

MORE TRIGONOMETRIC IDENTITIES

MAIN SUBJECT CLASSIFICATION: MATHEMATICS EDUCATION 97 G60 PLANE AND SPHERICAL TRIGONOMETRY

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ABSTRACT

There are certain trigonometric identities which look formidable and intimidating from the point of view of their solutions and appear tedious and cumbersome also. In ordinary ways, these seem practically unsolvable until we know the exact method to solve these. Most of the identities being considered here, according to me, are not known earlier and do not find mention in textbooks but attempt has been made to present and solve these by applying unique and different methods. Trigonometric ratios of sine and cosine are expandable in series expansion and also their multiple angles are expressible in single angle but their finite product is hardly found in textbooks, however, infinite product is commonly given. Attempt has been made to factorise these ratios and express these as finite product. In doing so, it turns out that such multiplication leads to Chebyshev polynomials. Hence roots of Chebyshev polynomials can be found out which are otherwise difficult to determine considering the degree n of the polynomials. To start with, a few such identities are given below, thereafter, their General formulae are derived and then sine and cosine of multiple angles are factorised and used in solving Chebyshev polynomials.

1. $(\cos^2 \pi/3 - \cos^2 \pi/250) \cdot (\cos^2 \pi/3 - \cos^2 3\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 7\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 9\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 11\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 13\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 17\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 19\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 21\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 23\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 27\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 29\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 31\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 33\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 37\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 39\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 41\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 43\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 47\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 49\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 51\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 53\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 57\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 59\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 61\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 63\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 67\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 69\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 71\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 73\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 77\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 79\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 81\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 83\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 87\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 89\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 91\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 93\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 97\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 99\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 101\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 103\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 107\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 109\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 111\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 113\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 117\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 119\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 121\pi/250) \cdot (\cos^2 \pi/3 - \cos^2 123\pi/250) = (1/2)^{100}$
2. $(\cos \pi/3 - \cos \pi/131) \cdot (\cos \pi/3 + \cos 2\pi/131) \cdot (\cos \pi/3 - \cos 3\pi/131) \cdot (\cos \pi/3 + \cos 4\pi/131) \cdot (\cos \pi/3 - \cos 5\pi/131) \cdot (\cos \pi/3 + \cos 6\pi/131) \cdot (\cos \pi/3 - \cos 7\pi/131) \cdot (\cos \pi/3 + \cos 8\pi/131) \cdot (\cos \pi/3 - \cos 9\pi/131) \cdot (\cos \pi/3 + \cos 10\pi/131) \cdot (\cos \pi/3 - \cos 11\pi/131) \cdot (\cos \pi/3 + \cos 12\pi/131) \cdot (\cos \pi/3 - \cos 13\pi/131) \cdot (\cos \pi/3 + \cos 14\pi/131) \cdot (\cos \pi/3 - \cos 15\pi/131) \cdot (\cos \pi/3 + \cos 16\pi/131) \cdot (\cos \pi/3 - \cos 17\pi/131) \cdot (\cos \pi/3 + \cos 18\pi/131) \cdot (\cos \pi/3 - \cos 19\pi/131) \cdot (\cos \pi/3 + \cos 20\pi/131) \cdot (\cos \pi/3 - \cos 21\pi/131) \cdot (\cos \pi/3 + \cos 22\pi/131) \cdot (\cos \pi/3 - \cos 23\pi/131) \cdot (\cos \pi/3 + \cos 24\pi/131) \cdot (\cos \pi/3 - \cos 25\pi/131) \cdot (\cos \pi/3 + \cos 26\pi/131) \cdot (\cos \pi/3 - \cos 27\pi/131) \cdot (\cos \pi/3 + \cos 28\pi/131) \cdot (\cos \pi/3 - \cos 29\pi/131) \cdot (\cos \pi/3 + \cos 30\pi/131) \cdot (\cos \pi/3 - \cos 31\pi/131) \cdot (\cos \pi/3 + \cos 32\pi/131) \cdot (\cos \pi/3 - \cos 33\pi/131) \cdot (\cos \pi/3 + \cos 34\pi/131) \cdot (\cos \pi/3 - \cos 35\pi/131) \cdot (\cos \pi/3 + \cos 36\pi/131) \cdot (\cos \pi/3 - \cos 37\pi/131) \cdot (\cos \pi/3 + \cos 38\pi/131) \cdot (\cos \pi/3 - \cos 39\pi/131) \cdot (\cos \pi/3 + \cos 40\pi/131) \cdot (\cos \pi/3 - \cos 41\pi/131) \cdot (\cos \pi/3 + \cos 42\pi/131) \cdot (\cos \pi/3 - \cos 43\pi/131) \cdot (\cos \pi/3 + \cos 44\pi/131) \cdot (\cos \pi/3 - \cos 45\pi/131) \cdot (\cos \pi/3 + \cos 46\pi/131) \cdot (\cos \pi/3 - \cos 47\pi/131) \cdot (\cos \pi/3 + \cos 48\pi/131) \cdot (\cos \pi/3 - \cos 49\pi/131) \cdot (\cos \pi/3 + \cos 50\pi/131) \cdot (\cos \pi/3 - \cos 51\pi/131) \cdot (\cos \pi/3 + \cos 52\pi/131) \cdot (\cos \pi/3 - \cos 53\pi/131) \cdot (\cos \pi/3 + \cos 54\pi/131) \cdot (\cos \pi/3 - \cos 55\pi/131) \cdot (\cos \pi/3 + \cos 56\pi/131) \cdot (\cos \pi/3 - \cos 57\pi/131) \cdot (\cos \pi/3 + \cos 58\pi/131) \cdot (\cos \pi/3 - \cos 59\pi/131) \cdot (\cos \pi/3 + \cos 60\pi/131) \cdot (\cos \pi/3 - \cos 61\pi/131) \cdot (\cos \pi/3 + \cos 62\pi/131) \cdot (\cos \pi/3 - \cos 63\pi/131) \cdot (\cos \pi/3 + \cos 66\pi/131) \cdot (\cos \pi/3 - \cos 67\pi/131) = (1/2)^{65}$

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Formulae on which these are based are given below. In addition, other identities are also derived using simple but unconventional methods of factorisation of sine and cosine of multiple angle. Symbols Π denotes multiplication of terms and symbol k denotes power to which the base is raised. That is $2^3 = 8$.

A. $(1/2 + \cos 2\pi/n)(1/2 + \cos 3.2\pi/n)(1/2 + \cos 9.2\pi/n)(1/2 + \cos 27.2\pi/n) \dots (1/2 + \cos 81.2\pi/n) \dots$
 $[1/2 + \cos (3^{k-1}).2\pi/n] = (1/2)^k$

In other words, $\prod_{j=1}^k \{1/2 + \cos 3^{j-1}.2\pi/n\} = (1/2)^k$,

also by putting $x = \pi/n$, $\prod_{j=1}^k \{1/2 + \cos 3^{j-1}.2x\} = (1/2)^k$,

where integers m and k are given by relation $(3^k + 1)/n = m$ and m is odd. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/n = m$ and m is even. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

B. $(1/2 + \cos 2\pi/n)(1/2 + \cos 3.2\pi/n)(1/2 + \cos 9.2\pi/n)(1/2 + \cos 27.2\pi/n) \dots (1/2 + \cos 81.2\pi/n) \dots [1/2 + \cos (3^{k-1}).2\pi/n] = -(1/2)^k$

In other words, $\prod_{j=1}^k \{1/2 + \cos 3^{j-1}.2\pi/n\} = -(1/2)^k$,

also by putting $x = \pi/n$, $\prod_{j=1}^k \{1/2 + \cos 3^{j-1}.2x\} = -(1/2)^k$,

where integers m and k are given by relation $(3^k + 1)/n = m$ and m is even. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/n = m$ and m is odd. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

C. When $(2^k + 1)/n = m$ and m is odd

or

when $(2^k - 1)/n = m$ and m is even.

$\cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n \cdot 2^{k-1} = (1/2)^k$

or

$\prod_{j=1}^k \cos 2^{j-1}.\pi/n = (1/2)^k$,

when integers k and m are given by relation $(2^k + 1)/n = m$ and m is odd or relation $(2^k - 1)/n = m$ and m is even for all odd n .

And also by putting $\pi/n = x$,

$\prod_{j=1}^k \cos 2^{j-1}.x = (1/2)^k$,

when integers m and k are given by relation $(2^k + 1).x = m.\pi$ and m is odd

or

when integers m and k are given by relation $(2^k - 1).x = m.\pi$ and m is even. And n is odd in all cases.

For cases where $(2^k + 1)/n = m$ and m is even

or

where $(2^k - 1)/n = m$ and m is odd.

$\prod_{j=1}^k \cos 2^{j-1}.\pi/n = -(1/2)^k$,

when integers k and m are given by relation $(2^k + 1)/n = m$ and m is even or relation $(2^k - 1)/n = m$ and m is odd and also n is odd.

It can be written in angle x form by putting $\pi/n = x$ as

$\prod_{j=1}^k \cos 2^{j-1}.x = -(1/2)^k$,

when integers m and k are given by relation $(2^k + 1).x = m.\pi$ and m is odd

or

when integers m and k are given by relation $(2^k - 1)/n = m.\pi$ and m is even. And n is odd in all cases.

D.1). $\cos n.x = 2^{n-1} \cdot \{\cos x - \cos \pi/2n\} \cdot \{\cos x - \cos 3\pi/2n\} \cdot \{\cos x - \cos 5\pi/2n\} \dots \dots \{\cos x - \cos (2n-1)\pi/2n\}$

Or

$$\cos n.x = 2^{n-1} \cdot \prod_{k=0}^{n-1} \{\cos x - \cos (2k+1)\pi/2n\}$$

(2). $\prod_{k=0}^{n-1} \{1 - \cos (2k+1)\pi/2n\} = (1/2)^{n-1}$

or

$$\prod_{k=0}^{n-1} \sin (2k+1)\pi/4n = (1/2)^{(n-1)/2}$$

(3). When n is even,

$$\prod_{k=0}^{n-1} \{\cos (2k+1)\pi/2n\} = (-1)^{n/2} \cdot (1/2)^{(n-1)}$$

E. (1) When n is odd and more than 1.

$$\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{n-1}\} (\sin x) \cdot (\sin^2 x - \sin^2 \pi/n) \cdot (\sin^2 x - \sin^2 2\pi/n) \cdot (\sin^2 x - \sin^2 3\pi/n) \dots \dots \dots \cdot (\sin^2 x - \sin^2 (n-1)\pi/2n)$$

Or

$$\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{n-1}\} (\sin x) \prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)$$

Or

$$\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{n-1}\} \prod_{k=-(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)$$

(2). When n is odd and more than 1,

$$\prod_{k=-(n-1)/2}^{(n-1)/2} \{1 - \sin k\pi/n\} = (1/2)^{n-1}$$

Or

$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (1/2)^{(n-1)/2}$$

F. When n is even and more than 2,

1. $\sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \sin x \cdot \prod_{k=1}^{n/2-1} (\sin^2 x - \sin^2 k\pi/n)$

Also when n is even and more than 2,

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \prod_{k=-n/2+1}^{n/2-1} (\sin x - \sin k\pi/n)$$

2. $\prod_{k=1}^p \cos k\pi/(4p+1) = (1/2)^{4p}$

or $\cos \pi/(4p+1) \cdot \cos 2\pi/(4p+1) \cdot \cos 3\pi/(4p+1) \dots \dots \dots \cos 4p\pi/(4p+1) = (1/2)^{4p}$

3. $\prod_{k=1}^{4p-2} \cos k\pi/(4p-1) = -(1/2)^{(4p-2)}$

$\cos \pi/(4p-1) \cdot \cos 2\pi/(4p-1) \cdot \cos 3\pi/(4p-1) \dots \dots \dots \cos (4p-2)\pi/(4p-1) = -(1/2)^{(4p-2)}$

Keywords: Trigonometry Identities, Multiple Angles, Finite Product, Recursive Relation, Factorisation, Multiplication. Chebyshev Polynomial.

1. INTRODUCTION

Cosine and sine of an angle x are ratios of base to hypotenuse, perpendicular to hypotenuse respectively in a right angled triangle with angle x between base and hypotenuse and these can be written as

$$\cos x = \{(1 + \cos 2x)/2\}^{1/2} \quad (1)$$

$$\sin x = \{(1 - \cos 2x)/2\}^{1/2} \quad (2)$$

on the basis of formula, $\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

by putting $x = y$. Sign $^{\wedge}$ means, it is raised to power, for example $\sin^2 x = (\sin x) \cdot (\sin x)$.

Similarly, $\cos 3x$ and $\sin 3x$ can be written as

$$\cos 3x = 4 \cos^3 x - 3 \cos x \quad (3)$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x \quad (4)$$

by expanding $\cos(x + 2x)$ and $\sin(x + 2x)$ so as to reduce the angle from $3x$ to x .

Before proceeding further, it would be appropriate to be familiar with recursive relation as it will be used repeatedly in this paper. A recursive relation is that where in an equation, quantity on left side also appears in right hand side. In this way, quantity on RHS can be successively substituted by the quantity in left hand side. Such a relation forms self generating and never ending loop. For example the recursive relation $x = 1 + \sqrt{x}$ has x in the left hand side as well as in the right side. Therefore, $x = 1 + \sqrt{x}$ can be substituted for x in right hand side and that makes $x = 1 + (1 + \sqrt{x})^{1/2}$. On successive substitution we get,

$$x = x = 1 + [1 + \{1 + \dots\dots\dots(1 + \sqrt{x})^{1/2} \dots\dots\}^{1/2}]^{1/2}$$

We will be using the recursive relation, angle doubling and tripling formula to achieve the desired result. With this introduction, we proceed to main derivations.

An angle can be mentioned in degrees or radians and can also be converted from degrees to radians and vice versa by the relation π written as $\pi = 180$ degrees. Therefore one radian equals $180/\pi$, it is also defined as an angle subtended by an arc equal in length to radius at the centre of the circle. In this way, an angle x can be written in radians as π/n and also, it can be transformed to this form by rearranging it. n can be any integer except 0, n can be a fraction also of the form p/q . Using recursive relations, value of finite product of trigonometric identities will be determined. At the same time, $\cos n.x$ and $\sin n.x$ will also be expressed as finite product.

2. THEORY AND CONCEPT

2.1 Part 1.

First, those cases where n is any integer except multiple of 3 or is a fraction p/q where p is any integer except multiple of 3, will be taken up.

