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NEIGHBORHOOD SETS AND NEIGHBORHOOD POLYNOMIAL OF A PATH

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ABSTRACT

A *set S of vertices in a graph G is a neighborhood set of G if $G = \cup_{v\in S}\langle N[v]\rangle$, where is the $\langle N[v]\rangle$ subgraph of G induced by v and all vertices adjacent to. The neighborhood number $n_0(G)$ of G is the minimum number of vertices in a neighborhood of G [3]. Let $P_n^{\,i}$ be the family of neighborhood sets of a Path P_n with cardinality i. In this paper we construct family of neighborhood sets of Paths $P_n^{\,i}$ and its polynomial of a path.*

Keywords*: Neighborhood set, neighborhood number and neighborhood polynomials.*

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1. INTRODUCTION

Let G be a simple graph with vertex set $V = \{v_1, v_2, v_3, .., v_n\}$ and the Edge set $E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\}$. A neighborhood set $S \subseteq V(G)$ is a neighborhood set of G if $G = \cup_{v\in S}\langle N[v]\rangle$ where $\langle N[v]\rangle$ is induced subgraph of G. The neighborhood number $n_0(G)$. Let P_n^i be the family of neighborhood sets of a Paths P_n^i with cardinality i and $n_0(P_n, i) = |P_n^i|$ and the Polynomials are $N_0(P_n, x) = \sum_{i=n_0}^{n} n(P_n, i)x^i$ the polynomial of the path.

2. NEIGHBORHOOD SETS OF PATHS

Let $P_n, n \geq 4$ be the path with n vertices $V(P_n) = \{1,2,3 \ldots n\}$ and $E(P_n) = \{(1,2), (2,3) \ldots (n-1, n)\}$. Let P_n^i be the family of neighborhood sets of P_n with cardinality i. Every path P_n consist a simple path. The following lemmas and theorems are needed for the construction of family Neighborhood sets with different cardinality.

Lemma 2.1: For a graph $= P_n$, $n \geq 3$ The following Properties are true

$$P_n^i = \emptyset \text{ if and only if } i > n \text{ or } i < \left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem 2.2: If $X \in P_{n-3}^{i-1}$ and there exists $x \in [n]$ such that $X \cup \{x\} \in P_n^i$ then $X \in P_{n-2}^{i-1}$.

Proof: Suppose that $X \notin P_{n-2}^{i-1}$ since $X \in P_{n-3}^{i-1}$, X contains at least one vertex label $n-4$ *or* $n-3$. If $n-4 \in X$, then $X \in P_{n-3}^{i-1}$ a contradiction. Hence, $n-4 \in X$, but in this case, $X \cup \{x\} \notin P_n^i$ for any $x \in [n]$ also a contradiction. This $X \in P_{n-2}^{i-1}$.

Lemma 2.3: Let $P_n, n \geq 2$ be a path. Then
- (i) $X \in P_{n-2}^{i-1} = \emptyset$ then $P_{n-1}^{i-1} = \emptyset$..
- (ii) $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$ then $P_n^i = \emptyset$.

Proof:
- (i) Let $P_{n-2}^{i-1} = \emptyset \Rightarrow i-1 < n-2 \Rightarrow i-1 < n-1$ therefore $P_{n-1}^{i-1} \neq \emptyset$ which is a contradiction. Since, By lemma 2.1. $P_{n-2}^{i-1} = \emptyset$ then $P_{n-1}^{i-1} = \emptyset$.
- (ii) By the result (i), $i-1 < n-2 \Rightarrow i-1 < n$, therefore $P_n^{i-1} \neq \emptyset$ which is a contradiction. Hence $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$ then $P_n^i = \emptyset$.

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Lemma 2.4: If $P_n^i \neq \emptyset$ then the following properties are true

(i) $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} = \emptyset$ if and only if $n = 2k+1$ and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$.
(ii) $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ if and only if $i = n$.
(iii) $P_{n-1}^{i-1} \neq \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ if and only if $n = 2k, k = i$.

Proof:

(i) Let $P_{n-1}^{i-1} = \emptyset \Rightarrow i-1 > n-1$ or $i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. If $i-1 > n-1$ then $i > n$ by lemma 2.1, $P_n^i = \emptyset$ which is a contradiction. Therefore $-1 < \left\lceil \frac{n-1}{2} \right\rceil + 1$. Since $P_n^i \neq \emptyset$ and $\left\lfloor \frac{n}{2} \right\rfloor \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil + 1$, we get $n = 2k+1$ and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose $n = 2k + 1$and $i = k = \left\lfloor \frac{n}{2} \right\rfloor$ for some $\in N$. Then by lemma 2.1 , if $P_{n-1}^{i-1} = \emptyset$ then $P_{n-2}^{i-1} \neq \emptyset$.

