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DUAL OF A FUZZY MEASURE- A NOTE

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ABSTRACT

The objective of this paper is to construct the dual of the fuzzy measure. For this purpose conjugate of the fuzzy measure is formulated and checked for duality. Self dual fuzzy measures exist. Example of self dual is given. The properties of the fuzzy measure are formulated. The same properties hold good for the conjugate fuzzy measure also.

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1. INTRODUCTION

This paper gives a notion of the conjugate of the fuzzy measure and paves way to get the dual of the fuzzy measure. This paper is actually a continuation of [4] where the fuzzy measure is redefined.

This concept is an extension of the measure in the sense that the additivity in the normal measure is replaced by monotonicity. The new idea in [4] is that there exists a membership(function) grade for the fuzzy measure associated with every member of the family of sets. This membership function enhances the notion of fuzzy measure.

Consequently, we formulate the conjugate of the fuzzy measure. If this conjugate is normalised and monotone it is called its dual. Henceforth there exists a membership function for this measure which is monotone.

The function 'm' is defined on a family of sets of the universe X. We say 'm' is normalized if and

$$\max\{\mu_m(A_i, m(A_i)) \mid A_i \subset 2^X\} = 1$$

For a set function 'm' defined on a family (class) of sets such that $m(\phi) = 0$. Its conjugate set function $\overline{m}$is defined as for all .

Notation:

μ_m - membership grade for measure 'm'
$x = m(A), A \subset A$
$y = m(B), B \subset A$
$\mathcal{A} = \{A_1, A_2, \ldots\}$ - Class of Sets
$\Re$ - Set of all positive real numbers
$\mu_{\overline{m}}$ - membership grade for measure '$\overline{m}$'

2. A NEW DEFINITION OF CONJUGATE FUZZY MEASURE

Let X be any set. '$\mathcal{A}$' be a class of subsets of X and $(X, \mathcal{A})$ be a measurable space. A fuzzy relation $m: \mathcal{A} \to R$ is said to be a fuzzy measure if the following conditions are satisfied

1. $\mu_m(\phi, 0) = 1$

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2. a) If $A \subseteq B \Rightarrow \underset{m(A)=x}{Sup}\ x \leq \underset{m(B)=y}{Sup}\ y$
   b) If $A \subseteq B;\ A \neq \phi \Rightarrow \mu_m(A, Sup(x)) \leq \mu_m(B, Sup(y))$

   then 'm' is said to be a fuzzy measure. This fuzzy measure is said to have a conjugate $\overline{m}$ if $\overline{\overline{m}} = m$ and for all .

correspondingly there exists a membership function $\mu_{\overline{m}}$ for every $A \subset A$ associated with the $\overline{m}(A)$ which is monotone. Then this $\overline{m}$ is called conjugate fuzzy measure.

Note 2.1: If μ_m is normalized and monotone and $\mu_{\overline{m}}$ is also normalized and monotone then $\overline{m}$ is called dual of m.

Note 2.2: Fuzzy measure 'm' does not have conjugate whenever is monotonicity is not satisfied (or) $\overline{\overline{m}} \neq m$ or $\mu_{\overline{m}}(A_1, \overline{m}(A_1)) < 0$ (or) $\mu_{\overline{m}}(A, \overline{m}(A)) = \pm\infty$ for any $A \subset A$.

Note 2.3: When $\overline{m} = m$. That is $\overline{m}(A) = m(A)$ for all $A \subset A$ and $\mu_{\overline{m}}(A, \overline{m}(A)) = \mu_m(A, m(A))$ then 'm' is self conjugate.

Example 2.1: The example given in [4] is taken to find the conjugate fuzzy measure.

Let $X = \{1,2,3\}$

$$\mathcal{A} = P(X) = \{\{1,2,3$$
$$= \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}\ (say)$$

Define $m = \frac{|A_i|}{i}$, $\mu_m(A_i, m(A_i)) = \begin{cases} \frac{x^2}{1+x^2} & for\ A_i \neq \phi \\ 1 & for\ A_i = \phi \end{cases}$

We get the following values

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
m	3	1	2/3	2/4	1/5	1/6	1/7	0
$\mu_m(A_i, m(A_i))$	9/10	1/2	4/3	1/6	1/26	1/37	1/50	1

To find the conjugate $\overline{m}$ of the fuzzy measure in 'm'.