That means angle $x = \pi/n$ must not have n multiple of 3. Angle x may be of the form for examples, $\pi/62, \pi/8, \pi/7, \pi/31, \pi/(8/3), \pi/(56/3)$ or π divided by any integer except multiple of 3 or $\pi/(p/q)$ where p is any integer not multiple of 3 and q is odd.

Equation (4) can be written as

$$\sin x \cdot (3 - 4 \sin^2 x) = \sin 3x \text{ or}$$

$$\sin x = \sin 3x / (3 - 4 \sin^2 x) \quad (5)$$

This relation is recursive in nature as $\sin x$ appears on left hand side and also as $\sin 3x$ in right hand side. $\sin x$ in LHS and $\sin 3x$ in RHS in numerator are being considered for recurrence.

From recursive relation (5), $\sin 3x$ can also be written as

$$\sin 3x = \sin 9x / (3 - 4 \sin^2 3x),$$

$$\sin 9x = \sin 27x / (3 - 4 \sin^2 9x)$$

.....so on.

$$\sin x \cdot 3^{k-1} = \sin 3^k / [3 - 4 \sin^2 \{x \cdot 3^{k-1}\}]$$

On substituting value of $\sin 3x = \sin 9x / (3 - 4 \sin^2 3x)$, relation (5) can be written as

$$\sin x = \sin 9x / (3 - 4 \sin^2 x) \cdot (3 - 4 \sin^2 3x).$$

On successively substituting values of $\sin 9x, \sin 27x, \dots\dots\dots \sin x \cdot 3^{k-1}$, equation takes the form

$$\sin x = \sin x \cdot 3^k / [(3 - 4 \sin^2 x) \cdot (3 - 4 \sin^2 3x) \cdot (3 - 4 \sin^2 9x) \cdot (3 - 4 \sin^2 27x) \dots\dots\dots \{3 \cdot \sin^2 \{x \cdot 3^{k-1}\}\}].$$

Putting $x = \pi/n$, this equation is written as

$$\sin \pi/n = \sin \pi/n \cdot 3^k / [(3 - 4 \sin^2 \pi/n) \cdot (3 - 4 \sin^2 3\pi/n) \cdot (3 - 4 \sin^2 9\pi/n) \cdot (3 - 4 \sin^2 27\pi/n) \dots\dots\dots \{3 \cdot \sin^2 \{\pi/n \cdot 3^{k-1}\}\}] \quad (6)$$

Next task is to find out k so that equation (6) simplifies. For that a proposition is given that follows.

2.2. Proposition

If n is any integer but not multiple of three or is of form p/q where q is not multiple of 3 then $(3^k + 1)$ or $(3^k - 1)$ will always be divisible by n for some integer value of k. It is obvious $3^k + 1$ and $3^k - 1$ will always be even on account of the fact that odd integer plus 1 or - 1 is always even. Also division of even integer by some other even integer is possible.

Here k can assume any value from 0 and then proceeding to 1, then 2, then 3 so on till that value of k is reached where it gets divided by n without any remainder. There are thus infinite positive integers available as value of k. Since even can be divided by even, then there is complete possibility for division of $(3^k + 1)$ or $(3^k - 1)$ by n.

Let $m = (3^k + 1)/n$ or
 $m = (3^k - 1)/n$

where m is that minimum number where $(3^k + 1)$ or $(3^k - 1)$ is just divisible by n.

2.3 Application of Proposition

Applying this proposition that there exists a value of k that makes $(3^k + 1)$ or $(3^k - 1)$ divisible by n, we can easily say that denominator of equation (6) has factors which are k in number. First factor pertains to angle $\pi/n \cdot 3^0$, second to $\pi/n \cdot 3^1$, third to $\pi/n \cdot 3^2$, fourth to $\pi/n \cdot 3^3$so on and kth to $\pi/n \cdot 3^{(k-1)}$.

From equation (6), it is clear, it has numerator $\sin \pi/n \cdot 3^k$ and this pertains to angle $\pi/n \cdot 3^k$. It is already proved

$$(3^k + 1)/n = m, \text{ or}$$

$$(3^k - 1)/n = m.$$

First taking the case where $(3^k + 1)/n = m$ and on multiplying both sides by π , makes

$$\pi/n \cdot 3^k = (m \pi - \pi/n)$$

and numerator of RHS of equation (6) becomes

$$\sin \pi/n \cdot 3^k = \sin (m \pi - \pi/n) = \sin \pi/n \text{ when } m \text{ is odd or equals } - \sin \pi/n \text{ when } m \text{ is even.}$$

For the cases where $(3^k - 1)/n = m$, numerator of RHS of equation (6) becomes

$$\sin \pi/n \cdot 3^k = \sin (m \pi + \pi/n) = \sin \pi/n \text{ when } m \text{ is even or it equals } - \sin \pi/n \text{ when } m \text{ is odd.}$$

When $(3^k + 1)/n = m$ and m is odd but n is not multiple of 3

or

when when $(3^k - 1)/n = m$ and also m is even but n is not multiple of 3, equation (6) takes the form,

$$\sin \pi/n = \sin \pi/n [(3 - 4 \sin^2 \pi/n) \cdot (3 - 4 \sin^2 3 \pi/n) \cdot (3 - 4 \sin^2 9 \pi/n) \cdot (3 - 4 \sin^2 27 \pi/n) \dots \{3 \sin^2 \{\pi/n \cdot 3^{(k-1)}\}\}].$$

On cancelling $\sin \pi/n$ on both sides, we get

$$[(3 - 4 \sin^2 \pi/n) \cdot (3 - 4 \sin^2 3 \pi/n) \cdot (3 - 4 \sin^2 9 \pi/n) \cdot (3 - 4 \sin^2 27 \pi/n) \dots \{3 \sin^2 \{\pi/n \cdot 3^{(k-1)}\}\}] = 1 \tag{7}$$

We know from equation (3) that

$$\sin x = (1 - \cos 2x)/2 \text{ } ^{1/2}. \text{ This can also be written as}$$

$$\cos 2x = 1 - 2 \sin^2 x.$$

Using this identity in identity (7), we get

$$(1 + 2 \cos 2\pi/n) \cdot (1 + 2 \cos 6\pi/n) \cdot (1 + 2 \cos 18\pi/n) \cdot (1 + 2 \cos 54\pi/n) \dots \{1 + 2 \cos \{2\pi/n \cdot 3^{(k-1)}\}\} = 1 \tag{8}$$

This can also be written as

$$(1/2 + \cos 2\pi/n) \cdot (1/2 + \cos 6\pi/n) \cdot (1/2 + \cos 18\pi/n) \cdot (1/2 + \cos 54\pi/n) \dots \{1/2 + \cos \{2\pi/n \cdot 3^{(k-1)}\}\} = (1/2)^k \tag{9}$$

In other words, $\prod_{j=1}^k \{1/2 + \cos 3^{(j-1)} \cdot 2\pi/n\} = (1/2)^k$, (10)

also by putting $x = \pi/n$, $\prod_{j=1}^k \{1/2 + \cos 3^{(j-1)} \cdot 2x\} = (1/2)^k$, (11)

When $(3^k + 1)/n = m$ and m is even but n is not multiple of 3,

or

when when $(3^k - 1)/n = m$ and also m is odd but n is not multiple of 3, right hand side of identities (7), (8) and (9) will be negative.

Therefore, in such cases,

$$[(3- 4.\sin^2 \pi/n).(3- 4.\sin^2 3 \pi/n).(3- 4.\sin^2 9 \pi/n).(3- 4.\sin^2 27 \pi/n).....\{3.\sin^2 \{\pi/n.3^{(k-1)}\}\}] = -1 \quad (7/1)$$

$$(1+ 2.\cos 2\pi/n).(1+ 2.\cos 6\pi/n).(1+ 2.\cos 18\pi/n).(1+ 2.\cos 54\pi/n).....1+ 2.\cos \{2\pi/n.3^{(k-1)}\} = -1 \quad (8/1)$$

$$(1/2+ \cos 2\pi/n).(1/2+ \cos 6\pi/n).(1/2+ \cos 18\pi/n).(1/2+ \cos 54\pi/n).....\{1/2+ \cos \{2\pi/n.3^{(k-1)}\}\} = -(1/2)^k. \quad (9/1)$$

In other words,
$$\prod_{j=1}^k \{1/2+ \cos 3^{(j-1)}.2\pi/n\} = - (1/2)^k. \quad (10/1)$$

also by putting $x = \pi/n$,
$$\prod_{j=1}^k \{1/2+ \cos 3^{(j-1)}.2x\} = - (1/2)^k, \quad (11/1)$$

2.4 Examples 1:

Sin $\pi/5$

Here n is 5 and let us find out m and k by relation

$$m = (3^k + 1)/n \text{ or}$$

$$m = (3^k - 1)/n.$$

It satisfies the relation

$m = (3^k + 1)/n$ when $k = 2$, that makes m equals 2. That means there are two factors as $k = 2$ and their product is -1 from identities (9) and 11 since m is even. Therefore,

$$(1 + 2 \cos 2.\pi/5). (1 + 2 \cos 6.\pi/5) = -1,$$

$$\text{Or } (1 + 2 \cos 2.\pi/5). (1 - 2 \cos \pi/5) = -1$$

$$\text{Or } (1/2 + \cos 2.\pi/5). (1/2 - \cos \pi/5) = -(1/2)^2 = -1/4.$$

From this, value of $\cos \pi/5$ can also be found out.

$$(1 + 2 \cos 2.\pi/5). (1 - 2 \cos \pi/5) = -1$$

$$1 + 2 \cos 2.\pi/5 - 2 \cos \pi/5 - 4.\cos \pi/5.\cos 2.\pi/5 = -1$$

$$\text{Or } 1 + 2 \cos 2.\pi/5 - 2 \cos \pi/5 - 2.(\cos 3\pi/5 + \cos \pi/5) = -1$$

$$\text{Or } 1 + 2 \cos 2.\pi/5 - 2 \cos \pi/5 + 2.\cos 2\pi/5 - 2.\cos \pi/5 = -1$$

$$\text{Or } 2 (\cos \pi/5 - \cos 2\pi/5) = 2$$

$$4 \cos^2 \pi/5 - 2.\cos \pi/5 - 1 = 0.$$

This is a quadratic in $\cos \pi/5$ and its roots are

$$(1 + 5^{1/2})/4 \text{ and } (1 - 5^{1/2})/4.$$

Ignoring negative value, we get

$$\cos \pi/5 = (1 + 5^{1/2})/4 \text{ or Golden ratio } \phi = 2.\cos \pi/5 = (1 + 5^{1/2})/2.$$

Thus the identity is useful in derivation of value of $\cos \pi/5$ or $\sin \pi/5$.

Example 2:

Sin $\pi/10$

Here n is 10 and let us find out m and k by relation

$$m = (3^k + 1)/n \text{ or}$$

$$m = (3^k - 1)/n.$$

It satisfies the relation

$m = (3^k + 1)/n$ when $k = 2$, that makes m equals 1. That means there are two factors and their product is 1 from identity (9) since m is odd. Therefore,

$$(1 + 2 \cos \pi/5). (1 + 2 \cos 3.\pi/5) = 1.$$

Example 3:

Sin $\pi/80$

Here n is 80 and let us find out m and k by relation

$$m = (3^k + 1)/n \text{ or}$$

$$m = (3^k - 1)/n.$$

It satisfies the relation

$m = (3^k - 1)/n$ when $k = 4$, that makes m equals 1. That means there are four factors and their product is -1 from identities (7/1) and (8/1) since m is odd and relation $m = (3^k - 1)/n$ is satisfied. Therefore,

$$(1 + 2 \cos \pi/40). (1 + 2 \cos 3\pi/40).(1 + 2 \cos 9\pi/40). (1 + 2 \cos 27\pi/40) = -1.$$

On simplification,

$$(1 + 2 \cos \pi/40). (1 + 2 \cos 3\pi/40).(1 + 2 \cos 9\pi/40). (1 - 2 \cos 13\pi/40) = -1.$$

Example 5:

Sin $\pi/100$

Here n is 100 and let us find out m and k by relation

$$m = (3^k + 1)/n \text{ or}$$

$$m = (3^k - 1)/n.$$

It satisfies the relation

$m = (3^k - 1)/n$ when $k = 20$, that makes m equals 34867844 which is even. That means there are 20 factors and their product is 1 from identities (8) and (10) since m is even and relation $m = (3^k - 1)/n$ is satisfied. Therefore,

$$(1 + 2 \cos \pi/50). (1 + 2 \cos 3\pi/50).(1 + 2 \cos 9\pi/50). (1 + 2 \cos 27\pi/50).(1 + 2 \cos 3^4\pi/50). (1 + 2 \cos 3^5\pi/50). (1 + 2 \cos 3^6\pi/50). (1 + 2 \cos 3^7\pi/50).(1 + 2 \cos 3^8\pi/50). (1 + 2 \cos 3^9\pi/50).(1 + 2 \cos 3^{10}\pi/50). (1 + 2 \cos 3^{11}\pi/50).(1 + 2 \cos 3^{12}\pi/50). (1 + 2 \cos 3^{13}\pi/50).(1 + 2 \cos 3^{14}\pi/50). (1 + 2 \cos 3^{15}\pi/50). (1 + 2 \cos 3^{16}\pi/50). (1 + 2 \cos 3^{17}\pi/50).(1 + 2 \cos 3^{18}\pi/50). (1 + 2 \cos 3^{19}\pi/50) = 1$$

$$\text{Or } (1 + 2 \cos \pi/50). (1 + 2 \cos 3\pi/50).(1 + 2 \cos 9\pi/50). (1 - 2 \cos 23\pi/50).(1 + 2 \cos 19\pi/50). (1 - 2 \cos 7\pi/50). (1 - 2 \cos 21\pi/50). (1 + 2 \cos 13\pi/50).(1 - 2 \cos 11\pi/50). (1 + 2 \cos 17\pi/50).(1 - 2 \cos \pi/50). (1 - 2 \cos 3\pi/50). (1 - 2 \cos 9\pi/50). (1 + 2 \cos 23\pi/50).(1 - 2 \cos 19\pi/50). (1 + 2 \cos 7\pi/100). (1 + 2 \cos 21\pi/50). (1 - 2 \cos 13\pi/50). (1 + 2 \cos 11\pi/50). (1 - 2 \cos 17\pi/50) = 1$$

$$\text{Or } (1 - 4 \cos^2 \pi/50). (1 - 4 \cos^2 3\pi/50).(1 - 4 \cos^2 7\pi/50).(1 - 4 \cos^2 9\pi/50).(1 - 4 \cos^2 11\pi/50)$$

$$(1 - 4 \cos^2 13\pi/50). (1 - 4 \cos^2 17\pi/50). (1 - 4 \cos^2 19\pi/50)(1 - 4 \cos^2 21\pi/50). (1 - 4 \cos^2 23\pi/50) = 1$$

$$\text{Or } (1/4 - \cos^2 \pi/50). (1/4 - \cos^2 3\pi/50).(1/4 - \cos^2 7\pi/50).(1/4 - \cos^2 9\pi/50).(1/4 - \cos^2 11\pi/50). (1/4 - \cos^2 13\pi/50).(1/4 - \cos^2 17\pi/50).(1/4 - \cos^2 19\pi/50).(1/4 - \cos^2 21\pi/50).(1/4 - \cos^2 23\pi/50) = (1/2)^{20}.$$

2.3b How to avoid excessive and difficult to calculate 3^k while calculating m in relations

$$m = (3^k + 1)/n \text{ or}$$

$$m = (3^k - 1)/n.$$

A practical difficulty is observed while finding m particularly when power k reaches higher values. To avoid such difficulty, basic trigonometric identities may be applied to contain value of k . Basic identities which can be utilised are

$$\sin(\pi - x) = \sin x, \quad \cos(\pi - x) = -\cos x$$

$$\sin(\pi + x) = -\sin x. \quad \cos(\pi + x) = -\cos x$$

$$\sin(2\pi - x) = -\sin x \quad \cos(2\pi - x) = \cos x$$

$$\sin(2\pi + x) = \sin x \quad \cos(2\pi + x) = \cos x.$$

These identities can be extended as

$$\sin \{(2k + 1)\pi - x\} = \sin x, \quad \cos \{(2k + 1)\pi - x\} = -\cos x,$$

$$\sin \{(2k + 1)\pi + x\} = -\sin x \quad \cos \{(2k + 1)\pi + x\} = -\cos x,$$

$$\sin (2.k.\pi + x) = \sin x \quad \cos (2.k.\pi + x) = \cos x.$$

$$\sin (2.k.\pi - x) = -\sin x \quad \cos (2.k.\pi - x) = \cos x$$

where k is any number 0, 1, 2, 3.....

