- \sin^2 \pi/6). (\sin^2 x- \sin^2 2\pi/6).$

$= 2^5. \cos x. \sin x. (\sin^2 x- 3/4). (\sin^2 x- 1/4).$

Similarly,

$\sin 8x = - 2^7. \cos x. \sin x. (\sin^2 x- \sin^2 \pi/8). (\sin^2 x- \sin^2 \pi/4). (\sin^2 x- \sin^2 3\pi/8).$

A generalised form can be written as

$\sin n.x = (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \sin x. (\sin^2 x- \sin^2 \pi/n). (\sin^2 x- \sin^2 2\pi/n). (\sin^2 x- \sin^2 3\pi/n)$………….. $(\sin^2 x- \sin^2 (n-2)\pi/2n.$

Also

$\sin n.x= (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \sin x. (\sin x- \sin \pi/n). (\sin x+ \sin \pi/n) (\sin x- \sin 2\pi/n) (\sin x+ \sin 2\pi/n)$ ………… $.\{\sin x- \sin (n-2)\pi/2n\}.\{\sin x- \sin (n-2)\pi/2n\}$

Or $$\sin n.x= (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \sin x. \prod_{k=1}^{n/2-1} (\sin^2 x- \sin^2 k\pi/n) \tag{19}$$

Or

$$\sin n.x= (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \prod_{k= -n/2+1}^{n/2-1} (\sin x- \sin k\pi/n) \tag{20}$$

when n is even and more than 2.

Since $\sin n.x = (-1)^{(n/2-1)}.\cos x.U_{n-1}(\sin x)$ where U_{n-1} is Chebyshev polynomial of second kind of degree n-1, Therefore,

$$U_{n-1}(\sin x) = 2^{(n-1)}.\cos x.\prod_{k=-n/2+1}^{n/2-1}(\sin x - \sin k\pi/n)$$

Also $\sin n.x = \sin x.\, U_{n-1}(\cos x)$ for all value of x, therefore,

$$U_{n-1}(\cos x) = (-1)^{(n/2+1)}.\, 2^{(n-1)}.\sin x.\prod_{k=1}^{n/2-1}(\cos^2 k\pi/n - \sin^2 x)$$

When x is equal to $\pi/4$,

$$\sin n\pi/4 = (-1)^{(n/2+1)}.\, 2^{(n-1)}.\cos \pi/4 \sin \pi/4.\prod_{k=1}^{n/2-1}(\sin^2 \pi/2 - \sin^2 k\pi/n)$$

$$\sin n\pi/4 = (-1)^{(n/2+1)}.\, 2^{(n-2)}\prod_{k=1}^{n/2-1}(\sin^2 \pi/2 - \sin^2 k\pi/n)$$

If n is not multiple of 4 or of form 2(4p+1) where p is any integer 1, 2, 3, ……… then

$$1 = 2^{8p}.\prod_{k=1}^{4p}(1/2 - \sin^2 k\pi/(8p+2)),$$

$$\prod_{k=1}^{4p}\cos k\pi/(4p+1) = (½)^{4p} \qquad (21)$$

or $\cos \pi/(4p+1).\cos 2\pi/(4p+1).\cos 3\pi/(4p+1)\ldots\ldots\cos 4p\pi/(4p+1) = (½)^{4p}$.

When n is of form 2(4p-1) and x is $\pi/4$, then

$$\prod_{k=1}^{4p-2}\cos 2k\pi/n = -(½)^{(4p-2)} \qquad (22)$$

$\cos \pi/(4p-1).\cos 2\pi/(4p-1).\cos 3\pi/(4p-1)\ldots\ldots\cos (4p-2)\pi/(4p-1) = -(½)^{(4p-2)}$

3. CONCLUSIONS AND RESULTS

Finite product of factors *{1/2+ cos 3^(j-1).2pi/n}* where j varies from 0 to k is solvable as equal to $(1/2)^k$ or $-(1/2)^k$ where k is given by relation $(3^k + 1)/n = m$ or $(3^k - 1)/n = m$. k and m are positive integers, however n may be any integer not multiple of 3 and it can be fraction of the form p/q where p is not multiple of 3. Positive or negative sign of $(1/2)^k$ depends upon whether m is even or odd and whether relation satisfied is $(3^k + 1)/n$ or $(3^k - 1)/n$.
When $(3^k + 1)/n = m$ and m is odd OR $(3^k - 1)/n = m$ and m is even, then the product is $(½)^k$.
When $(3^k + 1)/n = m$ and m is even OR $(3^k - 1)/n = m$ and m is odd, then the product is $-(½)^k$.

Term *$(3^k + 1)$ or $(3^k - 1)$* is always even and divisible by n where n is not multiple of 3. If n is multiple of 3, then this identity is not applicable.

