COMMON FIXED POINT THEOREMS FOR WEAKLY COMMUTING MAPPINGS
IN GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES

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ABSTRACT

In this paper, we prove a common fixed point theorems for compatible and weakly commuting maps in generalized intuitionistic fuzzy metric spaces.

Keywords: Intuitionistic fuzzy metric spaces, S- Fuzzy metric spaces, Compatible and weakly commuting maps.

AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION


In this paper, we define weakly commuting and compatible maps in generalized intuitionistic fuzzy metric spaces and prove common fixed point theorem for weakly commuting and compatible maps in generalized intuitionistic fuzzy metric spaces.

2. PRELIMINARIES

Definition 2.1: The 5- tuple (X, S, T, *, ◊) is said to be generalized intuitionistic fuzzy metric space if X is an arbitrary set. * is a continuous t- norm, ◊ is a continuous t-conorm and S, T are fuzzy sets on X³× (0, ∞) satisfying the following conditions: for every x, y, z, a ∈ X and t, s > 0.

(i) S(x, y, z, t) + T(x, y, z, t) ≤ 1,
(ii) S(x, y, z, t) > 0,
(iii) S(x, y, z, t) = 1 iff x = y = z,
(iv) S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t) = . . . . ,
(v) S(x, y, z, r + s + t) ≥ S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t),
(vi) S(x, y, z, t): (0, ∞) → [0, 1] is continuous,
(vii) T(x, y, z, t) < 0,
(viii) T(x, y, z, t) = 0 iff x = y = z,
(ix) T(x, y, z, t) = T(y, z, x, t) = T(z, y, x, t) = . . . . ,
(x) T(x, y, z, r + s + t) ≤ T(x, y, w, r) ◊ T(x, w, z, s) ◊ T(w, y, z, t),
(xi) T(x, y, z, t): (0, ∞) → [0, 1] is continuous.
Definition 2.2: Let (X, S, T, *, ◊) be a generalized intuitionistic fuzzy metric space, then
i) A sequence \( \{x_n\} \) in X is said to be convergent to \( x \) if
\[
\lim_{n \to \infty} S(x_n, x, x, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} T(x_n, x, x, t) = 0.
\]
ii) A sequence \( \{x_n\} \) in X is said to be a Cauchy sequence if
\[
\lim_{n, m, p \to \infty} S(x_n, x_m, x_p, t) = 1 \quad \text{and} \quad \lim_{n, m, p \to \infty} T(x_n, x_m, x_p, t) = 0
\]
for all \( x, y, z \in X \) and for each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[
S(x_n, x_m, x_p, t) > 1 - \epsilon \quad \text{and} \quad T(x_n, x_m, x_p, t) < \epsilon
\]
for \( n, m, p \geq n_0 \).
iii) A generalized intuitionistic fuzzy metric space \((X, S, T, *, ◊)\) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.3: Two self maps A and B of a generalized intuitionistic fuzzy metric space \((X, S, T, *, ◊)\) are said to be weakly commuting if
\[
S(ABx, BAx, y, t) \geq S(Ax, Bx, z, t) \quad \text{and} \quad T(ABx, BAx, y, t) \leq T(Ax, Bx, z, t)
\]
where \( y = ABx \) or \( BAx \) and \( z = Ax \) or \( Bx \) for all \( x \in X \).

Definition 2.4: Two self mappings A and B of a generalized intuitionistic fuzzy metric space \((X, S, T, *, ◊)\) are said to be compatible if
\[
\lim_{n \to \infty} S(ABx_n, BAx_n, z, t) = 1 \quad \text{and} \quad \lim_{n \to \infty} T(ABx_n, BAx_n, z, t) = 0
\]
where \( z = ABx_n \) or \( BAx_n \), whenever \( \{x_n\} \) is a sequence in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y \) for some \( y \in X \).

Clearly, commutativity implies weak commutativity and weak commutativity implies compatibility, but neither implication is always reversible. This can be seen in following examples.