As and when value of angle $(3^k)/n$ increases above $1/2$, it can be brought down by the above identities, k is any number for calculating m . However, signs positive and negative must be kept in mind. The process should be continued till value of angle reaches $\pi.(n + 1)/n$ or $\pi.(n - 1)/n$. Sign (positive or negative) of $\sin \pi.(n + 1)/n$ or $\sin \pi.(n - 1)/n$ as the case may be, found out. This sign of $\sin \pi.(n + 1)/n$ or $\sin \pi.(n - 1)/n$ then will be the sign of right hand side of the identity. For illustration, I take up the following example.

Example 4:

Sin $\pi/500$.

This can be expanded as

$$\sin \pi/500 = \sin \pi/500 / [(3 - 4 \sin^2 \pi/500). (3 - 4 \sin^2 3\pi/500).(3 - 4 \sin^2 9\pi/500).(3 - 4 \sin^2 27\pi/500). (3 - 4 \sin^2 81\pi/500).(3 - 4 \sin^2 243\pi/500). (3 + 4 \sin^2 229\pi/500) (3 - 4 \sin^2 187\pi/500). (3 + 4 \sin^2 61\pi/500). (3 + 4 \sin^2 183\pi/500). (3 - 4 \sin^2 49\pi/500). (3 - 4 \sin^2 147\pi/500). (3 - 4 \sin^2 59\pi/500). (3 - 4 \sin^2 177\pi/500). (3 + 4 \sin^2 31\pi/500). (3 + 4 \sin^2 93\pi/500). (3 + 4 \sin^2 221\pi/500). (3 - 4 \sin^2 163\pi/500). (3 - 4 \sin^2 11\pi/500).$$

(3- 4.sin² 33pi/500) (3- 4.sin² 99pi/500). (3- 4.sin² 203pi/500). (3+ 4.sin² 109pi/500). (3+ 4.sin² 173pi/500). (3- 4.sin² 19pi/500). (3- 4.sin² 57pi/500). (3- 4.sin² 171pi/500). (3+ 4.sin² 13pi/500). (3+ 4.sin² 39pi/500). (3+4.sin² 117pi/500). (3+ 4.sin² 149pi/500). (3+ 4.sin² 53pi/500). (3+ 4.sin² 159pi/500). (3+ 4.sin² 23pi/500). (3+ 4.sin² 69pi/500). (3+ 4.sin² 207pi/500). (3- 4.sin² 121pi/500). (3- 4.sin² 137pi/500). (3- 4.sin² 89pi/500). (3- 4.sin² 233pi/500). (3+ 4.sin² 199pi/500). (3- 4.sin² 97pi/500). (3- 4.sin² 209pi/500). (3+ 4.sin² 127pi/500). (3+ 4.sin² 119pi/500). (3+ 4.sin² 143pi/500). (3+ 4.sin² 71pi/500). (3+ 4.sin² 213pi/500). (3- 4.sin² 139pi/500). (3- 4.sin² 83pi/500). (3- 4.sin² 249pi/500). (3+ 4.sin² 247pi/500). (3- 4.sin² 241pi/500). (3+ 4.sin² 223pi/500). (3- 4.sin² 169pi/500). (3+ 4.sin² 7pi/500). (3+ 4.sin² 21pi/500). (3+ 4.sin² 63pi/500). (3+ 4.sin² 189pi/500). (3- 4.sin² 67pi/500). (3- 4.sin² 201pi/500). (3+ 4.sin² 103pi/500). (3+ 4.sin² 191pi/500). (3- 4.sin² 73pi/500). (3- 4.sin² 219pi/500). (3+ 4.sin² 157pi/500). (3+ 4.sin² 29pi/500). (3+ 4.sin² 87pi/500). (3+ 4.sin² 239pi/500). (3- 4.sin² 217pi/500). (3+ 4.sin² 151pi/500). (3+ 4.sin² 47pi/500). (3+ 4.sin² 141pi/500). (3+ 4.sin² 77pi/500). (3+ 4.sin² 231pi/500). (3- 4.sin² 193pi/500). (3+ 4.sin² 79pi/500). (3+ 4.sin² 237pi/500). (3- 4.sin² 211pi/500). (3+ 4.sin² 133pi/500). (3+ 4.sin² 101pi/500). (3+ 4.sin² 197pi/500). (3- 4.sin² 91pi/500). (3- 4.sin² 227pi/500). (3+ 4.sin² 181pi/500). (3- 4.sin² 43pi/500). (3- 4.sin² 129pi/500). (3- 4.sin² 113pi/500). (3- 4.sin² 161pi/500). (3- 4.sin² 17pi/500). (3- 4.sin² 51pi/500). (3- 4.sin² 153pi/500). (3- 4.sin² 41pi/500). (3- 4.sin² 123pi/500). (3- 4.sin² 131pi/500). (3- 4.sin² 107pi/500). (3- 4.sin² 179pi/500). (3+ 4.sin² 37pi/500). (3+ 4.sin² 111pi/500). (3+ 4.sin² 167pi/500)]

Here we have obtained lastly, - sin 501pi/500 and that is - sin (pi+pi/500) = sin pi/500 which has positive sign. In this way, there does not appear necessity of finding 'm' to decide sign (positive or negative) on right hand side. Here sign obtained is positive sin pi/500 without finding value of 'm'. In this way we obtain identity as

[(3- 4.sin² 2pi/500). (3- 4.sin² 3pi/500).(3- 4.sin² 9pi/500).(3- 4.sin² 27pi/500). (3- 4.sin² 81pi/500). (3- 4.sin² 243pi/500). (3+ 4.sin² 229pi/500) (3- 4.sin² 187pi/500). (3+ 4.sin² 61pi/500). (3+ 4.sin² 183pi/500). (3- 4.sin² 49pi/500). (3- 4.sin² 147pi/500). (3- 4.sin² 59pi/500). (3- 4.sin² 177pi/500). (3+ 4.sin² 31pi/500). (3+ 4.sin² 93pi/500). (3+ 4.sin² 221pi/500). (3- 4.sin² 163pi/500). (3- 4.sin² 11pi/500). (3- 4.sin² 33pi/500) (3- 4.sin² 99pi/500). (3- 4.sin² 203pi/500). (3+ 4.sin² 109pi/500). (3+ 4.sin² 173pi/500). (3- 4.sin² 19pi/500). (3- 4.sin² 57pi/500). (3- 4.sin² 171pi/500). (3+ 4.sin² 13pi/500). (3+ 4.sin² 39pi/500). (3+4.sin² 117pi/500). (3+ 4.sin² 149pi/500). (3+ 4.sin² 53pi/500). (3+ 4.sin² 159pi/500). (3+ 4.sin² 23pi/500). (3+ 4.sin² 69pi/500). (3+ 4.sin² 207pi/500). (3- 4.sin² 121pi/500). (3- 4.sin² 137pi/500). (3- 4.sin² 89pi/500). (3- 4.sin² 233pi/500). (3+ 4.sin² 199pi/500). (3- 4.sin² 97pi/500). (3- 4.sin² 209pi/500). (3+ 4.sin² 127pi/500). (3+ 4.sin² 119pi/500). (3+ 4.sin² 143pi/500). (3+ 4.sin² 71pi/500). (3+ 4.sin² 213pi/500). (3- 4.sin² 139pi/500). (3- 4.sin² 83pi/500). (3- 4.sin² 249pi/500). (3+ 4.sin² 247pi/500). (3- 4.sin² 241pi/500). (3+ 4.sin² 223pi/500). (3- 4.sin² 169pi/500). (3+ 4.sin² 7pi/500). (3+ 4.sin² 21pi/500). (3+ 4.sin² 63pi/500). (3+ 4.sin² 189pi/500). (3- 4.sin² 67pi/500). (3- 4.sin² 201pi/500). (3+ 4.sin² 103pi/500). (3+ 4.sin² 191pi/500). (3- 4.sin² 73pi/500). (3- 4.sin² 219pi/500). (3+ 4.sin² 157pi/500). (3+ 4.sin² 29pi/500). (3+ 4.sin² 87pi/500). (3+ 4.sin² 239pi/500). (3- 4.sin² 217pi/500). (3+ 4.sin² 151pi/500). (3+ 4.sin² 47pi/500). (3+ 4.sin² 141pi/500). (3+ 4.sin² 77pi/500). (3+ 4.sin² 231pi/500). (3- 4.sin² 193pi/500). (3+ 4.sin² 79pi/500).(3+ 4.sin² 237pi/500). (3- 4.sin² 211pi/500). (3+ 4.sin² 133pi/500).(3+ 4.sin² 101pi/500). (3+ 4.sin² 197pi/500). (3- 4.sin² 91pi/500). (3- 4.sin² 227pi/500). (3+ 4.sin² 181pi/500). (3- 4.sin² 43pi/500). (3- 4.sin² 129pi/500). (3- 4.sin² 113pi/500). (3- 4.sin² 161pi/500). (3- 4.sin² 17pi/500). (3- 4.sin² 51pi/500). (3- 4.sin² 153pi/500). (3- 4.sin² 41pi/500). (3- 4.sin² 123pi/500). (3- 4.sin² 131pi/500). (3- 4.sin² 107pi/500). (3- 4.sin² 179pi/500). (3+ 4.sin² 37pi/500). (3+ 4.sin² 111pi/500). (3+ 4.sin² 167pi/500) = 1.

This can also be written as

(1 + 2 cos pi/250). (1 + 2 cos 3pi/250).(1+2 cos 9pi/250). (1 +2 cos 27pi/250).(1+2 cos 81.pi/250).(1-2 cos 7pi/250). (1 - 2 cos 21.pi/250). (1 - 2 cos 63.pi/250).(1 + 2 cos 61.pi/250).(1 - 2 cos 67.pi/250) (1+ 2 cos.49pi/250). (1 -2 cos 103pi/250).(1 + 2 cos 59.pi/250). (1 - 2 cos 73.pi/250).(1+ 2 cos 31.pi/250).(1 + 2 cos 93.pi/259). (1-2 cos 29.pi/250). (1 -2 cos 87.pi/250). (1 + 2 cos 11.pi/250). (1+ 2 cos 33.pi/250).(1 + 2 cos 99.pi/250). (1 - 2 cos 47pi/250). (1 + 2 cos 109.pi/250).(1 - 2 cos 77pi/250). (1+ 2 cos 19pi/250). (1+2 cos.57.pi/250). (1 - 2 cos 79pi/250).(1 + 2 cos 13.pi/250). (1+2 cos 39pi/250). (1+ 2 cos 117pi/250).(1- 2 cos 101/250). (1+2 cos 53/250).(1-2 cos 91pi/250).(1+ 2 cos 23pi/250).(1+2 cos 69pi/250). (1- 2 cos 43pi/250). (1+2 cos 121pi/250). (1- 2 cos 113.pi/250).(1+2 cos.89pi/250). (1- 2 cos 17pi/250).(1 - 2 cos 51pi/250). (1 + 2 cos 97pi/250). (1- 2 cos 41pi/250). (1 - 2 cos 123/259). (1+ 2 cos 119pi/250). (1- 2 cos 107pi/250). (1+2 cos 71.pi/250).(1- 2 cos 37pi/250). (1- 2 cos 111pi/250).(1+ 2 cos 83.pi/250).(1-2 cos.pi/250).(1-2 cos 3.pi/250). (1 - 2 cos9pi/250). (1- 2 cos 27pi/250). (1- 2 cos 81.pi/250). (1 + 2 cos 7pi/250).(1+ 2 cos 21.pi/250). (1+ 2 cos 63.pi/250).(1- 2 cos 61.pi/250).(1+ 2 cos 67.pi/250).(1-2 cos.49pi/250). (1+ 2 cos 103pi/250). (1- 2 cos 59.pi/250).(1 + 2 cos 73.pi/250). (1- 2 cos 31.pi/250). (1- 2 cos 93.pi/259). (1+ 2 cos 29.pi/250). (1+ 2 cos 87.pi/250).(1 - 2 cos 11.pi/250). (1- 2 cos 33.pi/250).(1 - 2 cos 99.pi/250).(1+ 2 cos 47pi/250). (1- 2 cos 109.pi/250).(1+ 2 cos 77pi/250). (1- 2 cos 19pi/250).(1 - 2 cos.57.pi/250). (1+ 2 cos 79pi/250). (1- 2 cos 13.pi/250).(1- 2 cos 39pi/250). (1- 2 cos 117pi/250).(1+ 2 cos 101/250). (1- 2 cos 53/250).(1+ 2 cos 91pi/250). (1- 2 cos 23pi/250).(1- 2 cos 69pi/250). (1+ 2 cos 43pi/250).(1- 2 cos 121pi/250). (1+ 2 cos 113.pi/250). (1- 2 cos.89pi/250). (1+ 2 cos 17pi/250).(1 + 2 cos 51pi/250). (1- 2 cos 97pi/250).(1+ 2 cos 41pi/250).