$$\overline{m}(A_i) = \underset{A_i \subset \mathcal{A}}{\underset{m(A_i)=x}{max}}\ x - m(A_i^c)$$

$$\overline{m}(A_1) = 3 - 0 = 3 \qquad A_1^c = \phi$$
$$\overline{m}(A_2) = 3 - \frac{1}{7} = \frac{20}{7} \qquad A_2^c = A_7, m(A_7) = \frac{1}{7}$$
$$\overline{m}(A_3) = 3 - \frac{1}{6} = \frac{17}{6} \qquad A_3^c = A_6, m(A_6) = \frac{1}{6}$$
$$\overline{m}(A_4) = 3 - \frac{1}{5} = \frac{14}{5} \qquad A_4^c = A_5, m(A_5) = \frac{1}{6}$$
$$\overline{m}(A_5) = 3 - \frac{1}{2} = \frac{5}{2} \qquad A_5^c = A_4, m(A_4) = \frac{1}{2}$$
$$\overline{m}(A_6) = 3 - \frac{2}{3} = \frac{7}{3} \qquad A_6^c = A_3, m(A_3) = \frac{2}{3}$$
$$\overline{m}(A_7) = 3 - 1 = 2 \qquad A_7^c = A_2, m(A_2) = 1$$
$$\overline{m}(A_8) = 3 - 3 = 0 \qquad A_8^c = A_1, m(A_1) = 3$$

$$\mu_{\overline{m}}(A_i, \overline{m}(A_i)) = \begin{cases} \frac{(\overline{m}(A_i))^2}{1 + (\overline{m}(A_i))^2} & for\ A_i \neq \phi \\ 1 & for\ A_i = \phi \end{cases}$$

$$\mu_{\overline{m}}(A_1, 3) = \frac{9}{10}$$

$$\mu_{\overline{m}}\left(A_2, \frac{20}{7}\right) = \frac{\left(\frac{20}{7}\right)^2}{1 + \left(\frac{20}{7}\right)^2} = \frac{400}{49} \times \frac{49}{449} = 0.89$$

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$$\mu_{\overline{m}}\left(A_3,\frac{17}{6}\right)=\frac{\left(\frac{17}{6}\right)^2}{1+\left(\frac{17}{6}\right)^2}=\frac{289}{16}\times\frac{256}{325}=0.889$$

$$\mu_{\overline{m}}\left(A_4,\frac{14}{5}\right)=\frac{\left(\frac{14}{5}\right)^2}{1+\left(\frac{14}{5}\right)^2}=\frac{196}{25}\times\frac{25}{221}=0.886$$

$$\mu_{\overline{m}}\left(A_5,\frac{5}{2}\right)=\frac{\left(\frac{5}{2}\right)^2}{1+\left(\frac{5}{2}\right)^2}=\frac{25}{4}\times\frac{4}{29}=0.862$$

$$\mu_{\overline{m}}\left(A_6,\frac{7}{3}\right)=\frac{\left(\frac{7}{3}\right)^2}{1+\left(\frac{7}{3}\right)^2}=\frac{49}{9}\times\frac{9}{56}=0.875$$

$$\mu_{\overline{m}}(A_7,2)=\frac{4}{5}=0.8$$

$$\mu_{\overline{m}}(A_8,0)=1$$

	$A_2,A_3,A_4,A_5,A_6,A_7\subset A_1$	$A_5,A_6\subset A_2$	$A_5,A_7\subset A_3$	$A_6,A_7\subset A_4$
$\overline{m}$	$\frac{20}{7},\frac{17}{6},\frac{14}{5},\frac{5}{2},\frac{7}{3},2\le 3$	$\frac{5}{2},\frac{7}{3}\le\frac{20}{7}$ $2.5,2.3<2.85$	$\frac{5}{2},2\le\frac{17}{6}$ $2.5,2\le 2.83$	$\frac{7}{3},2\le\frac{14}{5}$ $2.3,2<2.8$
$\mu_{\overline{m}}$	$0.89,0.89,0.89,0.86,0.87,0.8\le 0.9$	$0.89,0.87\le 0.89$	$0.86,0.8\le 0$	$0.87,0.8\le 0$