Example 2.5: Let \( X = [0, 1] \). Define \( S(x, y, z, t) = \min \{M(x, y, t), M(y, z, t), M(z, x, t)\} \) and
\[
T(x, y, z, t) = \max \{N(x, y, t), N(y, z, t), N(z, x, t)\},
\]
where \( M(x, y, t) = \frac{t}{t + d(x, y)} \), \( \frac{t}{t + d(x, y)} \), \( d(x, y) = | x – y | \) for all \( x, y \in X \). Also define self maps \( A \) and \( B \) of \( X \), by \( Ax = x^2 \), \( Bx = x^2/2 \) for all \( x \in X \).

Then we see that \( AB \neq BA \) and \( S(ABx, BAx, Ax, t) \geq S(Ax, BAx, Ax, t) \) and \( T(ABx, BAx, Ax, t) \leq T(Ax, BAx, Ax, t) \) for \( x \in [0, 1] \). This shows weak commutativity does not imply commutativity.

Example 2.6: Let \( X = \mathbb{R} \). Define \( S(x, y, z, t) = \min \{M(x, y, t), M(y, z, t), M(z, x, t)\} \) and
\[
T(x, y, z, t) = \max \{N(x, y, t), N(y, z, t), N(z, x, t)\},\]
where \( M(x, y, t) = \frac{t}{t + d(x, y)} \), \( N(x, y, t) = \frac{t}{t + d(x, y)} \) and \( d(x, y) = | x – y | \) for all \( x, y \in X \). Also define self maps \( A \) and \( B \) of \( X \), by \( Ax = x^2 \), \( Bx = x^3/3 \) for all \( x \in \mathbb{R} \) and \( x_n = 1/n, n = 1, 2, 3 \ldots \).

Here \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = 0 \in X \). \( S(ABx_n, BAx_n, ABx_n, t) \to 1 \) and \( T(ABx_n, BAXx_n, ABx_n, t) \to 0 \) as \( n \to \infty \).

But \( S(ABx, BAx, ABx, t) \geq S(Ax, BAx, ABx, t) \) and \( T(ABx, BAx, ABx, t) \leq T(Ax, BAx, ABx, t) \) are not true for \( x \in \mathbb{R} \) and \( AB \neq BA \). Thus we see that \( A \) and \( B \) are compatible, but neither commutative nor weakly commutative.

3. MAIN RESULTS

Theorem 3.1: Let \( A, B, P \) and \( T \) be self maps of a complete generalized intuitionistic fuzzy metric space \((X, S, T, *, ◊)\) with \( t \)-norm \( * \) defined by \( a * b = \min \{a, b\} \) and \( t \)-conorm \( ◊ \) defined by \( a ◊ b = \max \{a, b\} \), \( a, b \in [0, 1] \) satisfying the conditions.

3.1 \( A(X) \subseteq T(X), B(X) \subseteq P(X) \),
3.2 One of \( A, B, P \) or \( T \) is continuous,
3.3 \( (A, P) \) and \( (B, T) \) is weakly commuting pairs of maps,
3.4 For all \( x, y, z \in X \), \( 0 < k < 1, t > 0 \)
\[
S(Ax, By, z, kt) \geq \min \{S(Px, Ty, z, t), S(Ax, Ty, z, t), S(By, Px, z, t)\}
\]
and \( T(Ax, By, z, k) \leq \max \{T(Px, Ty, z, t), T(Ax, Ty, z, t), T(By, Px, z, t)\} \)
3.5 \( S(x, y, z, t) \to 1 \) and \( T(x, y, z, t) \to 0 \) as \( t \to \infty \).
Then \( A, B, P \) and \( T \) have a unique common fixed point in \( X \).
Proof: Let $x_0 \in X$ be arbitrary, construct a sequence $\{y_n\}$ in $X$ such that $y_{2n+1} = T_{2n+1}x_n = A_{2n}$ and $y_{2n} = P_{2n}x_{2n-1} = B_{2n}$; $n = 0, 1, 2, \ldots$ using (3.1.4), we have

$$S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) = S(A_{2n}, B_{2n-1}, y_{2n+1}, t) \geq \min\{S(P_{2n}, T_{2n+1}, y_{2n+1}, t), S(A_{2n}, T_{2n+1}, y_{2n+1}, t), S(B_{2n-1}, P_{2n}, y_{2n+1}, t)\}$$