$$(1+2 \cos 123/250).(1-2 \cos 119\pi/250).(1+2 \cos 107\pi/250). (1-2 \cos 71.\pi/250).(1+2 \cos 37\pi/250). (1+2 \cos 111\pi/250).(1-2 \cos 83.\pi/250) = 1$$

$$\text{Or } (1-4.\cos^2 \pi/250).(1-4.\cos^2 3.\pi/250).(1-4 \cos^2 29\pi/250).(1-4 \cos^2 27\pi/250).(1-4.\cos^2 81.\pi/250). (1-4.\cos^2 7\pi/250).(1-4 \cos^2 21.\pi/250). (1-4 \cos^2 63.\pi/250).(1-4 \cos^2 61.\pi/250). (1-4.\cos^2 67.\pi/250). (1-4.\cos^2.49\pi/250). (1-4.\cos^2 103\pi/250).(1-4.\cos^2 59.\pi/250). (1-4.\cos^2 73.\pi/250).(1-4.\cos^2 31.\pi/250). (1-4.\cos^2 93.\pi/250). (1-4.\cos^2 29.\pi/250). (1-4.\cos^2 87.\pi/250). (1-4.\cos^2 11.\pi/250). (1-4.\cos^2 33.\pi/250). (1-4.\cos^2 99.\pi/250).(1-4.\cos^2 47\pi/250). (1-4.\cos^2 109.\pi/250).(1-4.\cos^2 77\pi/250). (1-4.\cos^2 19\pi/250). (1-4.\cos^2 57.\pi/250). (1-4.\cos^2 79\pi/250).(1-4.\cos^2 13.\pi/250). (1-4.\cos^2 39\pi/250). (1-4.\cos^2 117\pi/250). (1-4.\cos^2 101/250). (1-4.\cos^2 53/250).(1-4.\cos^2 91\pi/250).(1-4.\cos^2 23\pi/250).(1-4.\cos^2 69\pi/250). (1-4.\cos^2 43\pi/250). (1-4.\cos^2 121\pi/250). (1-4.\cos^2 113.\pi/250).(1-4.\cos^2 89\pi/250). (1-4.\cos^2 17\pi/250). (1-4.\cos^2 251\pi/250). (1-4.\cos^2 97\pi/250).(1-4.\cos^2 41\pi/250). (1-4.\cos^2 123/250). (1-4.\cos^2 119\pi/250). (1-4.\cos^2 107\pi/250). (1-4.\cos^2 71.\pi/250). (1-4.\cos^2 37\pi/250).(1-4.\cos^2 111\pi/250).(1-4.\cos^2 83.\pi/250) = 1.$$

$$\text{Or } (1/4-\cos^2 \pi/250).(1/4-\cos^2 3.\pi/250).(1/4-\cos^2 9\pi/250). (1/4-\cos^2 27\pi/250).(1/4-\cos^2 81.\pi/250). (1/4-\cos^2 7\pi/250).(1/4-\cos^2 21.\pi/250). (1/4-\cos^2 63.\pi/250).(1/4-\cos^2 61.\pi/250). (1/4-\cos^2 67.\pi/250). (1/4-\cos^2.49\pi/250). (1/4-\cos^2 103\pi/250).(1/4-\cos^2 59.\pi/250). (1/4-\cos^2 73.\pi/250).(1/4-\cos^2 31.\pi/250). (1/4-\cos^2 93.\pi/250). (1/4-\cos^2 29.\pi/250). (1/4-\cos^2 87.\pi/250). (1/4-\cos^2 11.\pi/250). (1/4-\cos^2 33.\pi/250). (1/4-\cos^2 99.\pi/250).(1/4-\cos^2 47\pi/250). (1/4-\cos^2 109.\pi/250).(1/4-\cos^2 77\pi/250). (1/4-\cos^2 19\pi/250). (1/4-\cos^2 57.\pi/250). (1/4-\cos^2 79\pi/250).(1/4-\cos^2 13.\pi/250). (1/4-\cos^2 39\pi/250). (1/4-\cos^2 117\pi/250). (1/4-\cos^2 101/250). (1/4-\cos^2 53/250).(1/4-\cos^2 91\pi/250). (1/4-\cos^2 23\pi/250).(1/4-\cos^2 69\pi/250). (1/4-\cos^2 43\pi/250).(1/4-\cos^2 121\pi/250). (1/4-\cos^2 113.\pi/250).(1/4-\cos^2 89\pi/250). (1/4-\cos^2 17\pi/250). (1/4-\cos^2 51\pi/250). (1/4-\cos^2 97\pi/250).(1/4-\cos^2 41\pi/250).(1/4-\cos^2 123/250). (1/4-\cos^2 119\pi/250). (1/4-\cos^2 107\pi/250). (1/4-\cos^2 71.\pi/250). (1/4-\cos^2 37\pi/250).(1/4-\cos^2 111\pi/250).(1/4-\cos^2 83.\pi/250) = (1/2)^100.$$

On rearranging,

$$(1/4-\cos^2 \pi/250).(1/4-\cos^2 3.\pi/250).(1/4-\cos^2 7\pi/250).(1/4-\cos^2 9\pi/250).(1/4-\cos^2 11.\pi/250). (1/4-\cos^2 13.\pi/250). (1/4-\cos^2 17\pi/250).(1/4-\cos^2 19\pi/250).(1/4-\cos^2 21.\pi/250).(1/4-\cos^2 23\pi/250). (1/4-\cos^2 27\pi/250). (1/4-\cos^2 29.\pi/250).(1/4-\cos^2 31.\pi/250).(1/4-\cos^2 33.\pi/250).(1/4-\cos^2 37\pi/250). (1/4-\cos^2 39\pi/250). (1/4-\cos^2 41\pi/250).(1/4-\cos^2 43\pi/250).(1/4-\cos^2 47\pi/250).(1/4-\cos^2 49\pi/250). (1/4-\cos^2 51\pi/250). (1/4-\cos^2 53/250).(1/4-\cos^2 57.\pi/250).(1/4-\cos^2 59.\pi/250).(1/4-\cos^2 61.\pi/250). (1/4-\cos^2 63.\pi/250). (1/4-\cos^2 67.\pi/250).(1/4-\cos^2 69\pi/250).(1/4-\cos^2 71.\pi/250).(1/4-\cos^2 73.\pi/250). (1/4-\cos^2 77\pi/250). (1/4-\cos^2 79\pi/250).(1/4-\cos^2 81.\pi/250). (1/4-\cos^2 83.\pi/250).(1/4-\cos^2 87.\pi/250). (1/4-\cos^2 89\pi/250). (1/4-\cos^2 91\pi/250).(1/4-\cos^2 93.\pi/250).(1/4-\cos^2 97\pi/250).(1/4-\cos^2 99.\pi/250). (1/4-\cos^2 101/250). (1/4-\cos^2 103\pi/250) (1/4-\cos^2 107\pi/250). (1/4-\cos^2 109.\pi/250).(1/4-\cos^2 111\pi/250). (1/4-\cos^2 113.\pi/250). (1/4-\cos^2 117\pi/250).(1/4-\cos^2 119\pi/250).(1/4-\cos^2 121\pi/250).(1/4-\cos^2 123/250) = (1/2)^100.$$

On further rearranging,

$$(\cos^2 \pi/3-\cos^2 \pi/250).(\cos^2 \pi/3-\cos^2 3.\pi/250).(\cos^2 \pi/3-\cos^2 7\pi/250).(\cos^2 \pi/3-\cos^2 9\pi/250).(\cos^2 \pi/3-\cos^2 11.\pi/250).(\cos^2 \pi/3-\cos^2 13.\pi/250).(\cos^2 \pi/3-\cos^2 17\pi/250).(\cos^2 \pi/3-\cos^2 19\pi/250).(\cos^2 \pi/3-\cos^2 21.\pi/250).(\cos^2 \pi/3-\cos^2 23\pi/250).(\cos^2 \pi/3-\cos^2 27\pi/250).(\cos^2 \pi/3-\cos^2 29.\pi/250).(\cos^2 \pi/3-\cos^2 31.\pi/250).(\cos^2 \pi/3-\cos^2 33.\pi/250).(\cos^2 \pi/3-\cos^2 37\pi/250).(\cos^2 \pi/3-\cos^2 39\pi/250).(\cos^2 \pi/3-\cos^2 41\pi/250).(\cos^2 \pi/3-\cos^2 43\pi/250).(\cos^2 \pi/3-\cos^2 47\pi/250).(\cos^2 \pi/3-\cos^2 49\pi/250).(\cos^2 \pi/3-\cos^2 51\pi/250).(\cos^2 \pi/3-\cos^2 53/250).(\cos^2 \pi/3-\cos^2 57.\pi/250).(\cos^2 \pi/3-\cos^2 59.\pi/250).(\cos^2 \pi/3-\cos^2 61.\pi/250).(\cos^2 \pi/3-\cos^2 63.\pi/250).(\cos^2 \pi/3-\cos^2 67.\pi/250).(\cos^2 \pi/3-\cos^2 69\pi/250).(\cos^2 \pi/3-\cos^2 71.\pi/250).(\cos^2 \pi/3-\cos^2 73.\pi/250).(\cos^2 \pi/3-\cos^2 77\pi/250).(\cos^2 \pi/3-\cos^2 79\pi/250).(\cos^2 \pi/3-\cos^2 81.\pi/250).(\cos^2 \pi/3-\cos^2 83.\pi/250).(\cos^2 \pi/3-\cos^2 87.\pi/250).(\cos^2 \pi/3-\cos^2 89\pi/250).(\cos^2 \pi/3-\cos^2 91\pi/250).(\cos^2 \pi/3-\cos^2 93.\pi/250).(\cos^2 \pi/3-\cos^2 97\pi/250).(\cos^2 \pi/3-\cos^2 99.\pi/250).(\cos^2 \pi/3-\cos^2 101/250).(\cos^2 \pi/3-\cos^2 103\pi/250) (\cos^2 \pi/3-\cos^2 107\pi/250).(\cos^2 \pi/3-\cos^2 109.\pi/250).(\cos^2 \pi/3-\cos^2 111\pi/250).(\cos^2 \pi/3-\cos^2 113.\pi/250).(\cos^2 \pi/3-\cos^2 117\pi/250).(\cos^2 \pi/3-\cos^2 119\pi/250).(\cos^2 \pi/3-\cos^2 121\pi/250).(\cos^2 \pi/3-\cos^2 123/250) = (1/2)^100.$$

Thus the identity given in the abstract is solved.

Example 5:

Sin $7\pi/131$

Here n is of the form $p/q = 131/7$.

$$\sin 7\pi/131 = -\sin 138.\pi/131 / [(3-4.\sin^2 7\pi/131).(3-4.\sin^2 21.\pi/131) (3-4.\sin^2 63\pi/131) (3+4.\sin^2 58\pi/131) (3-4.\sin^2 43\pi/131) (3-4.\sin^2 2\pi/131) (3-4.\sin^2 6\pi/131) (3-4.\sin^2 18.\pi/131).(3-4.\sin^2 54\pi/131).(3+4.\sin^2 31\pi/131) (3+4.\sin^2 38\pi/131) (3+4.\sin^2 17\pi/131) (3+4.\sin^2 51\pi/131) (3-4.\sin^2$$

$$22\pi/131) (3- 4.\sin^2 65\pi/131) (3+ 4.\sin^2 64\pi/131) (3- 4.\sin^2 61\pi/131) .(3+ 4.\sin^2 52\pi/131) (3- 4.\sin^2 25\pi/131) (3- 4.\sin^2 56\pi/131) (3+ 4.\sin^2 37\pi/131) (3+ 4.\sin^2 20\pi/131) (3+ 4.\sin^2 60\pi/131) (3- 4.\sin^2 49\pi/131).(3+4.\sin^2 16\pi/131) (3+ 4.\sin^2 48\pi/131) (3- 4.\sin^2 13\pi/131) (3- 4.\sin^2 39\pi/131) (3- 4.\sin^2 14\pi/131) (3- 4.\sin^2 42\pi/131) (3- 4.\sin^2 5\pi/131) (3- 4.\sin^2 15\pi/131) (3- 4.\sin^2 45\pi/131) .(3+ 4.\sin^2 4\pi/131) (3+ 4.\sin^2 12\pi/131) (3+ 4.\sin^2 36\pi/131) (3+ 4.\sin^2 23\pi/131) (3+4.\sin^2 62\pi/131) (3- 4.\sin^2 55\pi/131) (3+ 4.\sin^2 34\pi/131).(3+ 4.\sin^2 29\pi/131) (3+ 4.\sin^2 44\pi/131) (3- 4.\sin^2 \pi/131) (3- 4.\sin^2 3\pi/131) (3- 4.\sin^2 9\pi/131) (3- 4.\sin^2 27\pi/131) (3- 4.\sin^2 50\pi/131) (3+ 4.\sin^2 19\pi/131) (3+ 4.\sin^2 57\pi/131) .(3- 4.\sin^2 40\pi/131) (3- 4.\sin^2 11\pi/131) (3- 4.\sin^2 33\pi/131) (3- 4.\sin^2 32\pi/131)(3- 4.\sin^2 35\pi/131) (3- 4.\sin^2 26\pi/131) (3- 4.\sin^2 53\pi/131).(3+ 4.\sin^2 28\pi/131) (3+ 4.\sin^2 47\pi/131) (3- 4.\sin^2 10\pi/131) (3- 4.\sin^2 30\pi/131) (3- 4.\sin^2 41\pi/131) (3- 4.\sin^2 8\pi/131) (3- 4.\sin^2 24\pi/131) (3- 4.\sin^2 59\pi/131) (3+ 4.\sin^2 46\pi/131).$$

