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GEOMETRICAL INTERPRETATION OF AN ANALYTIC HP-TRANSFORMATIONS IN ALMOST KAEHLERIAN SPACES

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ABSTRACT

In this paper, we have defined and studied on geometrical interpretation of an analytic holomorphically projective transformations in almost Kaehlerian spaces and several theorems have been obtained.

Key Words: *Kaehlerian space, H- Projective, Recurrent, Symmetric, Transformation.*

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1. INTRODUCTION

An almost Kaehlerian space is first of all an almost complex space, that is, a 2n-dimensional space with an almost complex structure $F^h{}_i$:

$$F^i{}_j\, F^h{}_i = -\delta^h{}_j, \tag{1.1}$$

And always admits a positive definite Riemannian metric tensor g_{ji} satisfying:

$$F^a{}_j\, F^b{}_i\, g_{ab} = g_{ji}, \tag{1.2}$$

From which

$$F_{ji} = -F_{ij}, \tag{1.3}$$

Where

$$F_{ji} \overset{\text{def}}{=} F^a{}_j\, g_{ai} \tag{1.4}$$

And finally has the property that the differential form $F_{ji}\, d_\xi{}^j \wedge d_\xi{}^i$ is closed, that is,

$$F_{jih} \overset{\text{def}}{==} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0$$

And finally has the property that the skew-symmetric F_{ih} is a Killing tensor

$$\nabla_j F_{ih} + \nabla_i F_{hj} = 0 \tag{1.5}$$

From which

$$\nabla j F^j{}_i + \nabla_i F^h{}_j = 0 \tag{1.6}$$

And $$F_i = -\nabla_j F^j{}_i = 0 \tag{1.7}$$

Here ∇ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{{}_j{}^h{}_i\}$.

The Nijenhuis tensor $N^h{}_{ji}$ is written in the form:

$$N^h{}_{ji} = -4\,(\nabla_j F^t{}_i)\, F^h{}_t + 2G^t{}_{ji}\, F^h{}_i + F^t{}_j\, G^h{}_{ti} - F^t{}_i\, G^h{}_{tj}\,. \tag{1.8}$$

A contravariant almost analytic vector field is defined as a vector field v^i, satisfying Tachibana (1959):

$$£_v F^h{}_i \equiv v^j \partial_j F^h{}_i - F^j{}_i\, \partial_j v^h + F^h{}_j\, \partial_i v^j = 0,$$

Where $£_v$ stands for the Lie-derivative with respect to v^i.

Let $R^h{}_{kji}$ be the Riemannian curvature tensor and put

$$R_{ji} = R^r{}_{rji}\,,\ R_{kjih} = R^r{}_{kji}\, g_{rh}\,,\ R = R_{ji}\, g^{ji} \ \text{ and }\ S_{ji} = F^r{}_j\, R_{ri},$$

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Then the following identities are satisfied (Yano 1957)

$$R^r{}_{kji}\, F^h{}_r = R^h{}_{kjr}\, F^r{}_i, \quad R_{kjir}\, F^r{}_h = R_{kjhr}\, F^r{}_i \tag{1.9}$$

$$R_{kjih} = R_{kjtr}\, F^t{}_i\, F^r{}_h, \; R_{ji} = R_{tr}\, F^t{}_j F^r{}_i \tag{1.10}$$

$$S_{ji} + S_{ij} = 0, \; S_{ji} = S_{tr}\, F^t{}_j\, F^r{}_i, \; S_{ji} = -\frac{1}{2}\, F^{tr} R_{trji}. \tag{1.11}$$

The holomorphically projective curvature tensor $P^h{}_{kji}$, which will be briefly called HP-curvature tensor, is given by

$$P^h{}_{kji} = R^h{}_{kji} + \frac{1}{n+2}\left(R_{ki}\, \delta^h{}_j - R_{ji}\, \delta^h{}_k + S_{ki}\, F^h{}_j - S_{ji}\, F^h{}_k + 2\, S_{kj}\, F^h{}_i\right) \tag{1.12}$$