Further using (3.1.4), we have,

$$T(y_{2n+1}, y_{2n+2}, y_{2n+3}, t) = T(B_{2n-1}, A_{2n}, y_{2n+1}, t) \leq \max\{T(P_{2n}, T_{2n+1}, y_{2n+1}, t), T(A_{2n}, T_{2n+1}, y_{2n+1}, t), T(B_{2n-1}, P_{2n}, y_{2n+1}, t)\}$$

This implies that,

$$S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \geq S(y_{2n}, y_{2n+2}, y_{2n+3}, t)$$

Further using (3.1.4), we have,

$$S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \geq S(y_{2n}, y_{2n+2}, y_{2n+3}, t)$$

Proceeding in the same way we get,

$$S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \geq S(y_{2n}, y_{2n+2}, y_{2n+3}, t)$$

Which implies that,

$$S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \geq S(y_{2n}, y_{2n+2}, y_{2n+3}, t)$$

Case-I: When $S(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \geq S(y_{2n}, y_{2n+2}, y_{2n+3}, t)$ and $T(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t) \leq T(y_{2n}, y_{2n+1}, y_{2n+2}, y_{2n+3}, t)$. Then for $p, q \in N$ and $t > 0$, we have

$$S(y_{2n}, y_{2n+p}, y_{2n+q}, t) \geq S(y_{2n}, y_{2n+p}, y_{2n+q}, t)$$

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Taking the limit as $n \to \infty$, we have,

$$\lim_{n \to \infty} S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq 1 \times 1 \times 1 \times \ldots \times 1 (2p - 1 \text{times}) \text{ and }$$

$$\lim_{n \to \infty} T(y_n, y_{n+p}, y_{n+p+q}, 3t) \leq 0 \times 0 \times 0 \times \ldots \times 0 (2p - 1 \text{times}),$$

which implies that $S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq 1$ and $T(y_n, y_{n+p}, y_{n+p+q}, 3t) \leq 0$ as $n \to \infty$.

**Case-II:**

When $S(y_n, y_{n+1}, y_m, t) \geq S(y_0, y_{n+1}, y_m, t/k^n)$ and $T(y_n, y_{n+1}, y_m, t) \leq T(y_0, y_{n+1}, y_m, t/k^n)$.

Then on the lines of Case I, we have,

$$S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq \{S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) * S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) * S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) * \ldots * S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) * S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) \}.$$
On letting $n \to \infty$, we have,
\[
S(Au, u, u, kt) \geq \min\{S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t)\}
\]
\[
= \min\{S(u, u, u, t), S(Au, u, u, t), S(Bv, Pu, u, t)\} \text{ or } S(Au, u, u, kt) \geq S(u, u, u, t)
\]
\[
T(Au, u, u, kt) \leq \max\{T(Pu, Tv, u, t), T(Au, Tv, u, t), T(Bv, Pu, u, t)\}
\]
Which implies that $Au = u$. Since $\Lambda(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Tv = Pu$.

Using (3.1.4) we have,
\[
S(u, Bv, u, kt) = S(Au, Bv, u, kt)
\]
\[
\geq \min\{S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t)\}
\]
\[
= \min\{S(u, u, u, t), S(Au, u, u, t), S(Bv, Pu, u, t)\} \text{ or } S(u, Bv, u, kt) \geq S(u, Bv, u, t)
\]
\[
T(u, Bv, u, kt) = T(Au, Bv, u, kt)
\]
\[
\leq \max\{T(Pu, Tv, u, t), T(Au, Tv, u, t), T(Bv, Pu, u, t)\}
\]
Which implies that $Bv = u$. Thus $u = Bv = Tv$. Since $(T, B)$ are weakly commuting, therefore $S(TBv, BTv, TBv, t) \geq S(Tv, Bv, Tv, t) = 1$ and $T(TBv, BTv, TBv, t) \leq T(Tv, Bv, Tv, t) = 0$.

Which implies that $TBv = BTv$ and so $Tu = Bu$. Using (3.1.4) we have,
\[
S(u, Tu, u, kt) = S(Au, Bu, u, kt)
\]
\[
\geq \min\{S(Pu, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t)\}
\]
\[
= \min\{S(u, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t)\} \text{ or } S(u, Tu, u, kt) \geq S(u, Tu, u, t)
\]
\[
T(u, Tu, u, kt) = T(Au, Bu, u, kt)
\]
\[
\leq \max\{T(Pu, Tu, u, t), T(Au, Tu, u, t), T(Bu, Pu, u, t)\}
\]
Which implies that $u = Tu = Bu$. Hence $u = Tu = Bu = Au = Pu$. Shows $u$ is a common fixed point of $A$, $B$, $P$ and $T$.