In this way numerator on right hand side comes as $-\sin 138\pi/131 = -\sin (\pi+ 7\pi/131) = \sin 7.\pi/131$ which is positive. Therefore, on cancelling $\sin 7.\pi/131$ from LHS and RHS, we get

$$(3- 4.\sin^2 7\pi/131).(3- 4.\sin^2 21.\pi/131) (3- 4.\sin^2 63\pi/131) (3+ 4.\sin^2 58\pi/131) (3- 4.\sin^2 43\pi/131) (3- 4.\sin^2 2\pi/131) (3- 4.\sin^2 6\pi/131) (3- 4.\sin^2 18.\pi/131).(3- 4.\sin^2 54\pi/131).(3+ 4.\sin^2 31\pi/131) (3+ 4.\sin^2 38\pi/131) (3+ 4.\sin^2 17\pi/131) (3+ 4.\sin^2 51\pi/131) (3- 4.\sin^2 22\pi/131) (3- 4.\sin^2 65\pi/131) (3+ 4.\sin^2 64\pi/131) (3- 4.\sin^2 61\pi/131) .(3+ 4.\sin^2 52\pi/131) (3- 4.\sin^2 25\pi/131) (3- 4.\sin^2 56\pi/131) (3+ 4.\sin^2 37\pi/131) (3+ 4.\sin^2 20\pi/131) (3+ 4.\sin^2 60\pi/131) (3- 4.\sin^2 49\pi/131).(3+ 4.\sin^2 16\pi/131) (3+ 4.\sin^2 48\pi/131) (3- 4.\sin^2 13\pi/131) (3- 4.\sin^2 39\pi/131) (3- 4.\sin^2 14\pi/131) (3- 4.\sin^2 42\pi/131) (3- 4.\sin^2 5\pi/131) (3- 4.\sin^2 15\pi/131) (3- 4.\sin^2 45\pi/131) .(3+ 4.\sin^2 4\pi/131) (3+ 4.\sin^2 12\pi/131) (3+ 4.\sin^2 36\pi/131) (3+ 4.\sin^2 23\pi/131) (3+ 4.\sin^2 62\pi/131) (3- 4.\sin^2 55\pi/131) (3+ 4.\sin^2 34\pi/131).(3+ 4.\sin^2 29\pi/131) (3+ 4.\sin^2 44\pi/131) (3- 4.\sin^2 \pi/131) (3- 4.\sin^2 3\pi/131) (3- 4.\sin^2 9\pi/131) (3- 4.\sin^2 27\pi/131) (3- 4.\sin^2 50\pi/131) (3+ 4.\sin^2 19\pi/131) (3+ 4.\sin^2 57\pi/131) .(3- 4.\sin^2 40\pi/131) (3- 4.\sin^2 11\pi/131) (3- 4.\sin^2 33\pi/131)(3- 4.\sin^2 32\pi/131) (3- 4.\sin^2 35\pi/131) (3- 4.\sin^2 26\pi/131) (3- 4.\sin^2 53\pi/131).(3+ 4.\sin^2 28\pi/131) (3+ 4.\sin^2 47\pi/131) (3- 4.\sin^2 10\pi/131) (3- 4.\sin^2 30\pi/131)(3- 4.\sin^2 41\pi/131) (3- 4.\sin^2 8\pi/131) (3- 4.\sin^2 24\pi/131) (3- 4.\sin^2 59\pi/131) (3+ 4.\sin^2 46\pi/131) = 1.$$

On applying identity $\cos 2x = 1- 2.\sin^2 x$, it can be rewritten as

$$(1+ 2.\cos 14\pi/131).(1+ 2.\cos 42\pi/131).(1- 2.\cos 5\pi/131). (1- 2.\cos 15\pi/131). (1-2.\cos 45\pi/131). (1+ 2.\cos 4\pi/131).(1+ 2.\cos 12\pi/131). (1+ 2.\cos 36\pi/131).(1-2.\cos 23\pi/131) (1+ 2.\cos 62\pi/131). (1- 2.\cos 55\pi/131). (1+ 2.\cos 34\pi/131). (1- 2.\cos 29\pi/131) (1+ 2.\cos 44\pi/131). (1- 2.\cos \pi/131).(1-2 \cos 3\pi/131) . (1- 2.\cos 9\pi/131). (1- 2.\cos 27\pi/131). (1+ 2.\cos 50\pi/131). (1-2.\cos 19\pi/131).(1- 2.\cos 57\pi/131). (1+ 2.\cos 40\pi/131). (1- 2.\cos 11\pi/131).(1- 2.\cos 33\pi/131).(1+ 2.\cos 32\pi/131). (1- 2.\cos 35\pi/131).(1+ 2.\cos 26\pi/131). (1- 2.\cos 53\pi/131). (1+ 2.\cos 28\pi//131). (1- 2.\cos 47\pi/131). (1+ 2.\cos 10\pi/131). (1+ 2.\cos 30\pi/131) (1- 2.\cos 41\pi/131). (1+ 2.\cos 8\pi/131). (1+ 2.\cos 24\pi/131). (1- 2.\cos 59\pi/131). (1+ 2.\cos 46\pi/131). (1- 2.\cos 7\pi/131). (1- 2.\cos 21\pi/131). (1- 2.\cos 63\pi/131). (1+ 2.\cos 58\pi/131). (1- 2.\cos 43\pi/131). (1+ 2.\cos 2\pi/131). (1+ 2.\cos 6\pi/131). (1+ 2.\cos 18\pi/131). (1+ 2.\cos 54\pi/131). (1- 2.\cos 31\pi/131). (1+ 2.\cos 38\pi/131). (1- 2.\cos 17\pi/131). (1- 2.\cos 51\pi/131). (1+ 2.\cos 22\pi/131). (1+ 2.\cos 66\pi/131). (1- 2.\cos 67\pi/131).(1- 2.\cos 61\pi/131). (1+ 2.\cos 52\pi/131).(1- 2.\cos 25\pi/131). (1+ 2.\cos 56\pi/131). (1- 2.\cos 37\pi/131). (1+ 2.\cos 20\pi/131). (1+ 2.\cos 60\pi/131). (1- 2.\cos 49\pi/131). (1+ 2.\cos 16\pi/131). (1+ 2.\cos 48\pi/131). (1- 2.\cos 13\pi/131). (1- 2.\cos 39\pi/131) = 1.$$

Rearranging

$$(1- 2.\cos \pi/131).(1+ 2.\cos 2\pi/131) (1-2 \cos 3\pi/131) (1+ 2.\cos 4\pi/131) (1- 2.\cos 5\pi/131).(1+ 2.\cos 6\pi/131) (1- 2.\cos 7\pi/131).(1+ 2.\cos 8\pi/131).(1- 2.\cos 9\pi/131).(1+ 2.\cos 10\pi/131).(1- 2.\cos 11\pi/131) (1+ 2.\cos 12\pi/131). (1- 2.\cos 13\pi/131) (1+ 2.\cos 14\pi/131).(1- 2.\cos 15\pi/131). (1+ 2.\cos 16\pi/131). (1- 2.\cos 17\pi/131) (1+ 2.\cos 18\pi/131).(1- 2.\cos 19\pi/131). (1+ 2.\cos 20\pi/131).(1- 2.\cos 21\pi/131).(1+ 2.\cos 22\pi/131).(1- 2.\cos 23\pi/131)(1+ 2.\cos 24\pi/131).(1- 2.\cos 25\pi/131).(1+ 2.\cos 26\pi/131).(1- 2.\cos 27\pi/131).(1+ 2.\cos 28\pi/131) (1- 2.\cos 29\pi/131) (1+ 2.\cos 30\pi/131) (1- 2.\cos 31\pi/131). (1+ 2.\cos 32\pi/131). (1- 2.\cos 33\pi/131).(1+ 2.\cos 34\pi/131). (1- 2.\cos 35\pi/131). (1+ 2.\cos 36\pi/131) (1- 2.\cos 37\pi/131).(1+ 2.\cos 38\pi/131).(1- 2.\cos 39\pi/131).(1+ 2.\cos 40\pi/131).(1- 2.\cos 41\pi/131).(1+ 2.\cos 42\pi/131).(1- 2.\cos 43\pi/131).(1+ 2.\cos 44\pi/131).(1- 2.\cos 45\pi/131).(1+ 2.\cos 46\pi/131).(1- 2.\cos 47\pi/131).(1+ 2.\cos 48\pi/131).(1- 2.\cos 49\pi/131). (1+ 2.\cos 50\pi/131).(1- 2.\cos 51\pi/131). (1+ 2.\cos 52\pi/131).(1- 2.\cos 53\pi/131). (1+ 2.\cos 54\pi/131).(1- 2.\cos 55\pi/131).(1+ 2.\cos 56\pi/131).(1- 2.\cos 57\pi/131).(1+ 2.\cos 58\pi/131).(1- 2.\cos 59\pi/131).(1+ 2.\cos 60\pi/131).(1- 2.\cos 61\pi/131).(1+ 2.\cos 62\pi/131).(1- 2.\cos 63\pi/131).(1+ 2.\cos 66\pi/131).(1- 2.\cos 67\pi/131) = 1.$$

It can also be written as

$$(1/2- \cos \pi/131).(1/2+ \cos 2\pi/131) (1/2- \cos 3\pi/131) (1/2+ \cos 4\pi/131) (1/2- \cos 5\pi/131).(1/2+ \cos 6\pi/131) (1/2- \cos 7\pi/131).(1/2+ \cos 8\pi/131).(1/2- \cos 9\pi/131).(1/2+ \cos 10\pi/131).(1/2- \cos 11\pi/131) (1/2+ \cos 12\pi/131). (1/2- \cos 13\pi/131) (1/2+ \cos 14\pi/131).(1/2- \cos 15\pi/131). (1/2+ \cos 16\pi/131). (1/2- \cos 17\pi/131) (1/2+ \cos 18\pi/131).(1/2- \cos 19\pi/131). (1/2+ \cos 20\pi/131).(1/2- \cos 21\pi/131).(1/2+ \cos 22\pi/131).(1/2- \cos 23\pi/131)(1/2+ \cos 24\pi/131).(1/2- \cos 25\pi/131).(1/2+ \cos 26\pi/131).(1/2- \cos 27\pi/131).(1/2+ \cos 28\pi/131) (1/2- \cos 29\pi/131).(1/2+ \cos 30\pi/131).(1/2- \cos 31\pi/131). (1/2+ \cos 32\pi/131).(1/2- \cos 33\pi/131).(1/2+ \cos 34\pi/131). (1/2- \cos 35\pi/131).(1/2- \cos$$

$$36\pi/131) (1/2-\cos 37\pi/131).(1/2+ \cos 38\pi/131).(1/2- \cos 39\pi/131).(1/2+ \cos 40\pi/131).(1/2- \cos 41\pi/131).(1/2+ \cos 42\pi/131).(1/2- \cos 43\pi/131).(1/2+ \cos 44\pi/131).(1/2- \cos 45\pi/131).(1/2+ \cos 46\pi/131).(1/2- \cos 47\pi/131).(1/2+ \cos 48\pi/131).(1/2- \cos 49\pi/131). (1/2+ \cos 50\pi/131).(1/2- \cos 51\pi/131). (1/2+ \cos 52\pi/131).(1/2- \cos 53\pi/131).(1/2+ \cos 54\pi/131).(1/2- \cos 55\pi/131).(1/2+ \cos 56\pi/131).(1/2- \cos 57\pi/131).(1/2+ \cos 58\pi/131).(1/2- \cos 59\pi/131).(1/2+ \cos 60\pi/131).(1/2- \cos 61\pi/131).(1/2+ \cos 62\pi/131).(1/2- \cos 63\pi/131).(1/2+ \cos 66\pi/131).(1/2- \cos 67\pi/131) = (1/2)^{65}.$$

Also

$$(\cos \pi/3- \cos \pi/131).(\cos \pi/3+ \cos 2\pi/131) (\cos \pi/3- \cos 3\pi/131) (\cos \pi/3+ \cos 4\pi/131) (\cos \pi/3- \cos 5\pi/131). (\cos \pi/3+ \cos 6\pi/131) (\cos \pi/3- \cos 7\pi/131).(\cos \pi/3+ \cos 8\pi/131).(\cos \pi/3- \cos 9\pi/131).(\cos \pi/3+ \cos 10\pi/131). (\cos \pi/3- \cos 11\pi/131) (\cos \pi/3+ \cos 12\pi/131). (\cos \pi/3- \cos 13\pi/131) (\cos \pi/3+ \cos 14\pi/131).(\cos \pi/3- \cos 15\pi/131). (\cos \pi/3+ \cos 16\pi/131). (\cos \pi/3- \cos 17\pi/131) (\cos \pi/3+ \cos 18\pi/131).(\cos \pi/3- \cos 19\pi/131). (\cos \pi/3+ \cos 20\pi/131).(\cos \pi/3- \cos 21\pi/131).(\cos \pi/3+ \cos 22\pi/131).(\cos \pi/3- \cos 23\pi/131)(\cos \pi/3+ \cos 24\pi/131).(\cos \pi/3- \cos 25\pi/131).(\cos \pi/3+ \cos 26\pi/131).(\cos \pi/3- \cos 27\pi/131).(\cos \pi/3+ \cos 28\pi/131) (\cos \pi/3- \cos 29\pi/131).(\cos \pi/3+ \cos 30\pi/131).(\cos \pi/3- \cos 31\pi/131). (\cos \pi/3+ \cos 32\pi/131).(\cos \pi/3- \cos 33\pi/131). (\cos \pi/3+ \cos 34\pi/131).(\cos \pi/3- \cos 35\pi/131).(\cos \pi/3+ \cos 36\pi/131) (\cos \pi/3- \cos 37\pi/131). (\cos \pi/3+ \cos 38\pi/131).(\cos \pi/3- \cos 39\pi/131).(\cos \pi/3+ \cos 40\pi/131).(\cos \pi/3- \cos 41\pi/131).(\cos \pi/3+ \cos 42\pi/131).(\cos \pi/3- \cos 43\pi/131).(\cos \pi/3+ \cos 44\pi/131).(\cos \pi/3- \cos 45\pi/131).(\cos \pi/3+ \cos 46\pi/131). (\cos \pi/3- \cos 47\pi/131).(\cos \pi/3+ \cos 48\pi/131).(\cos \pi/3- \cos 49\pi/131).(\cos \pi/3+ \cos 50\pi/131).(\cos \pi/3- \cos 51\pi/131).(\cos \pi/3+ \cos 52\pi/131).(\cos \pi/3- \cos 53\pi/131).(\cos \pi/3+ \cos 54\pi/131).(\cos \pi/3- \cos 55\pi/131). (\cos \pi/3+ \cos 56\pi/131).(\cos \pi/3- \cos 57\pi/131).(\cos \pi/3+ \cos 58\pi/131).(\cos \pi/3- \cos 59\pi/131).(\cos \pi/3+ \cos 60\pi/131).(\cos \pi/3- \cos 61\pi/131).(\cos \pi/3+ \cos 62\pi/131).(\cos \pi/3- \cos 63\pi/131).(\cos \pi/3+ \cos 66\pi/131). (\cos \pi/3- \cos 67\pi/131) = (1/2)^{65}.$$

Thus the identity mentioned in abstract is solved.