Monotonicity is satisfied. To show $\overline{\overline{m}}=m$ and $\overline{\overline{m}}(A_i)=\max\limits_{B\subset A}\overline{m}(B)-\overline{m}(B-A)$

$\overline{\overline{m}}(A_1)=\overline{m}(A_1)-\overline{m}(A_1-A_1)=3-0=3$

$\overline{\overline{m}}(A_2)=\overline{m}(A_1)-\overline{m}(A_1-A_2)=3-2=1$

$\overline{\overline{m}}(A_3)=\overline{m}(A_1)-\overline{m}(A_1-A_3)=3-\frac{7}{3}=\frac{2}{3}$

$\overline{m}(A_4)=\overline{m}(A_1)-\overline{m}(A_1-A_4)=3-\frac{5}{2}=\frac{1}{2}$

$\overline{\overline{m}}(A_5)=\overline{m}(A_1)-\overline{m}(A_1-A_5)=3-\frac{14}{5}=\frac{1}{5}$

$\overline{\overline{m}}(A_6)=\overline{m}(A_1)-\overline{m}(A_1-A_6)=3-\frac{17}{6}=\frac{1}{6}$

$\overline{\overline{m}}(A_7)=\overline{m}(A_1)-\overline{m}(A_1-A_7)=3-\frac{20}{7}=\frac{1}{7}$

$\overline{\overline{m}}(A_8)=\overline{m}(A_1)-\overline{m}(A_1-A_8)=3-3=0$

$\overline{\overline{m}}=m\Rightarrow\overline{m}$ is conjugate fuzzy measures of 'm'.

Note: It can be easily seen that the membership function of the conjugate fuzzy measure $\overline{m}$ associated with each member of $\mathcal{A}$ is very close to 1.This indicates that the degree of association of the conjugate measure with each member of $\mathcal{A}$ is perfect.

Example 2.2: The example given in [4] is taken to find the conjugate fuzzy measure.
Let $\mathcal{A}$ be a σ-algebra given by $X=\{a,b,c\};\mathcal{A}=\{\{a\},\{b,c\},\{a,b,c\},\phi\}$
$=\{A_1,A_2,A_3,A_4\}\,(say)$

Define $m(A_i)=\frac{|A_i|}{|X|};\mu_m(A_i,m(A_i))=\begin{cases}\frac{x}{1+ix} & for\,A_i\ne\phi\\ 1 & for\,A_i=\phi\end{cases}$

	A_1	A_2	A_3	A_4
m	$1/3$	$2/3$	1	0
μ_m	$1/4$	$2/7$	$1/4$	1

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To find the conjugate of m.

$\overline{m}(A) = \underset{B \subset A}{max}\, m(B) - m(B-A)$

$\bar{m}(A_1) = m(A_3) - \overline{m}(A_3 - A_1) = 1 - \frac{2}{3} = \frac{1}{3}$

$\bar{m}(A_2) = m(A_3) - \overline{m}(A_3 - A_2) = 1 - \frac{1}{3} = \frac{2}{3}$

$\bar{m}(A_3) = m(A_3) - \overline{m}(A_3 - A_3) = 1 - 0 = 1$

$\bar{m}(A_4) = m(A_3) - \overline{m}(A_3 - A_4) = 1 - 1 = 0$

$\mu_{\overline{m}}(A_i, \bar{m}(A_i)) = \frac{\bar{m}(A_i)}{1 + i\, m(A_i)}$; $\mu_{\overline{m}}(A_1, \bar{m}(A_1)) = \frac{\frac{1}{3}}{1+1\left(\frac{1}{3}\right)} = \frac{1}{4}$

$\mu_{\overline{m}}(A_2, \bar{m}(A_2)) = \frac{\frac{2}{3}}{1+2\left(\frac{2}{3}\right)} = \frac{2}{7}$

$\mu_{\overline{m}}(A_3, \bar{m}(A_3)) = \frac{1}{4}$

$\mu_{\overline{m}}(A_4, \bar{m}(A_4)) = 0$ (by definition)

Here $\overline{m} = m$ and $\mu_{\overline{m}}(A_i, \bar{m}(A_i)) = \mu_m(A_i, m(A_i))\ \forall\ i = 1, 2, 3\ \&\ 4$.

This measure $\bar{m}$ is monotone and normalized. Hence it is dual.