We can obtain the following identities

$$P^h{}_{(kj)j} = 0, \; P^h{}_{[kji]} = 0, \tag{1.13}$$

$$P^r{}_{rji} = 0, \tag{1.14}$$

$$P^r{}_{kji}\, F^h{}_r = P^h{}_{kjr}\, F^r{}_i, \; P^h{}_{rji}\, F^r{}_k == P^h{}_{rki}\, F^r{}_j \tag{1.15}$$

From which, we have

$$P^r{}_{kjr} = 0, \tag{1.16}$$

$$P^t{}_{rji}\, F^r{}_t = 0, \; P^t{}_{kjr}\, F^r{}_t = 0. \tag{1.17}$$

A necessary and sufficient condition for $P^h{}_{kji} = 0$, is that the space is a space of constant holomorphically curvature (Tashiro 1957), i.e., a space whose curvature tensor $R^h{}_{kji}$ takes the form

$$R^h{}_{kji} = -\frac{R}{n(n+2)}\left(g_{ki}\, \delta^h{}_i - g_{ji}\, \delta^h{}_k + F_{ki}\, F^h{}_j - F_{ji}\, F^h{}_k + 2\, F_{kj}\, F^h{}_j\right) \tag{1.18}$$

For a vector field V^i and a tensor field $\alpha^h{}_i$, the following identities are known (Yano 1957)

$$£_v\, \nabla_j\, \alpha^h{}_i - \nabla_j\, £_v\, \alpha^h{}_i = \alpha^r{}_i\; £_v\, \{{}_j{}^h{}_r\} - \alpha^h{}_r\; £_v\, \{{}_j{}^r{}_i\} \tag{1.19}$$

$$\nabla_k\, £_v\, \{{}_j{}^h{}_i\} - \nabla_j\, £_v\, \{{}_k{}^h{}_i\} = £_v\, R^h{}_{kji} \tag{1.20}$$

Where $£_v$ denotes the operator of Lie-differentiation with respect to V^i.

A Killing vector or an infinitesimal isometry V^i is defined by

$$£_v\, g_{ji} = \nabla_j\, V_i + \nabla_i\, V_j = 0.$$

Here we shall identify a contravariant vectors V^i with a covariant vector $V_i = g_{ir}\, V^r$. Hence we shall say V_i is a Killing vector, or that ρ^i is gradient, for example.

An infinitesimal affine transformation V^i is defined by

$$£_v\, \{{}_j{}^h{}_i\} = \nabla_j\, \nabla_i\, V^h + R^h{}_{rji}\, V^r = 0.$$

We shall say a vector field V^i an infinitesimal holomorphically projective transformation or, for simplicity, an HP-transformation, if it satisfies

$$£_v\, \{{}_j{}^h{}_i\} = \rho_j \delta^h{}_i + \rho_i\, \delta^h{}_j - \overline{\rho_j}\, F^h{}_i - \overline{\rho_i}\, F^h{}_j,$$

Where ρ_i is a certain vector and $\overline{\rho_i} = F^r{}_i\, \rho_r$. In this case, we shall called ρ_i the associated vector of the transformation, If ρ_i vanishes, then the HP-transformation reduces to an affine one.

Contracting the last equation with respect to h and i, we get

$$\nabla_j\, \nabla_r\, V^r = (n+2)\, \rho_j\,,$$

Which shows that the associated vector is gradient.

A vector field V^i is called Contravariant analytic or, for simplicity, analytic, if it satisfies

$$£_v\, F^h{}_i \equiv -F^r{}_i\, \nabla_r\, V^h + F^h{}_r\, \nabla_i\, V^r = 0.$$

2. GEOMETRICAL INTERPRETATION OF AN ANALYTIC HP-TRANSFORMATION

In a differentiable space M, we consider a tensor valued function V depending not only on a point P of M but also on k vectors $u_1, u_2, \ldots, u_k$ at the point and denote it by $V(P, u_1, u_2, \ldots, u_k)$. We assume that the value of this function V lies in the tensor space associated to the tangent space of M at P and that it depends differentially on its arguments.