(ii) Let $P_{n-2}^{i-1} = \emptyset$. By lemma 2.1, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$. If $i-1 < \left\lceil \frac{n-2}{2} \right\rceil$ then $-1 < \left\lceil \frac{n-1}{2} \right\rceil$. Therefore $P_{n-1}^{i-1} = \emptyset$ which is a contradiction. Hence, $i > n-1$. Since $P_{n-1}^{i-1} = \emptyset$, $-1 \leq n-1$. Therefore $= n$. Suppose $i = n$ then by lemma 2.1 if $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$.

(iii) Let $P_{n-1}^{i-1} = \emptyset$. Then $-1 > n-1 \Rightarrow i-1 < \left\lceil \frac{n-1}{2} \right\rceil$. If $i-1 > n-1$ then $i-1 > n-2$. Hence by lemma 2.1, $P_{n-1}^{i-1} = P_{n-2}^{i-1} = \emptyset$, which is a contradiction. Therefore $i-1 < \left\lceil \frac{n-1}{2} \right\rceil + 1$ and $P_{n-2}^{i-1} \neq \emptyset$. Hence $\left\lceil \frac{n-2}{2} \right\rceil + 1 \leq i \leq \left\lceil \frac{n-1}{2} \right\rceil$. Therefore, $n = 2k = i$. For some $k \in N$ then by lemma 2.1 $P_{n-1}^{i-1} = P_{2k}^{k} \neq \emptyset$ for some $k \in N$.

3. CONSTRUCTION OF FAMILIES OF NEIGHBORHOOD SETS OF PATHS

Theorem 3.1: For any path P_n^i, $n \geq 4$ and $i \geq \left\lfloor \frac{n}{2} \right\rfloor$, the following result are true.

(i) If $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ then $P_n^i = \{2,4,\ldots,n-5,n-3,n-1\}$.
(ii) If $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = \{[n]\}$.
(iii) If $P_{n-2}^{i-1} \neq \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = P_n^{n-1} = \{[n] - \{x\}/x \in [n]\}$.
(iv) If $P_{n-2}^{i-1} \neq \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$ then $P_n^i = \{X_1 \cup \{n\}/X_1 \in P_{n-1}^{i-1}\}, P_n^i = \{X_2 \cup \{n-1\}/1 \in X_2 \in P_{n-2}^{i-1}\}$,
$P_n^i = \{X_2 \cup \{n-1\}/X_2 \in P_{n-2}^{i-1}\}$ and $P_n^i = \{X_2 \cup \{n\}/1 \notin X_2 \in P_{n-2}^{i-1}\}$.

Proof:

(i) Let $P_{n-1}^{i-1} = \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$ **.** By lemma 2.4 (i), $n = 2k+1$ and $i = k$ for some $k = \left\lfloor \frac{n}{2} \right\rfloor \in N$. This imply that $P_n^i = P_n^{\lceil n/2 \rceil} = \{2,4,6,8,..,n-1\}$.

(ii) Let $P_{n-2}^{i-1} = \emptyset$ and $P_{n-1}^{i-1} \neq \emptyset$. By lemma 2.4 (ii) $i = n$. Hence $P_n^i = P_n^n = \{[n]\}$.

(iii) If $i = n-1$ in lemma 2.4 (iii), then we get $P_n^i = P_n^{n-1} = \{[n] - \{x\}/x \in [n]\}$.

(iv) $P_{n-1}^{i-1} \neq \emptyset$ and $P_{n-2}^{i-1} \neq \emptyset$. Let $X_1 \in P_{n-2}^{i-1}$ then there exists at least one vertex labled in $n-3$ *or* $n-2$ is in X_1. If $n-3$ *or* $n-2 \in X_1$, then $X_1 \cup \{n-1\} \in P_n^i$. Let $X_2 \in P_{n-1}^{i-1}$ then there exists one vertex labled as $n-1$ is in X_2. If $n-1 \in X_2$ then $X_1 \cup \{n\} \in P_n^i$.

3.1 NEIGHBORHOOD POLYNOMIAL OF PATHS

Definition 3.1.1: Let P_n^i be the family of neighborhood sets of a path P_n with cardinality *i* and let $n_0(P_n^i) = \left|P_n^i\right|$. Then the **neighborhood polynomial of** P_n is defined as

$$N(P_n, x) = \sum_{n_0 = \left\lfloor \frac{p}{2} \right\rfloor}^{n} n(P_n, x) x^i$$

We obtain a neighborhood polynomial of P_7 . $N(P_7, x) = x^3 + 10x^4 + 15x^5 + 7x^6 + x^7$.

Theorem 3.1.2: Let $= P_n$, $n \geq 3$ be a path. Then the following properties are true.

