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BLOCK DOUBLE DOMINATION IN GRAPHS

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ABSTRACT

For any graph $G = (V, E)$, the block graph $B(G)$ of a graph G is the graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. A subset D^d of $V[B(G)]$ is double dominating set of $B(G)$ if for every vertex $v \in V[B(G)]$, $|N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[B(G)] - D^d$ and has at least two neighbours in D^d. The block double dominating number $\gamma_{ddb}(G)$ is a minimum cardinality of block double dominating set. In this paper, we establish upper and lower bounds on $\gamma_{ddb}(G)$ in terms of elements of G and other dominating parameters of G are obtained.

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Keyword: *Block Graph/Dominating set/Strong split dominating set/Independent block domination/Total block domination/Strong split block domination/Double domination.*

1. INTRODUCTION

The graphs considered here are simple and finite. Let G be a graph with $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set of G. The neighbourhood of a vertex $v \in V$ is defined by $N(v) = \{ u \in V/uv \in E\}$. The close neighbourhood of a vertex v is $N[v] = N(v) \cup \{v\}$. The order $|V(G)|$ of G is denoted by p. The degree of v is $d(v) = |N(v)|$. The maximum degree of a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A block graph $B(G)$ of a graph G is the graph whose set of vertices is the union of the set of blocks of G in which two vertices are adjacent if and only if the corresponding blocks of G are adjacent. Let $G = (V, E)$ be a graph. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. A total dominating set of G is a subset S of V such that each vertex in V is adjacent to a vertex of S. The total domination number, denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set. A connected dominating set D to be a dominating set D whose induced subgraph $< D >$ is connected. The connected domination number $\gamma_c(G)$ of a connected graph G is minimum cardinality of a connected dominating set. A dominating set D of G is called strong split dominating set of G if $< V(G) - D >$ is totally disconnected with at least two vertices. The strong split domination number $\gamma_{ss}(G)$ is the minimum cardinality of minimal strong split dominating set. Introduction and study of $\gamma_{ss}(G)$ appears in [3]. Further strong split domination number of block graph $\gamma_{ssb}(G)$ is introduced and studied by Muddebihal *et al.* [5]. A domination set $I \subseteq V[B(G)$ is called an independent block dominating set if induced subgraph $< I >$ is independent. The independent block domination number is denoted by $\gamma_{ib}(G)$ is the minimum cardinality of minimal independent block dominating set is introduced by Muddebihal *et al.* [6]. Edge dominating set $F \subseteq E$ is such that every edge in $F - E$ must be adjacent to at least one edge in F. The edge domination number denoted as $\gamma^{'}(G)$ is the minimum cardinality of edge dominating set of G. Edge domination number was studied by S.L.Mitchell and Hedetniemi in [4]. A set D subset of $V[B(G)]$ is said to be a dominating set of $B(G)$, if every vertex not in D is adjacent to a vertex in D of $B(G)$. The domination number of $B(G)$ is denoted by $\gamma[B(G)]$ is the minimum cardinality of a dominating set. The domination in graphs with many variations is now well studied in graph theory. The recent book of Chartrand and

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Lesniak [2] includes a chapter on domination. A thorough study of domination appears in [2]. A subset D^d of $V[B(G)]$ is double dominating set of $B(G)$ if for every vertex $v \in V[B(G)], |N[v] \cap D^d| \geq 2$, that is v is in D^d and has at least one neighbour in D^d or v is in $V[B(G)] - D^d$ and has at least two neighbours in D^d and it is denoted by $\gamma_{ddb}(G)$. In this paper, we establish upper and lower bounds on $\gamma_{ddb}(G)$ in terms of elements of G and other dominating parameters of G are obtained.

2. LOWER BOUNDS FOR $\gamma_{ddb}(G)$.

Here we establish lower bounds for $\gamma_{ddb}(G)$ in terms of elements of G.

Theorem 2.1: For any tree T of order p, then $\gamma_{ddb}(T) \geq \gamma_{ss}(T)$.