After solving these example and taking the theory in view, it is concluded that

A. $(1/2+ \cos 2\pi/n).(1/2+ \cos 3.2\pi/n).(1/2+ \cos 9.2\pi/n).(1/2+ \cos 27.2\pi/n) (1/2+ \cos 81.2\pi/n) \dots [1/2+ \cos (3^k-1).2\pi/n] = (1/2)^k$

In other words
$$\prod_{j=1}^k \{1/2+ \cos 3^{j-1}.2\pi/n\} = (1/2)^k,$$

where integers m and k are given by relation $(3^k + 1)/n = m$ and m is odd. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/n = m$ and m is even. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

B. $(1/2+ \cos 2\pi/n).(1/2+ \cos 3.2\pi/n).(1/2+ \cos 9.2\pi/n).(1/2+ \cos 27.2\pi/n) (1/2+ \cos 81.2\pi/n) \dots [1/2+ \cos (3^k-1).2\pi/n] = -(1/2)^k$

In other words,
$$\prod_{j=1}^k \{1/2+ \cos 3^{j-1}.2\pi/n\} = -(1/2)^k$$

where integers m and k are given by relation $(3^k + 1)/n = m$ and m is even. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/n = m$ and m is odd. And n may be integer but not multiple of 3 and if fraction p/q , p is not multiple of 3.

On putting $\pi/n = x$ in radians, the above identities transform to

A. $(1/2+ \cos 2x).(1/2+ \cos 3.2x).(1/2+ \cos 9.2x).(1/2+ \cos 27.2x) (1/2+ \cos 81.2x) \dots [1/2+ \cos 3^{k-1}.2x] = (1/2)^k$, this can also be written as

$$\prod_{j=1}^k \{1/2+ \cos 3^{j-1}.2x\} = (1/2)^k,$$

where integers m and k are given by relation $(3^k + 1)/x = m$ and m is odd. And $1/x$ is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/x = m$ and m is even. And $1/x$ is not multiple of 3.

$$B. (1/2 + \cos 2x)(1/2 + \cos 3.2x)(1/2 + \cos 9.2x)(1/2 + \cos 27.2x) \dots (1/2 + \cos 81.2x) \dots [1/2 + \cos 3^{k-1}.2x] = (1/2)^k$$

$$\prod_{j=1}^k \{1/2 + \cos 3^{j-1}.2x\} = (1/2)^k,$$

where integers m and k are given by relation $(3^k + 1)/x = m.\pi$ and m is even. And $1/x$ is not multiple of 3.

OR

where integers m and k are given by relation $(3^k - 1)/n = m.\pi$ and m is odd and $1/x$ is not multiple of 3.

2.4 Cases where n is not multiple of 2 or is a fraction p/q where integer p is not multiple of 2 are taken up here.

In such cases, angle x is ‘ π divided by odd integer’ or π is divided by fraction p/q where p is odd integer. That is angle x may be of the form, for example, $\pi/9$ or $\pi/13$ or $\pi/31$ or $\pi/(21/8)$ or $\pi/(57/7)$ or π divided any integer except multiplier of 2 or $\pi/(p/q)$ where p is any integer not multiplier of 2.

Elementary identity

$$2.\sin x.\cos x = \sin 2x \text{ or}$$

$$\sin x = \sin 2x/2.\cos x \tag{12}$$

is recursive in nature as sine appears in left hand side as well as in right hand side. $\sin x$ in LHS and $\sin 2x$ in RHS in numerator are being considered for recurrence.

From this recursive relation, $\sin 2x$ can also be written as

$$\sin 2x = \sin 4x/(2.\cos 2x)$$

$$\sin 4x = \sin 8x/(2.\cos 4x)$$

.....

$$\sin x.2^{k-1} = \sin 2^k/[2.\cos 2^{k-1}]$$

On substituting value of $\sin 2x = \sin 4x/(2.\cos 2x)$, equation 12 can be written as

$$\sin x = \sin 4x/(2.\cos x.2.\cos 2x) = \sin 4x/(4.\cos x.\cos 2x)$$

On successively substituting value of $\sin 4x, \sin 8x, \dots, \sin x.2^{k-1}$, we get

$$\sin x = (2^k.\sin x)/[2^k.\cos x.\cos 2x.\cos 4x.\cos 8x.\dots.\cos x.2^{k-1}]$$

Putting $x = \pi/n$, this equation is written as

$$\sin \pi/n = \sin \pi/n.2^k/[2^k.\cos \pi/n.\cos 2\pi/n.\cos 4\pi/n.\cos 8\pi/n.\dots.\cos \pi/n.2^{k-1}] \tag{13}$$

If n is any integer except even integer or is form p/q where p is not even then $(2^k + 1)$ or $(2^k - 1)$ will always be divisible by n for some integer value of k . It is obvious $2^k + 1$ and $2^k - 1$ will always be odd as even integer plus or minus 1 always makes odd integer. Also division of odd integer by some other odd integer is possible.

Here k can assume any value starting from 0, and then proceeding to 1, then 2, then 3, so on till that value of k is reached where it gets divided by n without any remainder. There are thus many positive k integers available and since odd can be divided by odd then there is complete possibility for division of $(3^k + 1)$ or $(3^k - 1)$ by n .

Let $m = (2^k + 1)/n$ or

$$m = (2^k - 1)/n$$

where m is that minimum number where $(3^k + 1)$ or $(3^k - 1)$ is just divisible by n .

Applying this concept that there exists a value of k that makes $(2^k + 1)$ or $(2^k - 1)$ divisible by n , we can easily say that denominator of equation (13) has factors which are k in number. First factor pertains to angle $\pi/n.2^0$, second to $\pi/n.2^1$, third to $\pi/n.2^2$, fourth to $\pi/n.2^3, \dots$ so on and k th to $\pi/n.2^{k-1}$.

From equation (13), it is clear, RHS has numerator $\sin \pi/n.2^k$ and this pertains to angle $\pi/n.2^k$. It is already proved

$$(2^k + 1)/n = m, \text{ that makes}$$

$$\pi/n.2^k = (m\pi - \pi/n)$$

Similarly, for the case where $(2^k - 1)/n = m$,

$$\pi/n.2^k = (m\pi + \pi/n).$$

Therefore for the cases where $(2^k + 1)/n = m$, numerator of RHS of equation (13) becomes

$$\sin \pi/n.2^k = \sin (m\pi - \pi/n) = \sin \pi/n \text{ when } m \text{ is odd and } = -\sin \pi/n \text{ when } m \text{ is even.}$$

For the cases where $(2^k - 1)/n = m$, numerator of RHS of equation (13) becomes $\sin \pi/n \cdot 2^k = \sin (m\pi + \pi/n) = \sin \pi/n$ when m is even and it equals $-\sin \pi/n$ when m is odd.

When $(2^k + 1)/n = m$ and m is odd

or

when $(2^k - 1)/n = m$ and m is even.

Equation (13) becomes

$$\sin \pi/n = \sin \pi/n / [2^k \cdot \cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n, 2^{(k-1)}]$$

On cancelling $\sin \pi/n$ on both sides, we get

$$[2^k \cdot \cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n, 2^{(k-1)}] = 1$$

$$\text{Or } \cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n, 2^{(k-1)} = (1/2)^k \tag{14}$$

It can also be written as

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot \pi/n = (1/2)^k,$$

when integers k and m are given by relation $(2^k + 1)/n = m$ and m is odd or relation $(2^k - 1)/n = m$ and m is even for all odd n .

It can be written in angle x form by putting $\pi/n = x$ as

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot x = (1/2)^k,$$

when integers m and k are given by relation $(2^k + 1) \cdot x = m \cdot \pi$ and m is odd

or

when integers m and k are given by relation $(2^k - 1) \cdot x = m \cdot \pi$ and m is even. And n is odd in all cases. .

Similarly for cases

where $(2^k + 1)/n = m$ and m is even

or

where $(2^k - 1)/n = m$ and m is odd.

$$\cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n, 2^{(k-1)} = - (1/2)^k \tag{14/1}$$

It can also be written as

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot \pi/n = - (1/2)^k,$$

When integers k and m are given by relation $(2^k + 1)/n = m$ and m is even or relation $(2^k - 1)/n = m$ and m is odd for all odd n .

It can be written in angle x form by putting $\pi/n = x$ as

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot x = - (1/2)^k,$$

when integers m and k are given by relation $(2^k + 1) \cdot x = m \cdot \pi$ and m is odd

or

when integers m and k are given by relation $(2^k - 1) \cdot x = m \cdot \pi$ and m is even. And n is odd in all cases

2.5 Example

Example 1

$\sin \pi/63$

Here $n = 63$ and it satisfies relation

$$(2^k - 1)/n = m \text{ as } (2^6 - 1)/63$$

where $k = 6, m = 1$.

Hence equation (16) is satisfied, therefore,

$$\cos \pi/63 \cdot \cos 2\pi/63 \cdot \cos 4\pi/63 \cdot \cos 8\pi/63 \cdot \cos 16\pi/63 \cdot \cos 32\pi/63 = - (1/2)^6.$$

Example 2

Sin $\pi/81$

Cos $\pi/81$. Cos $2\pi/81$. Cos $4\pi/81$. Cos $8\pi/81$. Cos $16\pi/81$. Cos $32\pi/81$. Cos $17\pi/81$ Cos $34\pi/81$. Cos $13\pi/81$. Cos $26\pi/81$. Cos $29\pi/81$. Cos $23\pi/81$. Cos $35\pi/81$. Cos $11\pi/81$ Cos $22\pi/81$. Cos $37\pi/81$. Cos $7\pi/81$. Cos $14\pi/81$. Cos $28\pi/81$. Cos $25\pi/81$. Cos $31\pi/81$. Cos $19\pi/81$. Cos $38\pi/81$. Cos $5\pi/81$. Cos $10\pi/81$. Cos $20\pi/81$. Cos $40\pi/81 = (1/2)^{127}$.

Example 3

Sin $\pi/243$

Cos $\pi/243$. Cos $2\pi/243$. Cos $4\pi/243$. Cos $8\pi/243$. Cos $16\pi/243$. Cos $32\pi/243$ Cos $64\pi/243$ Cos $115\pi/243$ Cos $13\pi/243$ Cos $26\pi/243$ Cos $52\pi/243$. Cos $104\pi/243$ Cos $35\pi/243$ Cos $70\pi/243$ Cos $103\pi/243$ Cos $37\pi/243$ Cos $74\pi/243$ Cos $95\pi/243$ Cos $53\pi/243$ Cos $106\pi/243$ Cos $31\pi/243$. Cos $62\pi/243$ Cos $119\pi/243$ Cos $5\pi/243$ Cos $10\pi/243$ Cos $20\pi/243$ Cos $40\pi/243$ Cos $80\pi/243$ Cos $83\pi/243$ Cos $77\pi/243$ Cos $89\pi/243$. Cos $65\pi/243$ Cos $113\pi/243$ Cos $17\pi/243$ Cos $34\pi/243$ Cos $68\pi/243$ Cos $107\pi/243$ Cos $29\pi/243$ Cos $58\pi/243$ Cos $116\pi/243$ Cos $11\pi/243$. Cos $22\pi/243$ Cos $44\pi/243$ Cos $88\pi/243$ Cos $67\pi/243$ Cos $109\pi/243$ Cos $25\pi/243$ Cos $50\pi/243$ Cos $100\pi/243$ Cos $43\pi/243$ Cos $86\pi/243$. Cos $71\pi/243$ Cos $101\pi/243$ Cos $41\pi/243$ Cos $82\pi/243$ Cos $79\pi/243$ Cos $85\pi/243$ Cos $73\pi/243$ Cos $97\pi/243$ Cos $49\pi/243$ Cos $98\pi/243$. Cos $47\pi/243$ Cos $94\pi/243$ Cos $55\pi/243$ Cos $110\pi/243$ Cos $23\pi/243$ Cos $46\pi/243$ Cos $92\pi/243$ Cos $59\pi/243$ Cos $118\pi/243$ Cos $7\pi/243$ Cos $14\pi/243$ Cos $28\pi/243$ Cos $56\pi/243$ Cos $112\pi/243$ Cos $19\pi/243$ Cos $38\pi/243$ Cos $76\pi/243$ Cos $91\pi/243$ Cos $61\pi/243$ Cos $121\pi/243 = (1/2)^{81}$.

From these we can say that

A. $\cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n \cdot 2^{(k-1)} = (1/2)^k$,
 in other words product $\cos 2^j \cdot \pi/n = (1/2)^k$ and j varies from 0 to $k-1$
 where integers m and k are given by relation $(2^k + 1)/n = m$. And m and n are odd

or

where integers m and k are given by relation $(2^k - 1)/n = m$. And m is even and n is odd.

B. $\cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n \cdot 2^{(k-1)} = -(1/2)^k$

In other words, $\prod_{j=1}^k \cos 2^{(j-1)} \cdot \pi/n = -(1/2)^k$,

where integers m and k are given by relation $(2^k + 1)/n = m$. And m is even and n is odd.

or

where integers m and k are given by relation $(2^k - 1)/n = m$. And m and n are odd.

A. If we put $\pi/n = x$, we get

$$\cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \dots \dots \dots \cos 2^{(k-1)} \cdot x = (1/2)^k,$$

In other words,

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot x = (1/2)^k,$$

where integers m and k are given by relation $(2^k + 1) \cdot x = m \cdot \pi$ and m is odd and $1/x$ is not multiple of 2.

or

where integers m and k are given by relation $(2^k - 1)/n = m$ and m is even and $1/x$ is not multiple of 2.

B. $\cos \pi/n \cdot \cos 2\pi/n \cdot \cos 4\pi/n \cdot \cos 8\pi/n \dots \dots \dots \cos \pi/n \cdot 2^{(k-1)} = -(1/2)^k$ in other words,

$$\prod_{j=1}^k \cos 2^{(j-1)} \cdot x = -(1/2)^k,$$

where integers m and k are given by relation $(2^k + 1)/n = m$ and m is even and $1/x$ is not multiple of 2

or

where integers m and k are given by relation $(2^k - 1)/n = m$ and m is odd and $1/x$ is not multiple of 2.

2.6 Part 2

Factorisation Of Cosine Of An Angle

Now we come to factorisation of cosine of multiple angle. Let it be $\cos 3x$.