Now to prove uniqueness of $u$, let $w$ be another common fixed point of $A$, $B$, $P$ and $T$.

Then from (3.1.4) we have,
\[
S(w, u, u, kt) = S(Au, Bw, u, kt)
\]
\[
\geq \min\{S(Pw, Tw, u, t), S(Au, Tw, u, t), S(Bw, Pu, u, t)\}
\]
\[
= \min\{S(w, u, u, t), S(Au, u, u, t), S(Bw, Pu, u, t)\} \text{ or } S(w, u, u, kt) \geq S(w, u, u, t)
\]
\[
T(w, u, u, kt) = T(Au, Bw, u, kt)
\]
\[
\leq \max\{T(Pu, Tw, u, t), T(Au, Tw, u, t), T(Bw, Pu, u, t)\}
\]
Which implies that $u = w$.

Hence $u$ is a unique common fixed point of $A$, $B$, $P$ and $T$.

**Proposition 3.2:** Let $A$ and $B$ be compatible self mappings of a generalized intuitionistic fuzzy metric space $X$. If $Ay = By$ then $ABy = BAy$.

**Proof:** Let $Ay = By$ and $\{x_n\}$ be a sequence in $X$, such that $x_n = y$ for all $n$. Then $Ax_n, Bx_n \to Ay$.

Now by the compatibility of $A$ and $B$. We have $S(ABx_n, BAy, t) = S(ABx_n, BAx_n, ABx_n, t) \to 1$ and $T(ABy, BAy, ABy, t) = T(ABx_n, BAx_n, ABx_n, t) \to 0$ as $n \to \infty$, which yields $ABy = BAy$.  

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Theorem 3.3: Let A, B, P and T be self maps of a complete generalized intuitionistic fuzzy metric space \((X, S, T, *, ◊)\) with \(t\)-norm \(*\) defined by \(a * b = \min\{a, b\}\) and \(t\)-conorm \(◊\) defined by \(a ◊ b = \max\{a, b\}\), satisfying the conditions,

(3.3.1) \(A(X) \subseteq T(X), B(X) \subseteq P(X)\),

(3.3.2) One of A, B, P or T is continuous,

(3.3.3) \((A, P)\) and \((B, T)\) are compatible pairs of maps,

(3.3.4) For all \(x, y, z \in X, 0 < k < 1, t > 0\)

\[S(Ax, By, z, kt) \geq \min\{S(Px, Ty, z, t), S(Ax, Ty, z, t), S(By, Px, z, t), S(Ax, Px, z, t)\}\]

and

\[T(Ax, By, z, kt) \leq \max\{T(Px, Ty, z, t), T(Ax, Ty, z, t), T(By, Px, z, t), T(Ax, Px, z, t)\},\]

(3.3.5) \(S(x, y, z, t) \rightarrow 1\) and \(T(x, y, z, t) \rightarrow 0\) as \(t \rightarrow \infty\).

Then A, B, P and T have a unique common fixed point in X.

Proof: Let \(x_0 \in X\) be arbitrary. Construct a sequence \(\{y_n\}\) in X such that \(y_{2n+1} = Tx_{2n+1} = Ax_{2n}\) and \(y_{2n} = Px_{2n} = Bx_{2n-1}\), \(n = 0, 1, 2, \ldots\) using (3.3.4) we have,

\[S(y_1, y_2, y_m, kt) = S(Ax_0, Bx_1, y_m, kt) \geq \min\{S(Px_0, Tx_1, y_m, t), S(Ax_0, Px_0, y_m, t), S(Bx_1, Tx_1, y_m, t)\}\]

\[= \min\{S(y_0, y_1, y_m, t), S(y_1, y_1, y_m, t), S(y_2, y_1, y_m, t)\}\]

\[= \min\{S(y_0, y_1, y_m, t), S(y_2, y_1, y_m, t), S(y_0, y_2, y_m, t)\}\]

\[= \min\{S(y_0, y_1, y_m, t), S(y_2, y_1, y_m, t)\},\]

which implies that,

\[S(y_1, y_2, y_m, kt) \geq S(y_0, y_1, y_m, t) \text{ or } S(y_0, y_2, y_m, t) \text{ and } T(y_1, y_2, y_m, kt) \leq T(y_0, y_1, y_m, t) \text{ or } T(y_0, y_2, y_m, t).\]