(i) If P_n^i is the family of neighborhood sets with cardinality *i* of P_n then $\left|P_n^i\right| = \left|P_{n-1}^{i-1}\right| + \left|P_{n-2}^{i-1}\right|$.
(ii) For every $n \geq 4$, $N(P_n, x) = x[N(P_{n-1}, x) + N(P_{n-2}, x)]$ with the initial values,

$$N(P_1, x) = x, N(P_2, x) = x^2 + 2x, N(P_3, x) = x^3 + 3x^2 + 3x$$

4. COEFFICIENTS OF NEIGHBORHOOD POLYNOMIAL OF PATHS

The coefficients of $N(P_n, x)$ is determined for $1 \leq n \leq 12$. Let $n\,(P_n, i) = \left|P_n^i\right|$. Also there are some relationship exist between the coefficients $n(P_n, i)$ where $\frac{n}{2} \leq i \leq n$.

Theorem 4.1: Let $= P_n, n \geq 3$ be a path and Let $N(P_n, n)$ be the total number of neighborhood sets with size n. Then The following properties hold for coefficients of $N(P_n, x)$.

(i) For every $n \in N,\ n\,(P_{2n+1}, n) = 1$

(ii) For every $n \geq 4i \geq \left\lfloor \frac{n}{2} \right\rfloor, n(P_n, i) = n\,(P_{n-1}, i-1) + n\,(P_{n-2}, i-1)$

(iii) For every $n \in N,\ n\,(P_{2n+1}, n+1) = \frac{(n+1)(n+2)}{2}$

(iv) For every $n \in N, n\,(P_{2n+2}, n+1) = (n+2)$

(v) For every $n \in N, n(P_n, n) = 1$

(vi) For every $n \in N, n(P_n, n-1) = n$

(vii) For every $n \in N,\ n\,(P_{2n}, n+1) = \frac{n(n+1)(n+2)}{6}$

(viii) For every $n \in N, n(P_{2n}, n) = 1 + n, n \geq 2$

(ix) For every $n \in N, n\,(P_n, n-2) = \frac{(n-1)(n-2)}{2}$

(x) If $S_n = \sum_{i=\left\lfloor \frac{n}{2} \right\rfloor}^{n} n(P_n, i)$ then for every $n \geq 4$, $S_n = S_{n-1} + S_{n-2}$.

(xi) For every $n \in N$ and $k = 0,1,2, \dots, 2n-1$, then
$n\,(P_{n+1}, i+1) - n\,(P_n, i+1) = n(P_n, i) - n(P_{n-2}, i)$.

Proof:

(i) Since,$P_{2n+1}^{n} = \{\{2,4,6,8, \dots, n-1\}\}$, $|P_{2n+1}^{n}| = 1$, $P_{2n+1}^{n} = \{2,4,6,8, \dots, n-1\}$ therefore $P_{2n+1}^{n} = 1$, $n = 1,2,3, \dots$

(ii) From the theorem (3.8.2) $|P_n^i| = |P_{n-1}^{i-1}| + |P_{n-2}^{i-1}|, n(P_n, i) = n\,(P_{n-1}, i-1) + n\,(P_{n-2}, i-1)$.

(iii) This property is proved for $P_{2n+1}$by induction on . The result is true for $n = 1$ and $i = 1$ Similarly for $n = 3$ and $i = 2$, we get $n(P_3, 2) = n(P_2, 1) + n(P_1, 1) = 2 + 1 = 3$, therefore $P_3^2 = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. For $n = 2$ in $P_{2n+1} = P_5$ and $i = 2$ the Neighborhood sets are
$n(P_5, 3) = \{\{1,2,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\} = 6$ For $n = 3,4,5, \dots, n-1$ this result is true. Then by (i) and (ii), it is true for n in P_{2n+1}.
By property (ii),

$$n\,(P_{2n+1}, n+1) = n\,(P_{2n}, n) + n\,(P_{2n-1}, n)$$
$$= (n+1) + \frac{n(n+1)}{2}$$
$$n\,(P_{2n+1}, n+1) = \frac{(n+1)(n+2)}{2}$$

(iv) By induction on n. When $n = 1$ P_{2n+2} and $i = 2$ the neighborhood sets are
$n(P_4, 2) = n(P_3, 1) + n(P_2, 1) = 2 + 1 = 3$ $P_4^2 = \{\{1,3\}, \{2,3\}, \{2,4\}\}$, therefore $n(P_4, 2) = 3$. This result is true for $n = 2,3,4, \dots, n-1$. Then by the results (i), (ii) & (iii), this result is true for n

$$n\,(P_{2n+2}, n+1) = n\,(P_{2n+1}, n) + n\,(P_{2n}, n)$$
$$= (n+1) + 1$$
$$n\,(P_{2n+2}, n+1) = n + 2.$$

(v) For any Path P_n with n vertices the number of neighborhood sets of P_n of size $i = n$ is $n(P_n, n) = 1$.