By putting $\cos 3x = 0$, its roots correspond to angles that can be found as

$$3x = (2k + 1) \cdot \pi/2, \text{ where } k \text{ is } 0, 1, 2, \dots \dots \dots$$

Therefore, x is $\pi/6, \pi/2, 5\pi/6$. On putting $k = 0, 1, 2$ and its roots are found as $\cos \pi/6, \cos \pi/2, \cos 5\pi/6$. Now when roots are known, its equation can be written as

$$(\cos x - \cos \pi/6) \cdot (\cos x - \cos \pi/2) \cdot (\cos x - \cos 5\pi/6) = 0 \text{ and it has three factors.}$$

If we multiply this equation by 2^{j-1} , where j equals number of factors, it does not make any difference and equation becomes

$2^2 \cdot (\cos x - \cos \pi/6) \cdot (\cos x - \cos \pi/2) \cdot (\cos x - \cos 5\pi/6) = 0$ and these factors correspond to $\cos 3x$.

$$\begin{aligned} \cos 3x = \text{its factors} &= 2^2 \cdot (\cos x - \cos \pi/6) \cdot (\cos x - \cos \pi/2) \cdot (\cos x - \cos 5\pi/6) \\ &= 2^2 \cdot \cos x \cdot (\cos x - \cos \pi/6) \cdot (\cos x + \cos \pi/6) \\ &= 4 \cdot \cos x \cdot (\cos^2 x - 3/4) \end{aligned}$$

Similarly, it can be found that $\cos 4x$ has four roots, $\cos \pi/8, \cos 3\pi/8, \cos 5\pi/8, \cos 7\pi/8$ and

$$\begin{aligned} \cos 4x &= 2^3 \cdot (\cos x - \cos \pi/8) \cdot (\cos x - \cos 3\pi/8) \cdot (\cos x - \cos 5\pi/8) \cdot (\cos x - \cos 7\pi/8) \\ &= 2^3 \cdot (\cos x - \cos \pi/8) \cdot (\cos x - \cos 3\pi/8) \cdot (\cos x + \cos 3\pi/8) \cdot (\cos x + \cos \pi/8) \\ &= 2^3 \cdot (\cos^2 x - \cos^2 \pi/8) \cdot (\cos^2 x + \cos^2 3\pi/8) \end{aligned}$$

Similarly,

$$\begin{aligned} \cos 9x &= 2^8 \cdot (\cos x - \cos \pi/18) \cdot (\cos x - \cos 3\pi/18) \cdot (\cos x - \cos 5\pi/18) \cdot (\cos x - \cos 7\pi/18) \cdot (\cos x - \cos 9\pi/18) \\ &\cdot (\cos x - \cos 11\pi/18) \cdot (\cos x - \cos 13\pi/18) \cdot (\cos x - \cos 15\pi/18) \cdot (\cos x - \cos 17\pi/18) = 2^8 \cdot \cos x \cdot \\ &(\cos^2 x - \cos^2 \pi/18) \cdot (\cos^2 x - \cos^2 3\pi/18) \cdot (\cos^2 x - \cos^2 5\pi/18) \cdot (\cos^2 x - \cos^2 7\pi/18) \end{aligned}$$

By mathematical induction, it can be written as

$$\cos nx = 2^{n-1} \cdot \{\cos x - \cos \pi/2n\} \cdot \{\cos x - \cos 3\pi/2n\} \cdot \{\cos x - \cos 5\pi/2n\} \dots \dots \{\cos x - \cos (2n-1)\pi/2n\} \quad (15)$$

When n is even, it can also be written as

$$\cos nx = 2^{n-1} \cdot \{\cos^2 x - \cos^2 \pi/2n\} \cdot \{\cos^2 x - \cos^2 3\pi/2n\} \cdot \{\cos^2 x - \cos^2 5\pi/2n\} \dots \dots \{\cos^2 x - \cos^2 (n-1)\pi/2n\} \quad (15/4)$$

$$\cos nx = 2^{n-1} \cdot \prod_{k=0}^{n-1} \{\cos x - \cos (2k+1)\pi/2n\} \quad (15/1)$$

Since $\cos nx = T_n(\cos x)$ where $T_n(\cos x)$ is Chebyshev polynomial of first kind in $\cos x$, therefore,

$$T_n(\cos x) = 2^{n-1} \cdot \{\cos x - \cos \pi/2n\} \cdot \{\cos x - \cos 3\pi/2n\} \cdot \{\cos x - \cos 5\pi/2n\} \dots \dots \{\cos x - \cos (2n-1)\pi/2n\}.$$

That means Chebyshev polynomial of first kind in $\cos x$ has roots

$$\cos \pi/2n, \cos 3\pi/2n, \cos 5\pi/2n, \dots \dots \cos (2n-1)\pi/2n.$$

Also $\cos nx = (-1)^{n/2} \cdot T_n(\sin x)$ when n is even and identity (15/4) can be written in $\sin x$ as

$$\cos nx = (-1)^{n/2} \cdot 2^{n-1} \cdot \{\sin^2 x - \sin^2 \pi/2n\} \cdot \{\sin^2 x - \sin^2 3\pi/2n\} \cdot \{\sin^2 x - \sin^2 5\pi/2n\} \dots \dots \{\sin^2 x - \sin^2 (n-1)\pi/2n\},$$

Therefore,

$$T_n(\sin x) = 2^{n-1} \cdot \{\sin^2 x - \sin^2 \pi/2n\} \cdot \{\sin^2 x - \sin^2 3\pi/2n\} \cdot \{\sin^2 x - \sin^2 5\pi/2n\} \dots \dots \{\sin^2 x - \sin^2 (n-1)\pi/2n\}$$

when n is even.

When n is odd, identity (15) can be written as

$$\cos nx = 2^{n-1} \cdot \cos x \cdot \{\cos^2 x - \cos^2 \pi/2n\} \cdot \{\cos^2 x - \cos^2 3\pi/2n\} \cdot \{\cos^2 x - \cos^2 5\pi/2n\} \dots \dots \{\cos^2 x - \cos^2 (n-2)\pi/2n\}.$$

$$\text{Or } \cos nx = 2^{n-1} \cdot (-1)^{(n-1)/2} \cdot \cos x \cdot \{\sin^2 \pi/2n - \sin^2 x\} \cdot \{\sin^2 3\pi/2n - \sin^2 x\} \cdot \{\sin^2 5\pi/2n - \sin^2 x\} \dots \dots \dots \{\sin^2 \pi/2n - \sin^2 (n-2)\pi/2n\}.$$

$$\text{Also } \cos nx = (-1)^{(n-1)/2} \cdot \cos x \cdot U_{n-1}(\sin x).$$

$$\text{Therefore } U_{n-1}(\sin x) = 2^{n-1} \cdot \{\sin^2 x - \sin^2 \pi/2n\} \cdot \{\sin^2 x - \sin^2 3\pi/2n\} \cdot \{\sin^2 x - \sin^2 5\pi/2n\} \dots \dots \{\sin^2 x - \sin^2 (n-2)\pi/2n\}$$

and its roots are $\sin \pi/2n, -\sin \pi/2n, \sin 3\pi/2n, -\sin 3\pi/2n, \sin 5\pi/2n, -\sin 5\pi/2n, \dots \dots \dots \sin (n-1)\pi/2n, -\sin (n-1)\pi/2n$ when is odd. $U_{n-1}(\sin x)$ is Chebyshev polynomial of second kind in $\sin x$.

Coming to equation (15), when x is zero, equation (15) reduces to

$$1 = 2^{n-1} \cdot (1 - \cos \pi/2n) \cdot (1 - \cos 3\pi/2n) \cdot (1 - \cos 5\pi/2n) \dots \dots \dots \cdot \{1 - \cos (2n-1)\pi/2n\},$$

Or

$$\prod_{k=0}^{n-1} \{1 - \cos (2k+1)\pi/2n\} = (1/2)^{n-1} \quad (15/2)$$

Now since $1 - \cos(2k+1)\pi/2n = 2 \sin^2(2k+1)\pi/4n$, therefore above identity takes the form

$$2^n \prod_{k=0}^{n-1} \sin^2(2k+1)\pi/4n = (\frac{1}{2})^{(n-1)}$$

$$\text{or } \prod_{k=0}^{n-1} \sin^2(2k+1)\pi/4n = (\frac{1}{2})^{(2n-1)}$$

$$\text{or } \prod_{k=0}^{n-1} \sin(2k+1)\pi/4n = (\frac{1}{2})^{(n-1/2)} \tag{15/3}$$

When $x = \pi/2$, $\cos \pi/2 = 0$.

Equation (15/1) reduces to

$$\cos n\pi/2 = 2^{(n-1)} \cdot (-1)^n \prod_{k=0}^{n-1} \{\cos(2k+1)\pi/2n\}$$

Case-1:

Let n be odd and x is $\pi/2$, where p is any integer, then this identity takes the form

$$\cos(2p+1)\pi/2 = \{2^{2p}\} \cdot \{(-1)^{(2p+1)}\} \prod_{k=0}^{2p} \{\cos(2k+1)\pi/4p+2\},$$

$$\text{or } \prod_{k=0}^{2p} \{\cos(2k+1)\pi/(4p+2)\} = 0.$$

This product will always be zero as it contains one factor that pertains to integer $2p+1$ when k will be equal to p. In that case $\cos(2p+1)\pi/(4p+2) = \cos \pi/2 = 0$.

Case-2:

Let n is even of the form $4p$, where p is any integer and $x = \pi/2$, then the identity 15/1 takes the form

$$\cos 4p\pi/2 = \{2^{(4p-1)}\} \cdot \{(-1)^{4p}\} \prod_{k=0}^{4p-1} \{\cos(2k+1)\pi/8p\}$$

$$\text{or } \prod_{k=0}^{4p-1} \{\cos(2k+1)\pi/8p\} = (\frac{1}{2})^{(4p-1)}$$

Case-3:

Let n be even and of the form $2(2p+1)$ where p is any integer and $x = \pi/2$, the identity takes the form,

$$\cos 2(2p+1)\pi/2 = 2^{(4p+1)} \cdot \{(-1)^{(4p+2)}\} \prod_{k=0}^{4p+1} \{\cos(2k+1)\pi/(8p+4)\}$$

$$\text{or } \prod_{k=0}^{4p+1} \{\cos(2k+1)\pi/(8p+4)\} = -(\frac{1}{2})^{(4p+1)}$$

From above, the identities can be generalised as

$$\prod_{k=0}^{n-1} \{\cos(2k+1)\pi/2n\} = (-1)^{n/2} \cdot (\frac{1}{2})^{(n-1)}$$

Factorisation Of Sine Of An Angle

Now factorisation of sine of multiple angle is taken up when angle is odd. That is it is of form $2k+1$ where k is 0, 1, 2, 3..... Let it be $\sin 3x$.

By putting $\sin 3x = 0$, its roots correspond to angles that can be found as

$$3x = +k\pi,$$

$$3x = -k\pi,$$

where k is 0, 1, 2, ...therefore on putting k = 1 and also k = -1, x equals $\pi/3$ and $-\pi/3$. On putting k =0, x equals 0. Therefore roots of $\sin 3x$ are $\sin \pi/3, -\sin \pi/3, \sin 0$. Now when roots are known, its equation can be written as $(\sin x - \sin \pi/3)(\sin x + \sin \pi/3)(\sin x - \sin 0) = 0$.

If we multiply this equation by $(-1)^{(n-1)/2} \cdot 2^{(j-1)}$, where j is number of factors and n is multiplier of x, here it is 2, it does not make any difference and equation becomes

$$(-1)^{(n-1)/2} \cdot 2^{(j-1)} (\sin x - \sin \pi/3)(\sin x + \sin \pi/3)(\sin x - \sin 0) = 0$$

and these factors correspond to $\sin 3x$.

$$\begin{aligned} \sin 3x &= \text{its factors} = (-1)^{(n-1)/2} \cdot 2^{(j-1)} (\sin x - \sin \pi/3)(\sin x + \sin \pi/3)(\sin x - \sin 0) \\ &= 2^2 (\sin \pi/3 - \sin x)(\sin \pi/3 + \sin x) \sin x \\ &= 4(3/4 - \sin^2 x) \sin x \\ &= \sin x(3 - 4 \sin^2 x) \end{aligned}$$

Similarly, it can be found that $\sin 5x$ has five roots, $\sin \pi/5, -\sin \pi/5, \sin 3\pi/5, -\sin 3\pi/5, 0$ and it can be factorised as

$$\begin{aligned} \sin 5x &= (-1)^{(5-1)/2} \cdot 2^4 (\sin x - \sin \pi/5)(\sin x + \sin \pi/5)(\sin x - \sin 2\pi/5)(\sin x + \sin 2\pi/5)(\sin x) \\ &= 2^4 \sin x (\sin^2 x - \sin^2 \pi/5)(\sin^2 x - \sin^2 2\pi/5) \end{aligned}$$

$$\begin{aligned} \sin 17x &= (-1)^{(17-1)/2} \cdot 2^{16} \sin x (\sin^2 x - \sin^2 \pi/17)(\sin^2 x - \sin^2 2\pi/17)(\sin^2 x - \sin^2 3\pi/17) \\ &(\sin^2 x - \sin^2 4\pi/17)(\sin^2 x - \sin^2 5\pi/17)(\sin^2 x - \sin^2 6\pi/17)(\sin^2 x - \sin^2 7\pi/17)(\sin^2 x - \sin^2 \pi/17) \end{aligned}$$

By mathematical induction, it can be written as

$$\sin n = (-1)^{(n-1)/2} \cdot 2^{(n-1)} (\sin x)(\sin^2 x - \sin^2 \pi/n)(\sin^2 x - \sin^2 2\pi/n)(\sin^2 x - \sin^2 3\pi/n) \dots \dots \dots (\sin^2 x - \sin^2 (n-1)\pi/2n), \text{ when } n \text{ is odd.}$$

$$(n-1)/2$$

$$\text{In other words, } \sin n.x = (-1)^{(n-1)/2} \{2^{(n-1)} (\sin x) \prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)\}$$

when n is odd.

This can also be written as

$$\sin n.x = (-1)^{(n-1)/2} \{2^{(n-1)} \prod_{k=-(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)\} \tag{16}$$

when n is odd.

Since $\sin n.x = (-1)^{(n-1)/2} T_n(\sin x)$ where $T_n(\sin x)$ is Chebyshev polynomial of first kind in $\sin x$, therefore,

$$T_n(\sin x) = 2^{(n-1)/2} (\sin x) \prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)$$

Coming to equation (16), its three cases will be considered.

Case-1:

When $x = 0$ and n is odd, the identity (16) transforms to

$$\prod_{k=-(n-1)/2}^{(n-1)/2} \sin k\pi/n = 0$$

It is trivial due to the fact, above product contains term $\sin 0$ and $\sin 0$ being equal to 0, makes the product 0.