Further using (3.3.4) we have,

\[S(y_2, y_3, y_m, kt) = S(Bx_1, Ax_2, y_m, kt) \geq \min\{S(Px_2, Tx_1, y_m, t), S(Ax_2, Px_2, y_m, t), S(Bx_1, Tx_1, y_m, t)\}\]

\[= \min\{S(y_2, y_1, y_m, t), S(y_1, y_2, y_m, t), S(y_2, y_2, y_m, t)\},\]

\[= \min\{S(y_2, y_1, y_m, t), S(y_2, y_2, y_m, t)\},\]

which implies that,

\[S(y_2, y_3, y_m, kt) \geq S(y_1, y_2, y_m, t) \text{ or } S(y_1, y_3, y_m, t) \text{ and } T(y_2, y_3, y_m, kt) \leq T(y_1, y_2, y_m, t) \text{ or } T(y_1, y_3, y_m, t).\]

Again with the similar process as in Theorem (3.1) we can show \(\{y_n\}\) is a Cauchy sequence.

By the completeness of X, sequence \(\{y_n\}\) and its subsequences \(\{Ax_{2n}\}, \{Bx_{2n-1}\}\), \(\{Px_{2n}\}\) and \(\{Tx_{2n+1}\}\) converge to some \(u\) in X. Now if we suppose that P is continuous then \(PAX_{2n} \rightarrow Pu\).

Since (A, P) are compatible, therefore \(\lim_{n \rightarrow \infty} S(PAX_{2n}, APX_{2n-1}, PAX_{2n-1}) = 1\) and \(\lim_{n \rightarrow \infty} T(PAX_{2n}, APX_{2n-1}, PAX_{2n-1}) = 0\), where \(\{x_n\}\) is a sequence such that \(\lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Px_{2n} = u.\)

Thus, we have \(S(Pu, lim_{n \rightarrow \infty} APX_{2n}, Pu, t) = 1\) and \(T(Pu, lim_{n \rightarrow \infty} APX_{2n}, Pu, t) = 0.\)

Which implies that \(lim_{n \rightarrow \infty} APX_{2n} = Pu\). Now using (3.3.4) we have,

\[S(APX_{2n}, Bx_{2n-1}, u, kt) \geq \min\{S(PPX_{2n}, TX_{2n+1}, u, t), S(APX_{2n}, TX_{2n+1}, u, t), S(Bx_{2n-1}, PPX_{2n}, u, t), S(APX_{2n}, PPX_{2n}, u, t)\},\]

\[T(APX_{2n}, Bx_{2n-1}, u, kt) \leq \max\{T(PPX_{2n}, TX_{2n+1}, u, t), T(APX_{2n}, TX_{2n+1}, u, t), T(Bx_{2n-1}, PPX_{2n}, u, t), T(APX_{2n}, PPX_{2n}, u, t)\},\]
On letting $n \to \infty$ we have,

\[ S(Pu, u, u, kt) \geq \min \{ S(Pu, u, u, t), S(u, Pu, u, t), S(Pu, u, Pu, t), S(u, u, u, t) \} = S(Pu, u, u, t) \text{ and} \]
\[ T(Pu, u, u, kt) \leq \max \{ T(Pu, u, u, t), T(Pu, u, u, t), T(u, Pu, u, t), T(Pu, Pu, u, t), T(u, u, u, t) \} = T(Pu, u, u, t), \]
which implies that,
\[ S(Pu, u, u, kt) \geq S(Pu, u, u, t) \text{ and } T(Pu, u, u, kt) \leq T(Pu, u, u, t). \]