Proof: Let $D = \{v_1, v_2, \dots, v_n\} \subseteq V(T)$ set of all non-end vertices. Suppose $D' \subseteq D$ such that $< V(T) - D' >$ is totally disconnected with at least two vertices then D' is minimal strong split dominating set of T. For the double dominating set of block graph of a tree, we consider $E_1 = \{e_1, e_2, \dots, e_n\} \subseteq E(T)$, the non-end edges which form the set $C = \{c_1, c_2, \dots, c_n\} \subseteq V[B(T)]$ be the cut set, since each block in $B(T)$ is complete, $E_2 = \{e_1, e_2, \dots, e_m\} \subseteq \{E(T) - E\}$ are the block vertices in $B(T)$. Let $E_2' \subseteq E_2$ and $C_1 \subseteq C$ such that $\forall v \in V[B(G)] - \{E_2' \cup C_1\}$ is dominated by at two vertices of $\{E_2' \cup C_1\}$. Then $\{E_2' \cup C_1\}$ is double dominated set of $B(T)$. Thus $|D'| \leq |E_2' \cup C_1|$, which gives $\gamma_{ddb}(T) \geq \gamma_{ss}(T)$.

Theorem 2.2: For any tree T with maximum degree $\Delta(T)$, then $p - \Delta(T) \leq \gamma_{ddb}(T)$.

Proof: Let $A = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of T and $S = \{b_1, b_2, b_3, \dots, b_n\}$ be the set of vertices in $B(T)$ corresponding to the set A. Without loss of generality since $|A| = |S|$, let $M = \{b_1, b_2, b_3, \dots, b_i\}$ $1 \leq i \leq n$ be the set of cut vertices in $B(T)$ and $M' = \{b_1 b_2 b_3, \dots, b_j\} \subseteq V[B(T)]$. Suppose $D^d = \{b_1, b_2, b_3, \dots, b_k\} \subseteq S$ set of $B(T)$ such that $|N[b] \cap D^d| \geq 2$ $\forall v \in V[B(T)] - D^d$. Since for any tree T, there exists at least one vertex v, $\deg(v) = \Delta(T)$, $\Delta(T) < p$ which gives $p - \Delta(T) > 0$. It follows that $p - \Delta(T) \leq \gamma_{ddb}(T)$.

Theorem 2.3: For any connected (p, q) graph G, then $\gamma_{ssb}(G) \leq \gamma_{ddb}(G)$.

Proof: Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks in G and $M = \{b_1, b_2, \dots, b_n\}$ be the set of vertices which corresponds to the blocks of B in $B(G)$. Let $C = \{b_1, b_2, \dots, b_i\}$ be the set of cutvertices in $B(G)$. Since each block in $B(G)$ is complete and each cutvertex is incident with at least two blocks. Let $D = V[B(G)] - C$ and consider a set $D' \subset C$ such that $V[B(G)] - \{D \cup D'\} = F$ where $\forall$ $b_i \in F$ is an isolates. Hence $|F| = \gamma_{ssb}(G)$. Let $D^d = D \cup D'', D'' \subseteq C$ such that $\forall$ $v \in V[B(G)] - \{D \cup D''\}$ is dominated by at least two vertices of $\{D \cup D''\}$. Then $\{D \cup D''\}$ is double dominating set of $B(G)$. Thus $|D \cup D'| \leq |D \cup D''|$, which gives $\gamma_{ssb}(G) \leq \gamma_{ddb}(G)$.

Theorem 2.4: For any connected (p, q) graph G, then $\gamma_{ddb}(G) + \gamma_c(G) \geq p + \gamma(G) - \Delta(G)$.

Proof: Le G be a connected graph and $V = \{v_1, v_2, \dots, v_p\}$ be the set of vertices of G. Suppose that there exists a minimal set of vertices $V_1 = \{v_1, v_2, \dots, v_k\} \subseteq V(G)$ such that $N[v_i] = V(G)$ $\forall$ $v_i \in V_1, 1 \leq i \leq k$. Then V_1 forms a minimal dominating set of G. Further, if the subgraph $< V_1 >$ has exactly one component then V_1 is itself is a connected dominating set of G. Suppose V_1 has more than one component then attach the minimal set of vertices V_2 of $V(G) - V_1$ which are every in $u - w$ path $\forall$ $u, w \in V_1$ gives a single component $V_3 = V_1 \cup V_2$. Clearly V_3 form a minimal γ_c set of G. Let $B = \{B_1, B_2, \dots, B_n\}$ be the blocks of G and let $M = \{b_1, b_2, \dots, b_n\}$ be the vertices corresponding to the blocks of G. Let $M' = \{b_1, b_2, \dots, b_i\}$ be the set of cut vertices in $B(G)$ which are non-end blocks in G and $M'' = \{b_1, b_2, \dots, b_j\}$ be the set of all end vertices in $B(G)$. Let $D^d = H \cup M''$ where $H \subseteq M$ be the double dominating set of $B(G)$ such that $|N[b] \cap D^d| \geq 2$ $\forall$ $b \in V[B(G)] - D^d$. Since for any graph G, there exists at least one vertex $v \in V(G)$ with $\deg(v) = \Delta(G)$, it follows that $|D^d| \cup |V_3| \geq |V(G)| \cup |V_1| - \Delta(G)$. Hence $\gamma_{ddb}(G) + \gamma_c(G) \geq p + \gamma(G) - \Delta(G)$.