Assuming the space M to be affinely connected, we take an arbitrary curve C: $x^i = x^i\,(t)$ and denote its successive derivatives by

$$\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \; \frac{d^3x^i}{dt^3} \tag{2.1}$$

Then if we substitute (2.1) into the function V instead of $u_1, u_2, \ldots, u_k$. We have a family of tensors

$$V(C) = V\left(\dot{x}\,,\; \frac{dx}{dt}\;, \ldots\ldots,\; \frac{d^k x}{dt^k}\right)$$

along the curve C.

Let V^i be an infinitesimal transformation, i.e., a vector field, and $'x^i = x^i + \varepsilon v_i$ be the infinitesimal point transformation determined by v^i, ε being an infinitesimal constant. Given a curve C: $x^i = x^i(t)$, the image $'C$ of F is expressed by

$$x^i = x^i(t) + \varepsilon V^i(x(t)).$$

We shall call the limiting value

$$£_v V(C) \equiv \lim_{\varepsilon\to 0} \frac{V('C) - 'V(C)}{\varepsilon}$$

The Lie-derivative of V(C) with respect to V^i, where we have denoted by $'V(C)$ the family of tensors induced from V(C) by the transformation

$$'x^k = x^i + \varepsilon V^i$$

In a Almost Kaehlerian space, a curve $x^i = x^i(t)$ defined by

$$\frac{d^2x^h}{dt^2} + \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha \frac{dx^h}{dt} + \beta F_j^h \frac{dx^j}{dt} \tag{2.2}$$

is, by definition, a holomorphically planar curve, or an H-plane curve, where α and β are certain functions of t.

Let V^i be an infinitesimal transformation and assume that any ε the infinitesimal point transformation $'x^i = x^i\, \varepsilon\, V^i$ maps any H-plane curves.

Now we ask for the condition that V^i preserve that H-plane curves. For such a vector V^i taking account of (2.2), we have

$$£_v\Big[\frac{d^2x^h}{dt^2} + \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} \frac{dx^j}{dt}\frac{dx^i}{dt} - \alpha \frac{dx^h}{dt} - \beta F_j^h \frac{dx^j}{dt}\Big] = \gamma \frac{dx^h}{dt} + \delta F_j^h \frac{dx^j}{dt} \tag{2.3}$$

along any H-plane curve, where γ and δ are certain functions of t.

Denoting the Lie-derivative of the Christoffel's symbols and the complex structure F^h_i, respectively, by

$$t^h_{ji} = £_v \begin{Bmatrix} h \\ j \quad i \end{Bmatrix}, \quad \alpha^h_i = £_v F^h_i,$$

We have from (2.3)

$$t^h_{ji}\, \dot{x}^j \dot{x}^i + \mathrm{a}\, \dot{x}^h + \mathrm{b}\, F^h_j \dot{x}^j - \beta\, \alpha^h_j \dot{x}^j = 0 \tag{2.4}$$

Where we have put

$$\mathrm{a} = -(\gamma + £_v \alpha), \mathrm{b} = -(\delta + £_v \beta), \dot{x} = \frac{dx^i}{dt}$$

Since the relation (2.4) holds for any H-plane curve C, it must hold identically for any values of x^i and $\dot{x}^i$.

By means of the definition of the H-plane curve, we see further that the identity (2.4) holds for any value of the coefficient β.

Taking account of these arguments, we can easily see that relation

$$\alpha^h_j \dot{x}^j = f\, \dot{x}^h + g\, F^h_j \dot{x}^j, \tag{2.5}$$

$$t^h_{ji} \dot{x}^j \dot{x}^i = p\, \dot{x}^h + q\, F^h_j \dot{x}^j, \tag{2.6}$$

hold for any values x^i and $\dot{x}^i$, where f,g,p and q are certain functions of x^i and $\dot{x}^i$.