Hence $Pu = u$. Further using (3.3.4) we have,

\[ S(Au, Bx_{2n+1}, u, kt) \geq \min \{ S(Pu, Tx_{2n+1}, u, t), S(Au, Tx_{2n+1}, u, t), S(Bx_{2n+1}, Pu, u, t), S(Au, Pu, u, t), S(Bx_{2n+1}, Pu, u, t) \} \]
\[ = S(Pu, u, u, t) \text{ and} \]
\[ T(Au, Bx_{2n+1}, u, kt) \leq \max \{ T(Pu, Tx_{2n+1}, u, t), T(Au, Tx_{2n+1}, u, t), T(Bx_{2n+1}, Pu, u, t), T(Au, Pu, u, t), T(Bx_{2n+1}, Pu, u, t) \}. \]

On letting $n \to \infty$ we have,

\[ S(Au, u, u, kt) \geq \min \{ S(u, u, u, t), S(Au, u, u, t), S(u, u, u, t), S(Au, u, u, t), S(u, u, u, t) \} \]
\[ \text{and} \]
\[ T(Au, u, u, kt) \leq \max \{ T(u, u, u, t), T(Au, u, u, t), T(u, u, u, t), T(Au, u, u, t), T(u, u, u, t) \}. \]

This implies that,
\[ S(Au, u, u, kt) \geq S(Au, u, u, t) \text{ and } T(Au, u, u, kt) \leq T(Au, u, u, t). \]

Hence $Au = u$. Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $u = Tv = Pu$, using (3.3.4) we have,

\[ S(u, Bv, u, kt) = S(Au, Bv, u, kt) \geq \min \{ S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t), S(Au, Pu, u, t), S(Bv, Tv, u, t) \} \]
\[ = S(u, Bv, u, t) \text{ and} \]
\[ T(u, Bv, u, kt) = T(Au, Bv, u, kt) \leq \max \{ T(Pu, Tv, u, t), T(Au, Tv, u, t), T(Bv, Pu, u, t), T(Au, Pu, u, t), T(Bv, Tv, u, t) \}. \]

This implies that,
\[ S(u, Bv, u, kt) \geq S(u, Bv, u, t) \text{ and } T(u, Bv, u, kt) \leq T(u, Bv, u, t), \]
which implies that $Bv = u$. Thus, $u = Bv = Tv$.

By the compatibility of $(T, B)$ and from propositions (3.2), we have $TBv = BTv$ and so $Tu = Bu$.

Using (3.3.4) we have,

\[ S(u, Tu, u, kt) = S(Au, Bu, u, kt) \geq \min \{ S(Pu, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t), S(Au, Pu, u, t), S(Bu, Tu, u, t) \} \]
\[ \text{and} \]
\[ T(u, Tu, u, kt) = T(Au, Bu, u, kt) \leq \max \{ T(Pu, Tu, u, t), T(Au, Tu, u, t), T(Bu, Pu, u, t), T(Au, Pu, u, t), T(Bu, Tu, u, t) \}. \]

This implies that
\[ S(u, Tu, u, kt) \geq S(u, Tu, u, t) \text{ and } T(u, Tu, u, kt) \leq T(u, Tu, u, t), \]
which implies that $u = Tu = Bu$. Hence $u = Tu = Bu = Au = Pu$. Shows $u$ is a common fixed point of $A$, $B$, $P$ and $T$.

Now to prove uniqueness of $u$, let $w$ be another common fixed point of $A$, $B$, $P$ and $T$.

Then from (3.3.4) we have,

\[ S(u, w, u, kt) = S(Au, Bw, u, kt) \geq \min \{ S(Pu, Tw, u, t), S(Au, Tw, u, t), S(Bw, Pu, u, t), S(Au, Pu, u, t), S(Bw, Tw, u, t) \} \]
\[ \text{and} \]
\[ T(u, w, u, kt) = T(Au, Bw, u, kt) \leq \max \{ T(Pu, Tw, u, t), T(Au, Tw, u, t), T(Bw, Pu, u, t), T(Au, Pu, u, t), T(Bw, Tw, u, t) \}. \]

This implies that
\[ S(u, w, u, kt) \geq S(u, w, u, t) \text{ and } T(u, w, u, kt) \leq T(u, w, u, t). \]

Thus, $u$ is a unique common fixed point of $A, B, P$ and $T$. 

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CONFERECE PAPER

4. REFERENCES