Theorem 2.5: If G is a graph with $\Delta(G) \geq k \geq 2$, then $\gamma_{ddb}(G) \geq \gamma(G) + \Delta(G) - 2$.

Proof: Let D^d be a minimum double dominating set in $B(G)$, let $u \in V[B(G)] - D^d$ and let $D^d = \{v_1, v_2, \dots, v_k\}$ be distinct vertices in D^d which dominates u. Since $\Delta(G) \geq k \geq 2$ and $V[B(G)] - D^d \neq \varphi$, D^d is a double dominating set each vertex in $V[B(G)] - D^d$dominated by at least one vertex in $D^d - \{v_2, \dots, v_k\}$. Therefore, since u dominates each vertex in $\{v_2, \dots, v_k\}$, we know that the set $D' = D^d - \{v_2, \dots, v_k\} \cup \{u\}$ is a dominating set in $B(G)$. Therefore $\gamma(G) \leq |D'| = \gamma_{ddb}(G) - (k - 1) + 1 = \gamma_{ddb}(G) - k + 2$, which gives $\gamma_{ddb}(G) \geq \gamma(G) + \Delta(G) - 2$.

3. UPPER BOUNDS FOR $\gamma_{ddb}(G)$.

Here we establish upper bounds for $\gamma_{ddb}(G)$ in terms of elements of G.

Theorem 3.1: For any connected (p,q) graph G with n number of blocks, then $\gamma_{ddb}(G) \le n$.

Proof: Let G be a graph with $p \ge 3$ vertices and $V = \{v_1, v_2, \dots, v_p\}$ be the set of vertices of G. Let $A = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of G and let $M = \{b_1, b_2, b_3, \dots, b_n\}$ be the corresponding vertices in $B(G)$ respectively. Let $M' = \{b_1, b_2, b_3, \dots, b_i\}$ $1 \le i < n$ be the set of cut vertices in $B(G)$ and $M'' = \{b_1, b_2, b_3, \dots, b_j\}$ $1 \le i < n$ be the block vertex set in $B(G)$. For the double dominating set of $B(G)$, we consider $D^d = K \cup H$, $K \subseteq M'$, $H \subseteq M''$ be the sets such that for any vertex $v \in V[B(G)] - \{K \cup H\}$ is dominated by least two vertices in $B(G)$. Then $\{K \cup H\}$ is double dominating set of $B(G)$.Thus $|D^d| = |K \cup H| \le |M|$. Since $M = M' \cup M''$. Hence $\gamma_{ddb}(G) \le n$.

Theorem 3.2: For any connected (p,q) graph G, then $\gamma_{ddb}(G) + \gamma_{ib}(G) \le p + 1$.

Proof: Let $A = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of G and let $M = \{b_1, b_2, b_3, \dots, b_n\}$ be the set of vertices corresponding to the blocks of A in $B(G)$. Let $M' = \{b_1, b_2, b_3, \dots, b_i\}$ $1 \le i < n$ be the set of cut vertices in $B(G)$ and $M'' = \{b_1, b_2, b_3, \dots, b_j\}$ $1 \le j < n$ be the block vertices in $B(G)$. Where $M = M' \cup M''$. Let $I = H_1 \cup H_2$, $H_1 \subset M'$ and $H_2 \subset M''$ be the independent dominating set such that the induced subgraph $< H_1 \cup H_2 >$ is disconnected and also every $b \in I$ is at a distance at least 2 a part from other vertices of I. Clearly $|H_1 \cup H_2| = |I| = \gamma_{ib}(G)$. Let $D^d = \{b_1, b_2, b_3, \dots, b_k\} \subseteq K \cup H$, $K \subseteq M'$ and $H \subseteq M''$ in $B(G)$, which covers all the vertices in $B(G)$ and for every vertex $v \in V[B(G)] - D^d$ is adjacent to at least two vertices of D^d. Clearly $H_1 \cup H_2 \subset K \cup H$. Thus $|H_1 \cup H_2| < |K \cup H| < p$. It follows that $|H_1 \cup H_2| + |K \cup H| < p + 1$. Hence $\gamma_{ddb}(G) + \gamma_{ib}(G) \le p + 1$.