Let α^i_j be a tensor on V such that $F^r_j \alpha^i_r + \alpha^r_j F^i_r = 0$, we obtain by means of (2.5)

$$\alpha^h_i \equiv £_v F^h_i = 0. \tag{2.7}$$

On the other hand, If we substitute (2.7) and $\nabla_j F^i_j = 0$ into the identify

$$\nabla_j £_V F^h_i - £_V \nabla_j F^h_i = F^h_r £_V \{{}_j{}^r{}_i\} - F^r_i £_V \{{}_j{}^h{}_r\},$$

Then we get

$$t^r_{ji} F^h_r = t^h_{jr} F^r_i. \tag{2.8}$$

From (2.6) and (2.8), taking account of the fact that

$$t^h_{ji} = \alpha_j \delta^h_i + \alpha_i \delta^h_j - \overline{\alpha}_j F^j_i - \overline{\alpha}_i F^h_j,$$

Where α_i is certain vector and $\overline{\alpha}_I = F^r_i \alpha_r$, we get

$$t^h_{ji} = £_V\{{}_j{}^h{}_i\} = \rho_j \delta^h_i + \rho_i \delta^h_j - \overline{\rho}_j F^h_i - \overline{\rho}_i F^h_j, \tag{2.9}$$

Where ρ_i is a certain vector field. Therefore, the infinitesimal transformation V^i is an analytic HP-transformation.

Conversely, it is obvious that an analytic HP-transformation preserves the H-plane curves.

Thus we have the following:

Theorem 2.1: In an almost Kaehlerian space, an infinitesimal transformation preserves the H-plane curves, if and only if it is an analytic HP-transformation.

3. SOME PROPERTIES OF HP-TRANSFORMATIONS.

Let V^i be an HP-transformation, then it holds

$$£_V\{{}_j{}^h{}_i\} \equiv \nabla_j\nabla_i V^h + R^h_{rji}V^r = \rho_j\delta^h_i - \rho_i\delta^h_j - \bar{\rho}_j F^h_i - \bar{\rho}_i F^h_j. \quad (3.1)$$

Transvecting (3.1) with g^{ji}, we have

$$\nabla^r\nabla_r V'^i + R^h_r V^r = 0. \quad (3.2)$$

Hence, by virtue of the well known theorem on an analytic vectors, Yano (1957), Lichnerowiez (1957), we have the following:

Theorem 3.1: In a compact almost Kaehlerian space an HP-transformation is analytic.

In a compact almost Kaehlerian space, M, it holds that

$$\int_M (R_{ji}V^jV^i)d\sigma \geqq 0$$

For an analytic vector V^i, where $\boldsymbol{d\sigma}$ denote the volume element of M and the equality holds when and when only V^i is parallel. Therefore, if the Ricci's from $R_{ji}\xi^j\xi^i$ is negative definite, then there exists no non-trivial HP-transformation provided that the space is compact.

Taking account of the identity (1.19), we have for a vector field V^i

$$£_v\nabla_j F^h_i - \nabla_j £_v F^h_i = F^r_i £_v\{{}_j{}^h{}_r\} - F^h_r £_v\{{}_j{}^r{}_i\},$$

Which implies

$$\nabla_j £_v F^h_i = F^h_r £_v\{{}_j{}^r{}_i\} - F^r_i £_v\{{}_j{}^h{}_r\},$$

Because of $\nabla_j F^h_i = 0$. If the vector field V^i is an HP-transformation, it is easily verified that the right hand-side of the last equation vanishes. Thus we have the following theorems by the virtue of Obata's theorem, gives Obata (1956)

Theorem 3.2: In an irreducible almost Kaehlerian space admitting no quaternion structure, any HP-transformation is analytic.

Theorem 3.3: In an irreducible almost Kaehlerian space having non-vanishing Ricci tensor any HP-transformation is analytic.

Theorem 3.4: In an irreducible almost Kaehlerian Einstein space if its scalar curvature is non-vanishing, any HP-transformation is analytic.

Now, we shall find some formulae on analytic HP-transformation which will be useful in the further study.