Moreover $\bar{m} = m \Rightarrow \bar{\bar{m}} = \bar{m} = m.$

Hence 'm' is self conjugate fuzzy measure.

3. PROPERTIES OF CONJUGATE MEASURE

Property 1: $\boldsymbol{m(A_i \cup A_j) \leq m(A_i) + m(A_j)}$

Proof:

Let $A_i \cup A_j = A_k$ as m is monotone. $m(A_i \cup A_j) \leq m(A_k)$.....(1)

Also $A_i \subset A_k$ and $A_j \subset A_k$. Hence $m(A_i) \leq m(A_k)$....(2) and $m(A_j) \leq m(A_k)$....(3)

$(1) - (2) \Rightarrow m(A_i \cup A_j) - m(A_i) \leq 0$

$(1) - (3) \Rightarrow m(A_i \cup A_j) - m(A_j) \leq 0$

Adding $\Rightarrow 2[m(A_i \cup A_j)] - m(A_i) - m(A_j) \leq 0$

$\Rightarrow m(A_i \cup A_j) \leq m(A_i) + m(A_j)$

Property 2: $\boldsymbol{m(A_i \cap A_j) \leq m(A_i) + m(A_j)}$

Proof:

Let $A_i \cap A_j = A_l$ (say) as m is monotone. $m(A_i \cap A_j) \leq m(A_l)$.....(4)

Also $A_l \subset A_i$ and $A_l \subset A_j$. Hence $m(A_l) \leq m(A_i)$....(5)and $m(A_l) \leq m(A_j)$....(6)

$(4) + (5) \Rightarrow m(A_i \cap A_j) + m(A_l) \leq m(A_l) + m(A_i)$

$\Rightarrow m(A_i \cap A_j) - m(A_i) \leq 0$....(7)

$(4) + (6) \Rightarrow m(A_i \cap A_j) + m(A_l) \leq m(A_l) + m(A_j)$

$\Rightarrow m(A_i \cap A_j) - m(A_j) \leq 0$....(8)

$(7) + (8) \Rightarrow 2m(A_i \cap A_j) \leq m(A_i) + m(A_j)$

$\Rightarrow m(A_i \cap A_j) \leq m(A_i) + m(A_j)$

Property 3: $\boldsymbol{m(A_i - A_j) \geq m(A_j) - m(A_i)}$

Proof:
Let $A_i - A_j = A_k \Rightarrow m(A_i - A_j) \leq m(A_k)$
$\Rightarrow m(A_i - A_j) - m(A_k) \leq 0$.....(9)

Then $A_j \cup A_k = A_i \Rightarrow m(A_j \cup A_k) \leq m(A_i)$.

But $m(A_j \cup A_k) \leq m(A_j) + m(A_k) \leq m(A_i)$
$\Rightarrow m(A_j) - m(A_i) + m(A_k) \leq 0$....(10)

$(10) - (9) \Rightarrow m(A_j) - m(A_i) - m(A_i - A_j) \leq 0$
(or) $m(A_j) - m(A_i) \leq m(A_i - A_j)$
(or) $m(A_i - A_j) \geq m(A_j) - m(A_i)$

Note: The above three properties of measure hold good when m is replaced by its conjugate. In addition to these,

Property 4: $\overline{m}$ is unique.
Proof: If $\overline{m_1}$ and $\overline{m_2}$ are two conjugate fuzzy measures of m. Then by the definition
$\overline{m_1}(A) = \max_{B \subset A} m(B) - m(B - A)$
$\overline{m_2}(A) = \max_{B \subset A} m(B) - m(B - A)$

Therefore $\overline{m_1}(A) - \overline{m_2}(A) = 0 \Rightarrow \overline{m_1}(A) = \overline{m_2}(A) = m(A)$.

Hence $\overline{m}$ is unique.

This proof is trivial.

4. CONCLUSION

Fuzzy Measure is an interesting topic to explore and relish. While exploring this area the idea of changing the domain occurred. Then the idea of changing the domain to find the conjugate of the fuzzy measure arose. There by the dual of the fuzzy measure was formulated. It was intersecting to find the existence of self dual fuzzy measures. Some of the properties of the fuzzy measure was framed and the same properties was found to hold good for the conjugate fuzzy measure also.

Work can be continued on the domains $\sigma -$ ring $\sigma -$ algebra generated by a set A which shows there is lot of scope for further research.

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