In that case, another identity *ie finite product of factors cos $2^{(j-1)}.\pi/n$* where j varies from 0 to k is applicable and solvable as equal to $(1/2)^k$ or $-(1/2)^k$ where k is given by relation $(2^k + 1)/n = m$ or $(2^k - 1)/n = m$. k and m are positive integers, however n may be any integer not multiple of 2 and it can be fraction of the form p/q where p is not multiple of 2. Positive or negative sign of $(1/2)^k$ depends upon whether m is even or odd and whether relation satisfied is $(2^k + 1)/n$ or $(2^k - 1)/n$.
When $(2^k + 1)/n = m$ and m is odd OR $(2^k - 1)/n = m$ and m is even, then the product is $(½)^k$.
When $(2^k + 1)/n = m$ and m is even OR $(2^k - 1)/n = m$ and m is odd, then the product is $-(½)^k$.
Term *$(2^k + 1)$ or $(2^k - 1)$* is always odd and divisible by n where n is not multiple of 2.

Cosine of multiple angle is always factorable. When it is made equal to zero, it gives angle that correspond to its roots. Since n.x is multiple angle, then putting $\cos n.x = 0$ and n.x equal $(2k+1).\pi/2$ where k is integer 0, 1, 2, 3…………, value of angles x can be found as $\pi/2n, 3\pi/2n, 5\pi/2n\ldots,\ldots\ldots(2k+1).\pi/2n$ where k will have values 0, 1, 3, ….(n-1).

Hence its roots are $\cos \pi/2n$, $\cos 3\pi/2n$, $\cos 5\pi/2n$…,……$\cos (2k+1).\pi/2n$ and its factors are $(\cos x - \cos \pi/2n)$, $(\cos x - \cos 3\pi/2n)$, $(\cos x - \cos 5\pi/2n)$…,……$\{\cos x - \cos (2k+1).\pi/2n\}$. Therefore,
$\cos n.x = 2^{(n-1)}.(\cos x - \cos \pi/2n), (\cos x - \cos 3\pi/2n), (\cos x - \cos 5\pi/2n)\ldots,\ldots\ldots\{\cos x - \cos (2k+1).\pi/2n\}$.

Here term $2^{(n-1)}$ is multiplied to exact its value with that of $\cos n.x$.

From this, the identities,

(1). $\cos n.x = 2^{(n-1)}.\{\cos x - \cos \pi/2n\}.\{\cos x - \cos 3.\pi/2n. \{\cos x - \cos 5.\pi/2n\}\ldots\ldots\{\cos x - \cos (2n-1).\pi/2n\}$

Or

$$\cos n.x = 2^{(n-1)}. \prod_{k=0}^{n-1} \{\cos x - \cos (2k+1).\pi/2n\}$$

(2).
$$\prod_{k=0}^{n-1} \{1 - \cos (2k+1).\pi/2n\} = (½)^{(n-1)}$$

or

$$\prod_{k=0}^{n-1} \sin (2k+1).\pi/4n = (½)^{(n-1/2)}$$

(3). *When n is even,*

$$\prod_{k=0}^{n-1} \{\cos (2k+1).\pi/2n = (-1)^{n/2}. (½)^{(n-1)},$$

are derived.

Similarly, $\sin n.x$ when n is odd, can also be factorised and multiplication of its factors equated to $\sin n.x$ with precaution that factors are multiplied $(-1)^{(n-1)/2}. (2)^{(n-1)}$ multiplication so as to exact its value with that of $\sin n.x$.

1. $\sin n.x = \{(-1)^{(n-1)/2}.2^{(n-1)}\}(\sin x).(\sin^2 x - \sin^2 \pi/n).(\sin^2 x - \sin^2 2\pi/n).(\sin^2 x - \sin^2 3\pi/n)\ldots.(\sin^2 x - \sin^2 (n-1)\pi/2n)$, *when n is odd*

In other words,
$$\sin n.x = \{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}.(\sin x) \prod_{k=1}^{(n-1)/2}(\sin^2 x - \sin^2 k\pi/n)$$
when n is odd.

This can also be written as
$$\sin n.x = \{(-1)^{(n-1)/2}\}\{2^{(n-1)}\}. \prod_{k=-(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)$$
when n is odd.

2.
$$\prod_{k=-(n-1)/2}^{(n-1)/2} \{1 - \sin k\pi/n \} = (½)^{(n-1)}$$

and
$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (½)^{(n-1)/2}$$
when n is odd.

Sin n.x can also be factorised when n is even as
$$\sin n.x = (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \sin x. \prod_{k=1}^{n/2-1} (\sin^2 x - \sin^2 k\pi/n)$$
or
$$\sin n.x = (-1)^{(n/2+1)}. 2^{(n-1)}.\cos x. \prod_{k=-n/2+1}^{n/2-1} (\sin x - \sin k\pi/n)$$

Also

1.
$$\prod_{k=1}^{4p} \cos k\pi/(4p+1) = (½)^{4p}$$
or $\cos \pi/(4p+1). \cos 2\pi/(4p+1). \cos 3\pi/(4p+1)\ldots\ldots\ldots\ldots\ldots\ldots. \cos 4p\pi/(4p+1) = (½)^{4p}$
when n is even.

2. $$\prod_{k=1}^{4p-2} \cos k\pi/(4p-1) = -(½)^{(4p-2)}$$

$\cos \pi/(4p-1).\ \cos 2\pi/(4p-1).\ \cos 3\pi/(4p-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots \cos (4p-2)\pi/(4p-1) = -(½)^{(4p-2)}$

4. REFERENCES

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