Case-2:

When $x = \pi/2$ and $n = 4p+ 1$ where p is 0, 1, 2, 3....., the identity 16 transforms to

$$\sin (4p+1) \pi/2 = (-1)^{2p} \cdot (2^{4p}) \cdot \prod_{k=-2p}^{2p} \{1 - \sin k\pi/(4p+1)\}$$

$$\text{Or } \prod_{k=-2p}^{2p} \{1 - \sin k\pi/(4p+1)\} = (1/2)^{4p}$$

$$\text{Also } \prod_{k=1}^{2p} (1 - \sin^2 k\pi/(4p+1)) = (1/2)^{4p}$$

$$\text{Or } \prod_{k=1}^{2p} \cos k\pi/(4p+1) = (1/2)^{2p}$$

Case-3:

When $x = \pi/2$ and $n = 4p - 1$ where p is 1, 2, 3....., the identity 16 transforms to

$$\sin(4p-1)\pi/2 = (-1)^{2p-1} (2^{4p-2}) \prod_{k=-2p+1}^{2p-1} \{1 - \sin k\pi/(4p-1)\}$$

$$\prod_{k=-2p+1}^{2p-1} \{1 - \sin k\pi/(4p-1)\} = (1/2)^{4p-2}$$

Also $\prod_{k=1}^{2p-1} \{1 - \sin^2 k\pi/(4p-1)\} = (1/2)^{4p-2}$

or $\prod_{k=1}^{2p-1} \cos k\pi/(4p-1) = (1/2)^{2p-1}$

From above, it is clear that it does not matter whether n is of the form $4p + 1$ or of form $4p - 1$, the above identities therefore can be generalised as follows.

When n is odd and more than 1.

$$\prod_{k=-(n-1)/2}^{(n-1)/2} \{1 - \sin k\pi/n\} = (1/2)^{n-1} \tag{17}$$

and

$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (1/2)^{(n-1)/2} \tag{18}$$

Now factorisation of sine of multiple angle is taken up when angle is even. That is it is of form $2k$ where k is 1, 2, 3..... Let it be $\sin 2x$.

By putting $\sin 2x = 0$, its roots correspond to angles that can be found as

$$2x = +k\pi,$$

$$2x = -k\pi,$$

where k is 0, 1, -1, therefore on putting $k = 1$ and also $k = -1$, x equals $\pi/2$ and $-\pi/2$. Therefore, $\sin 2x$ should have factors $(\sin x - \sin \pi/2)(\sin x + \sin \pi/2)$ but it is $1 - \sin^2 x$ which equals $\cos^2 x$. However, $\sin 2x$ has factors $2 \sin x \cos x$. If we include, $k = 0$ also, then $\sin x$ can be considered its factor. That means if we consider three roots by putting $k = 1, 0, -1$, then its factors are $\sin x \cdot (\sin^2 x - 1)$ or $-\sin x \cdot \cos^2 x$. But $\sin 2x$ can have two factors, therefore whenever factor $(\cos^2 x - 1)$ appears and if it is assumed to correspond to $\cos x$, then the problem is solved and $\sin 2x$ will have two factors $\sin x$ and $\cos x$ and multiplying it with 2, it equates to $\sin 2x$. It can therefore be written as $\sin 2x = (-1)^{(2/2+1)} \cdot 2^1 \cdot \cos x \cdot \sin x$.

Likewise $\sin 4x$ will have factors corresponding to $k = -1, 0, 1, -2$ and 2 but as proposed above factor corresponding to $k = 4/2, -4/2$ will be considered pertaining to $-\cos x$, therefore,

$$\begin{aligned} \sin 4x &= (-1)^{(4/2+1)} \cdot 2^{4-1} \cdot \cos x \cdot \sin x \cdot (\sin^2 x - \sin^2 \pi/4) \\ &= -2^3 \cdot \cos x \cdot \sin x \cdot (\sin^2 x - 1/4). \end{aligned}$$

$$\begin{aligned} \text{Likewise } \sin 6x &= (-1)^{(6/2+1)} \cdot 2^{6-1} \cdot \cos x \cdot \sin x \cdot (\sin^2 x - \sin^2 \pi/6) \cdot (\sin^2 x - \sin^2 2\pi/6) \\ &= 2^5 \cdot \cos x \cdot \sin x \cdot (\sin^2 x - 3/4) \cdot (\sin^2 x - 1/4). \end{aligned}$$

Similarly,

$$\sin 8x = -2^7 \cdot \cos x \cdot \sin x \cdot (\sin^2 x - \sin^2 \pi/8) \cdot (\sin^2 x - \sin^2 \pi/4) \cdot (\sin^2 x - \sin^2 3\pi/8).$$

A generalised form can be written as

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \sin x \cdot (\sin^2 x - \sin^2 \pi/n) \cdot (\sin^2 x - \sin^2 2\pi/n) \cdot (\sin^2 x - \sin^2 3\pi/n) \dots \dots \dots (\sin^2 x - \sin^2 (n-2)\pi/2n).$$

Also

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \sin x \cdot (\sin x - \sin \pi/n) \cdot (\sin x + \sin \pi/n) \cdot (\sin x - \sin 2\pi/n) \cdot (\sin x + \sin 2\pi/n) \dots \dots \dots \cdot \{\sin x - \sin (n-2)\pi/2n\} \cdot \{\sin x - \sin (n-2)\pi/2n\}$$

$$\text{Or } \sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \sin x \cdot \prod_{k=1}^{n/2-1} (\sin^2 x - \sin^2 k\pi/n) \tag{19}$$

Or

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{n-1} \cdot \cos x \cdot \prod_{k=-n/2+1}^{n/2-1} (\sin x - \sin k\pi/n) \tag{20}$$

when n is even and more than 2.

Since $\sin nx = (-1)^{(n/2-1)} \cdot \cos x \cdot U_{n-1}(\sin x)$ where U_{n-1} is Chebyshev polynomial of second kind of degree n-1, Therefore,

$$U_{n-1}(\sin x) = 2^{(n/2-1)} \cdot \cos x \cdot \prod_{k=-n/2+1}^{n/2-1} (\sin x - \sin k\pi/n)$$

Also $\sin nx = \sin x \cdot U_{n-1}(\cos x)$ for all value of x, therefore,

$$U_{n-1}(\cos x) = (-1)^{(n/2+1)} \cdot 2^{(n-1)} \cdot \sin x \cdot \prod_{k=1}^{n/2-1} (\cos^2 k\pi/n - \sin^2 x)$$

When x is equal to $\pi/4$,

$$\sin n\pi/4 = (-1)^{(n/2+1)} \cdot 2^{(n-1)} \cdot \cos \pi/4 \cdot \sin \pi/4 \cdot \prod_{k=1}^{n/2-1} (\sin^2 \pi/2 - \sin^2 k\pi/n)$$

$$\sin n\pi/4 = (-1)^{(n/2+1)} \cdot 2^{(n-2)} \cdot \prod_{k=1}^{n/2-1} (\sin^2 \pi/2 - \sin^2 k\pi/n)$$

If n is not multiple of 4 or of form $2(4p+1)$ where p is any integer 1, 2, 3, then

$$1 = 2^{4p} \cdot \prod_{k=1}^{4p} (1/2 - \sin^2 k\pi/(8p+2))$$

$$\prod_{k=1}^{4p} \cos k\pi/(4p+1) = (1/2)^{4p} \tag{21}$$

or $\cos \pi/(4p+1) \cdot \cos 2\pi/(4p+1) \cdot \cos 3\pi/(4p+1) \dots \dots \dots \cos 4p\pi/(4p+1) = (1/2)^{4p}$.

When n is of form $2(4p-1)$ and x is $\pi/4$, then

$$\prod_{k=1}^{4p-2} \cos 2k\pi/n = -(1/2)^{(4p-2)} \tag{22}$$

$\cos \pi/(4p-1) \cdot \cos 2\pi/(4p-1) \cdot \cos 3\pi/(4p-1) \dots \dots \dots \cos (4p-2)\pi/(4p-1) = -(1/2)^{(4p-2)}$

3. CONCLUSIONS AND RESULTS

Finite product of factors $\{1/2 + \cos 3^{j-1} \cdot 2\pi/n\}$ where j varies from 0 to k is solvable as equal to $(1/2)^k$ or $-(1/2)^k$ where k is given by relation $(3^k + 1)/n = m$ or $(3^k - 1)/n = m$. k and m are positive integers, however n may be any integer not multiple of 3 and it can be fraction of the form p/q where p is not multiple of 3. Positive or negative sign of $(1/2)^k$ depends upon whether m is even or odd and whether relation satisfied is $(3^k + 1)/n$ or $(3^k - 1)/n$.

When $(3^k + 1)/n = m$ and m is odd OR $(3^k - 1)/n = m$ and m is even, then the product is $(1/2)^k$.

When $(3^k + 1)/n = m$ and m is even OR $(3^k - 1)/n = m$ and m is odd, then the product is $-(1/2)^k$.

Term $(3^k + 1)$ or $(3^k - 1)$ is always even and divisible by n where n is not multiple of 3. If n is multiple of 3, then this identity is not applicable.

In that case, another identity ie finite product of factors $\cos 2^{j-1} \cdot \pi/n$ where j varies from 0 to k is applicable and solvable as equal to $(1/2)^k$ or $-(1/2)^k$ where k is given by relation $(2^k + 1)/n = m$ or $(2^k - 1)/n = m$. k and m are positive integers, however n may be any integer not multiple of 2 and it can be fraction of the form p/q where p is not multiple of 2. Positive or negative sign of $(1/2)^k$ depends upon whether m is even or odd and whether relation satisfied is $(2^k + 1)/n$ or $(2^k - 1)/n$.

When $(2^k + 1)/n = m$ and m is odd OR $(2^k - 1)/n = m$ and m is even, then the product is $(1/2)^k$.

When $(2^k + 1)/n = m$ and m is even OR $(2^k - 1)/n = m$ and m is odd, then the product is $-(1/2)^k$.

Term $(2^k + 1)$ or $(2^k - 1)$ is always odd and divisible by n where n is not multiple of 2.

Cosine of multiple angle is always factorable. When it is made equal to zero, it gives angle that correspond to its roots. Since n.x is multiple angle, then putting $\cos nx = 0$ and n.x equal $(2k+1) \cdot \pi/2$ where k is integer 0, 1, 2, 3, value of angles x can be found as $\pi/2n, 3\pi/2n, 5\pi/2n, \dots, (2k+1) \cdot \pi/2n$ where k will have values 0, 1, 3,(n-1).

Hence its roots are $\cos \pi/2n, \cos 3\pi/2n, \cos 5\pi/2n, \dots, \dots, \cos (2k+1)\pi/2n$ and its factors are $(\cos x - \cos \pi/2n), (\cos x - \cos 3\pi/2n), (\cos x - \cos 5\pi/2n), \dots, \dots, \{\cos x - \cos (2k+1)\pi/2n\}$. Therefore,
 $\cos n.x = 2^{(n-1)} \cdot (\cos x - \cos \pi/2n) \cdot (\cos x - \cos 3\pi/2n) \cdot (\cos x - \cos 5\pi/2n) \cdot \dots \cdot \{\cos x - \cos (2k+1)\pi/2n\}$.

Here term $2^{(n-1)}$ is multiplied to exact its value with that of $\cos n.x$.

From this, the identities,

(1). $\cos n.x = 2^{(n-1)} \cdot \{\cos x - \cos \pi/2n\} \cdot \{\cos x - \cos 3\pi/2n\} \cdot \{\cos x - \cos 5\pi/2n\} \cdot \dots \cdot \{\cos x - \cos (2n-1)\pi/2n\}$

Or

$$\cos n.x = 2^{(n-1)} \cdot \prod_{k=0}^{n-1} \{\cos x - \cos (2k+1)\pi/2n\}$$

(2). $\prod_{k=0}^{n-1} \{1 - \cos (2k+1)\pi/2n\} = (1/2)^{(n-1)}$

or

$$\prod_{k=0}^{n-1} \sin (2k+1)\pi/4n = (1/2)^{(n-1)/2}$$

(3). When n is even,

$$\prod_{k=0}^{n-1} \{\cos (2k+1)\pi/2n\} = (-1)^{n/2} \cdot (1/2)^{(n-1)}$$

are derived.

Similarly, $\sin n.x$ when n is odd, can also be factorised and multiplication of its factors equated to $\sin n.x$ with precaution that factors are multiplied $(-1)^{(n-1)/2} \cdot (2)^{(n-1)}$ multiplication so as to exact its value with that of $\sin n.x$.

1. $\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{(n-1)}\} (\sin x) \cdot (\sin^2 x - \sin^2 2\pi/n) \cdot (\sin^2 x - \sin^2 2\pi/n) \cdot (\sin^2 x - \sin^2 3\pi/n) \cdot \dots \cdot (\sin^2 x - \sin^2 (n-1)\pi/2n)$, when n is odd

In other words, $\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{(n-1)}\} \cdot (\sin x) \cdot \prod_{k=1}^{(n-1)/2} (\sin^2 x - \sin^2 k\pi/n)$

when n is odd.

This can also be written as

$$\sin n.x = \{(-1)^{(n-1)/2} \cdot 2^{(n-1)}\} \cdot \prod_{k=-(n-1)/2}^{(n-1)/2} (\sin x - \sin k\pi/n)$$

when n is odd.

2. $\prod_{k=-(n-1)/2}^{(n-1)/2} \{1 - \sin k\pi/n\} = (1/2)^{(n-1)}$

and

$$\prod_{k=1}^{(n-1)/2} \{\cos k\pi/n\} = (1/2)^{(n-1)/2}$$

when n is odd.

$\sin n.x$ can also be factorised when n is even as

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{(n-1)} \cdot \cos x \cdot \sin x \cdot \prod_{k=1}^{n/2-1} (\sin^2 x - \sin^2 k\pi/n)$$

or

$$\sin n.x = (-1)^{(n/2+1)} \cdot 2^{(n-1)} \cdot \cos x \cdot \prod_{k=-n/2+1}^{n/2-1} (\sin x - \sin k\pi/n)$$

Also

1. $\prod_{k=1}^{4p} \cos k\pi/(4p+1) = (1/2)^{4p}$

or $\cos \pi/(4p+1) \cdot \cos 2\pi/(4p+1) \cdot \cos 3\pi/(4p+1) \cdot \dots \cdot \cos 4p\pi/(4p+1) = (1/2)^{4p}$
 when n is even.

2. $4p-2$

$$\prod_{k=1}^{4p-2} \cos k\pi/(4p-1) = -(\frac{1}{2})^{(4p-2)}$$

$$\cos \pi/(4p-1). \cos 2\pi/(4p-1). \cos 3\pi/(4p-1)..... \cos (4p-2)\pi/(4p-1) = -(\frac{1}{2})^{(4p-2)}$$

4. REFERENCES

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