Theorem 3.3: For any nontrivial tree T with $p \ge 3$ vertices and C cutvertices, then $\gamma_{ddb}(T) \le C + 1$.

Proof: Let $A = \{v_1, v_2, \dots, v_i\}$ be the set of all cut vertices in T with $|A| = C$. Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of T and $M = \{b_1, b_2, \dots, b_n\}$ be the corresponding block vertices of the set B in $B(T)$. Now let, $D^d = \{b_1, b_2, \dots, b_k\} \subseteq V[B(T)]$ in $B(T)$ be the minimal set of vertices which covers all the vertices in $B(T)$ such that for any vertex $v \in V[B(T)] - D^d$ is adjacent to at least two vertices of D^d, then D^d itself is a double dominating set of $B(T)$. Since any tree T contains at least one cutvertex, it follows that $|D^d| \le C + 1$. Hence $\gamma_{ddb}(G) \le C + 1$.

Theorem 3.4: For any connected (p,q) graph G, then $\gamma_{ddb}(G) \le diam(G)$.

Proof: Let any two vertices u and v belongs to $V(G)$ which constitutes the longest path in G.Then $dist(u,v) = diam(G)$. Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks in G and let $M = \{b_1, b_2, \dots, b_n\}$ be the set of vertices which corresponds to the blocks of B in $B(G)$. For the double dominating set of $B(G)$, we consider $D^d = \{b_1, b_2, \dots, b_k\} \subseteq V[B(G)]$. Suppose D^d covers all the vertices of $B(G)$ and $\forall v \in V[B(G)] - D^d$ is dominated by at least two vertices of D^d. Then D^d is double dominating set of $B(G)$. Since $|D^d| \ge 2$ and the diameteral path includes at least two vertices. It follows that $\gamma_{ddb}(G) \le diam(G)$.

Theorem 3.5: For any connected (p,q) graph G, with $p \ge 3$, then $\gamma_{ddb}(G) \le \gamma_t[B(G)] + \Delta(G)$.

Proof: Let $A = \{v_1, v_2, \dots, v_i\} \subseteq V(G)$ be the set of all vertices with degree $\ge 2, 1 \le i \le n$. Then there exists at least one vertex $v \in A$ of maximum degree $\Delta(G)$. Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of G and $M = \{b_1, b_2, \dots, b_n\}$ be the set of vertices which corresponds to the blocks of B in $B(G)$. Let $D = \{b_1, b_2, \dots, b_i\} \subseteq V[B(G)]$. Suppose that D covers all the vertices in $B(G)$ and if the subgraph $< D >$ has no isolated vertex, then D itself is a minimal total dominating set of $B(G)$. Now let $D^d = \{b_1, b_2, \dots, b_k\} \subseteq V[B(G)]$ be the minimal set of vertices which covers all the vertices in $B(G)$ and any vertex $v \in V[B(G)] - D^d$, is dominated by at least two vertices of D^d. Then D^d is double dominating set of $B(G)$. Thus $|D^d| \le |D| + \Delta(G)$, which gives $\gamma_{ddb}(G) \le \gamma_t[B(G)] + \Delta(G)$.

Theorem 3.6: For any tree T, then $\gamma_{ddb}(G) + \gamma(T) \le n(T) + \Delta(T)$.