(vi) For any Path P_n with n vertices the number of neighborhood sets of P_n of size $i = n-1$ is $(P_n, n-1) = n$.

(vii) By induction on n, the result is true for$n = 1$ in P_{2n} and $i = 2$ ie). $n(P_2, 2) = 3$. Then the result is true for all $n = 2,3, \dots, n-1$ in and $= n+1$. Therefore it is true for n. By the results (iii) and (iv) and the induction hypothesis, the number of neighborhood sets of P_{2n} and $i = n+1$ is

$$n\,(P_{2n}, n+1) = n\,(P_{2n-1}, n) + n\,(P_{2n-2}, n)$$
$$= \frac{n(n+1)}{2} + \frac{(n-1)n(n+1)}{6}$$
$$n(P_{2n}, n+1) = \frac{n(n+1)(n+2)}{6}$$

(viii) It is proved by induction on ≥ 2 . The result is true for $n = 2$. Then $(P_4, 2) = 3$. The result is true for all $n = 3, \dots, n-1$. and it is true for n by the result (v) and (vi), for n in P_{2n}

$$n(P_{2n}, n) = n(P_{2n-1}, n-1) + n(P_{2n-2}, n-1)$$
$$n(P_{2n}, n) = 1 + n$$

(ix) It is proved by induction on $n \geq 4$. The result is true $n = 4$ then $(P_4, 2) = 3$. The result is true for all $n = 4,5, \dots, n-1$.
In P_n with $i = n-2$. Then it is true for n in P_n with $i = n-2$

$$n(P_n, n-2) = n(P_{n-1}, n-3) + n(P_{n-2}, n-3)$$
$$= \frac{(n-2)(n-3)}{2} + n - 2$$
$$n(P_n, n-2) = \frac{(n-1)(n-2)}{2}$$

(x) It is proved from the result theorem 3.1.2 (i)

$$S_n = \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor}^{n} n(P_n, i)$$

$$= \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor}^{n} n(P_{n-1}, i-1) + n(P_{n-2}, i-1)$$

$$= \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor-1}^{n-1} n(P_{n-1}, i) + \sum_{i=\left\lfloor\frac{n}{2}\right\rfloor-1}^{n-2} n(P_{n-2}, i-1)$$

$$S_n = S_{n-1} + S_{n-2}.$$

(xi) From the result theorem 3.1.2 (i) for every $n \in N$ and $k = 0,1,2,\ldots,2n-1$ then

$$n(P_{n+1}, i+1) - n(P_n, i+1) = \Big(\big(n(P_n, i) + n(P_{n-1}, i)\big)\Big) - \big((n(P_{n-1}, i) + n(P_{n-2}, i))\big)$$

$$n(P_{n+1}, i+1) + n(P_n, i+1) = n(P_n, i) - n(P_{n-2}, i)$$

REFERENCES

1. J.Joseline Manora and I.Paulraj Jayasimman, Neighborhood sets polynomial of graph, International Journal of Applied Mathematical Sciences, 6(1) (2013) 91-97.
2. J. Joseline Manora and I.Paulraj Jayasimman, Neighborhood sets polynomial and Neighborhood Polynomial of a cycle Annals of Pure and Applied Mathematics, Vol. 7, No. 2, 2014, 45-51 ISSN: 2279-087X (P), 2279-0888 (online) Published on 11 September 2014.
3. E.Sampathkumar and Prabha S. Neeralagi, The neighbourhood number of a graph, Indian J. Pure Appl. Math., 16(2) (1985)126-132.
4. Saeid Alikhani and Yee-hock Peng, Introduction to domination polynomial of a graph to appear in Ars Combinatoria.
5. Saeid Alikhani and Yee-hock Peng, Dominating sets and domination polynomials of paths, International Journal of Mathematics and Mathematical Sciences, Volume 2009, Article Id 542040.
6. Saeid Alikhani and Yee-hock Peng, Dominating sets and domination polynomials of cycles, arxiv09053268v1 [math co] 20 May 2009.
7. Saeid Alikhani, On the domination polynomial of some graph operations, ISRN Combinatorics, Volume 2013, Article ID 146595. 6. F.Harary, Graph Theory Addison Wesley, Reading Mass (1969).
8. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Fundamentals of Domination in Graphs, 1998 by Marcel Dekker, Inc., New York. 8. E.Sampathkumar and Prabha S. Neeralagi, The neighbourhood number of a graph, Indian J. Pure Appl. Math., 16(2) (1985) 126-132.

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