Let V^i be an HP-transformation. Substituting (3.1) into the identity

$$£_V g_{ji} - £_V\nabla_k g_{ji} = g_{ri}£_V\{{}_k{}^r{}_j\} + g_{jr}£_V\{{}_k{}^r{}_i\},$$

We find

$$\nabla_k £_V g_{ji} = \rho_j g_{ki} + \rho_i g_{kj} - \bar{\rho}_j F_{ki} - \bar{\rho}_i F_{kj} + 2\rho_k g_{ji}. \quad (3.3)$$

If we substitute (3.1) into (1.20), then we have

$$£_V R^h_{kji} = \delta^h_j \nabla_k \rho_i - \delta^h_k \nabla_j \rho_i - F^h_j \nabla_k \bar{\rho}_i + F^h_k \nabla_j \bar{\rho}_i - (\nabla_k \bar{\rho}_j - \nabla_j \bar{\rho}_k) F^h_i, \quad (3.4)$$

Contracting the last equation with respect to h and k we find

$$£_V R_{ji} = -n\nabla_j\rho_i - 2F^r_j F^t_i \nabla_r \rho_t \quad (3.5)$$

Now we shall assume that V^i is an analytic HP-transformation. Then we have $£_V R_{ji} = £_V(R_{rt}F^r_j F^t_i)$

By virtue of (2.1). Hence from (3.5) it follows

$$\nabla_j \rho_i = F^r_j F^t_i \nabla_r \rho_t. \quad (3.6)$$

Since n > 2. The last equation also is written in the form:

$$£_V F^h_i \equiv -F^r_i \nabla_r \rho^h + F^h_r \nabla_i \rho^r = 0,$$

Which shows that ρ^i is analytic. Moreover, according to (3.6) we have

$$\nabla_j \bar{\rho}_i + \nabla_i \bar{\rho}_j = F_i^r (\nabla_j \rho_r - F_j^t F_r^s \nabla_t \rho_s) = 0, \tag{3.7}$$

Which means that $\bar{\rho}^i$ is a Killing vector. Thus we get the following:

Theorem 3.5: If a vector ρ_i is the associated vector of an analytic HP-transformation, then ρ^i is analytic and $\bar{\rho}^i$ is a Killing vector.

Now, from (3.5) and (3.6) it follows

$$£_V R_{ji} = -(n+2) \nabla_j \rho_i \,, \tag{3.8}$$

From which we have

$$£_V S_{ji} = (n+2) \nabla_j \bar{\rho}_i \,. \tag{3.9}$$

On the other hand, from (3.4) and (3.7) we get

$$£_V R_{kji}^h = \delta_j^h \nabla_k \rho_i - \delta_k^h \nabla_j \rho_i - F_j^h \nabla_k \bar{\rho}_I + F_k^h \nabla_j \rho_i - 2F_i^h \nabla_k \bar{\rho}_j. \tag{3.10}$$

If we substitute (3.8) and (3.9) into (3.10). Then we can verify Ishihara (1957)

$$£_V P_{kji}^h = 0. \tag{3.11}$$

In the next place, substitute (3.1) and (3.8) into the identify

$$£_V \nabla_k R_{ji} - \nabla_k £_V R_{ji} = -R_{rt}\, £_V \{ {}_k{}^r{}_j \} - R_{jr}\, £_V \{ {}_k{}^r{}_i \},$$

We have

$$\nabla_k R_{ji} = -(n+2) \nabla_k \nabla_j \rho_i - R_{ki} \rho_j - R_{kj} \rho_i + S_{ki} \bar{\rho}_j + S_{kj} \bar{\rho}_i - 2R_{ji} \rho_k. \tag{3.12}$$

Hence we put

$$P_{kji} = \frac{1}{n+2} (\nabla_k R_{ji} - \nabla_j R_{ki}). \tag{3.13}$$

It holds

$$£_V P_{kji} = P_{kji}^r \ \rho_r. \tag{3.14}$$

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