Proof: Let $V = \{v_1, v_2, \dots, v_p\}$ be the set of vertices of T. Let $D = \{v_1, v_2, \dots, v_i\}, 1 \le i \le p$ be a minimal dominating set of T such that $|D| = \gamma(G)$. Now we consider $M = \{b_1, b_2, \dots, b_n\}$ be the set of vertices of $B(T)$ corresponding to the blocks $B = \{B_1, B_2, \dots, B_n\}$ of T. Since for any tree T, there exists at least one vertex v, $\deg(v) = \Delta(G)$. Let $M' = \{b_1, b_2, \dots, b_i\}, 1 \le i \le n$ such that $M' \subseteq M$ and $\forall\, b_i \in M'$ are the non-end block in T which gives cutvertices in $B(T)$ corresponding to end blocks in T and $M'' \subseteq M$. Now we consider $K \subseteq M$ and M''. Since $K \cup M'' \subseteq V[B(T)]$ then $\forall\, v \in V[B(T) - \{K \cup M''\}$ is dominated by at least two vertices of $\{K \cup M''\}$. Clearly $\{K \cup M''\}$ forms a double dominating set of $B(T)$. Therefore it follows that $|K \cup M''| + |D| \le |M''| + \Delta(T)$, which gives $\gamma_{ddb}(G) + \gamma(T) \le n(T) + \Delta(T)$.

Theorem 3.7: If every non-end vertex of a tree T is adjacent to at least one end vertex, then $\gamma_{ddb}(T) \le 2p - 2m(T)$, where $m(T)$ is the number of end vertices in T.

Proof: Let $F = \{v_1, v_2, \dots, v_m\} \subseteq V(T)$ be the set of all end vertices with $|F| = m$. Let $B = \{B_1, B_2, \dots, B_n\}$ be the blocks of T and $M = \{b_1, b_2, \dots, b_n\}$ be the block vertices in $B(T)$. Let $M^{'} = \{b_1, b_2, \dots, b_i\} \subseteq V[B(T)]$ be the cut set and $M^{''} \subseteq V[B(T)] - M^{'}$ be the set of end block vertices. Since $\{B\} = V[B(T)]$. Let $D^d = K \cup M^{''} = \{b_1, b_2, \dots, b_k\} \subseteq V[B(T)], K \subseteq M$ be the minimal set of vertices which covers all the vertices in $B(T)$ such that if any vertex $v \in V[B(T)] - D^d$ there exists at least two vertices of $\{b_i, b_j\} \in D^d$ which are adjacent to at least one vertex of D and at least two vertices of $V[B(T)] - D^d$. Therefore D^d forms a double dominating set of $B(T)$. If $M^{''} = \varphi$ then $D^d = \{b_1, b_2, \dots, b_i\} \subseteq K \subseteq M$ forms γ_{ddb}-set in $B(T)$. If $M^{''} \neq \varphi$ then $D^d = K \cup M^{'}$ forms γ_{ddb}-set in $B(T)$. Hence in all cases $\left|K \cup M^{'}\right| \leq 2|V(T)| - 2\left|M^{''}\right|$ which gives $\gamma_{ddb}(T) \leq 2p - 2m(T)$.

Theorem 3.8: If v be an end vertex of a connected block graph $B(G)$, then v is in every γ_{ddb} set of $B(G)$.

Proof: Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of G and $M = \{b_1, b_2, \dots, b_n\}$ be the corresponding block vertices in $B(G)$. Suppose D^d be a minimum block double dominating set of $B(G)$. Assume that there exists an end vertex $v \in V[B(G)] - D^d$. Then v should be adjacent to at least one vertex of $V[B(G)]-D^d$ and at least one vertex of D^d this implies that $|N(v)| > 1$, which gives a contradiction. Hence $v \notin V[B(G)] - D^d$ and v is in every γ_{ddb}-set of $B(G)$.

Theorem 3.9: If D^d is a γ_{ddb} set of a graph G, then every vertex in $V[B(G)] - D^d$ is dominated by at least two vertices in D^d.

Proof: Let D^d be a minimum double dominating set in $B(G)$ and assume that every vertex in $B-D^d$ is dominated by three or more vertices. Le $u \in V[B(G)] - D^d$ and let v and w be two vertices in D^d which dominate u. It follows from our assumption that every vertex in $V[B(G)] - D^d$ is dominated by at least one vertex in $D^d - \{v, w\}$. Therefore the set $D^{'} = D^d - \{v, w\} \cup \{u\}$ is a dominating set. But since $\left|D^{'}\right| < |D^d|$, which is contradiction to the assumption that D^d is a minimum dominating set.

Theorem 3.10: Let D^d is a γ_{ddb} set of P_n with n number of blocks, then $\gamma_{ddb}(P_n) \leq \frac{2(n+1)}{3}$.

Proof: Let D^d be a double dominating set of $B(P_n)$. For every vertex v of degree 2, either v or its two neighbours are in D^d. So $V[B(G)] - D^d$ is an independent set, by definition of double dominating set every vertex of D^d has exactly one neighbour in $V[B(G)] - D^d$. Thus $|D^d| - 2 = 2|V[B(P_n)] - D^d| = 3|V[B(P_n)] - D^d| + 2$. Hence $\gamma_{ddb}(P_n) \leq \frac{2(n+1)}{3}$.

Theorem 3.11: For any connected (p, q) graph G, then $\gamma_{ddb}(G) + \gamma(G) \leq p + \gamma_c(G) - 1$.

Proof: Let G be a connected graph with $V = \{v_1, v_2, \dots, v_p\}$, the set of vertices of G. Suppose $V_1 = \{v_1, v_2, \dots, v_i\} \subseteq V(G)$ be the set of all non end vertices in G and assume there exists a minimal set of vertices $V_2 = \{v_1, v_2, \dots, v_j\} \subseteq V_1$ such that $N[v_k] = V_1(G), \forall\, v_k \in V_2, 1 \leq k \leq n$. Then V_2 forms a minimal dominating set of G. Suppose V_2 has more than one component then attached the minimal set of vertices V_2 of $V_1 - V_2$, which are in every $u - w$ path $\forall\, u, w \in V_2$ gives a single component $V_3 = V_2 \cup V_1$. Clearly V_3, forms a minimal γ_c set of G. Let $B = \{B_1, B_2, \dots, B_n\}$ be the set of blocks of G and $M = \{b_1, b_2, \dots, b_n\}$ be the corresponding block vertices in $B(G)$ Let $M^{'} = \{b_1, b_2, \dots, b_i\} \subseteq V[B(T)]$ be the cut set and $M^{''} \subseteq V[B(G)] - M^{'}$ be the set of end block vertices. Let $D^d = K \cup M^{''} = \{b_1, b_2, \dots, b_k\} \subseteq V[B(T)],\ K \subseteq M$ be the minimal set of vertices which covers all the vertices in $B(T)$ such that $|N[b] \cap D^d| \geq 2\ \forall\, b \in V[B(G)] - D^d$. It follows that $|D^d| \cup |V_1| \leq |V| \cup |V_3| - 1$. Hence $\gamma_{ddb}(G) + \gamma(G) \leq p + \gamma_c(G) - 1$.

Theorem 3.12: For any(p, q) graph G with C be the number of cut vertices, then $\gamma_{ddb}(G) \leq \gamma(G) + \gamma^{'}(G) + \left\lfloor\frac{C}{2}\right\rfloor$

Proof: Let be G a connected graph with $p \geq 3$ vertices of G and $V = \{v_1, v_2, \dots, v_p\}$ be the set of vertices of G. Let $B = \{B_1, B_2, \dots, B_n\}$ be the blocks of G and $M = \{b_1, b_2, \dots, b_n\}$ be the block vertices in $B(T)$. Let $M^{'} = \{b_1, b_2, \dots, b_i\} \subseteq V[B(G)]$ be the cut set and $M^{''} \subseteq V[B(G)] - M^{'}$ be the set of end block vertices. Since $\{B\} = V[B(G)]$. $D = \{v_1, v_2, \dots, v_m\}$ where $m < p$ be a dominating set of G such that $\gamma(G) = |D|$. Let F be minimal edge dominating set of G. Suppose $E - F$ is not an edge dominating set. Then there exists an edge f such that $f \in F$ is adjacent to any edge in $E - F$. Since G has no isolated edges then f is dominated by at least one edge in $F - \{f\}$. Thus $F - \{f\}$ is an edge dominating set, a contradiction to the minimality of F. Therefore F is edge dominating set, such that $|F| = \gamma^{'}(G)$. Let $D^d = K \cup M^{''}$ where $K \subseteq M$ be the double dominating set of $B(G)$ such that $|N[b] \cap D^d| \geq 2\ \forall\, b \in V[B(G)]-D^d$ and which covers all the vertices in $B(G)$. Then by the definition of $B(G)$ which gives $\left|K \cup M^{''}\right| \leq |D| + |F| + \left\lfloor\frac{C}{2}\right\rfloor$. Hence $\gamma_{ddb}(G) \leq \gamma(G) + \gamma^{'}(G) + \left\lfloor\frac{C}{2}\right\rfloor$.

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