International Journal of Mathematical Archive-9(1), 2018, 86-89
Available online through www.ijma.info ISSN 2229 – 5046

OSCILLATION OF THIRD ORDER DELAY DIFFERENTIAL EQUATION WITH SUBLINEAR NEUTRAL TERM

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(Received On: 07-12-17; Revised & Accepted On: 18-12-17)

ABSTRACT

The aim of this paper is to study the oscillatory and asymptotic behaviour of solutions of the equation

$$\left(a(t)\left(b(t)\left(x(t)+p(t)x^{\alpha}\big(\sigma(t)\big)\right)'\right)'\right)'+q(t)x\big(\tau(t)\big)=0.$$

The obtained results generalize and complement to the existing results in the literature. An example is given to illustrate the main result.

2010 Mathematics Subject Classification: *34C10, 34K11.*

Key words: *Third order neutral differential equation, sublinear neutral term, oscillation, nonoscillation.*

1. INTRODUCTION

In this paper, we are concern with the oscillation and asymptotic behaviour of the solutions of the third order neutral differential equation of the form

$$\left(a(t)\left(b(t)\left(x(t)+p(t)x^{\alpha}\big(\sigma(t)\big)\right)'\right)'\right)'+q(t)x\big(\tau(t)\big)=0, t\geq t_0 \tag{1}$$

subject to the following conditions:
(H$_1$) $a(t), b(t), p(t), q(t)\in C([t_0,\infty),\mathbb{R})$ with $a(t)>0, b(t)>0, q(t)>0$ and $0\leq p(t)\leq p<1$ for all $t\geq t_0$.
(H$_2$) $\tau(t),\sigma(t)\in C([t_0,\infty),\mathbb{R})$ such that $\tau(t)\leq t,\sigma(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$.
(H$_3$) α is a ratio of odd positive integers such that $0<\alpha\leq 1$.

Let $z(t)=x(t)+p(t)x\big(\sigma(t)\big)$. By a solution of equation (1), we mean a nontrivial function $x(t)\in C([T_x,\infty),\mathbb{R})$, $T_x\geq t_0$, with $z(t)\in C^1([T_x,\infty),\mathbb{R})$, $b(t)z'(t)\in C^1([T_x,\infty),\mathbb{R})$, $a(t)\big(b(t)z'(t)\big)'\in C^1([T_x,\infty),\mathbb{R})$ and satisfies equation (1) on $[T_x,\infty)$. We assume that equation (1) possesses such solutions. A solution of equation (1) is called oscillatory if it has infinitely many zeros on $[T_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1) is said to be almost oscillatory if all its solutions are either oscillatory or convergent to zero asymptotically.

In recent years there has been great interest in studying the oscillatory and asymptotic behaviour of solutions of neutral type differential equations since such type of equations have many applications in science and technology, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

In [10], the authors investigate the oscillatory behaviour of solutions of equation (1) when $\alpha=1$ and

$$\int_{t_0}^{\infty}\frac{1}{a(t)}\,dt=\infty,\quad \int_{t_0}^{\infty}\frac{1}{b(t)}\,dt=\infty. \tag{2}$$

Motivated by this observation, in this paper we further investigate the oscillatory behaviour of solutions of equation (1).

Due to the assumptions and the form of the equation (1), it is enough to deal with only positive solutions of equation (1) since the proof for the negative case is similar. In the following, all functional inequalities considered are assumed to hold eventually, that is, they are satisfied for all t large enough.

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2. MAIN RESULTS

In this section, we state and prove the main results. We begin with following lemma.

Lemma 1: Let $x(t)$ be a positive solution of equation (1). Then the corresponding function $z(t)$ satisfies one of the following two cases:
(I) $z(t) > 0, z'(t) > 0, \left(b(t)z'(t)\right)' > 0, (a(t)(b(t)z'(t))')' < 0;$
(II) $z(t) > 0, z'(t) < 0, \left(b(t)z'(t)\right)' > 0, (a(t)(b(t)z'(t))')' < 0;$
for $t \geq t_1$, where t_1 is sufficiently large.

Proof: The proof is similar to that of Lemma 1 of [3], and hence it is omitted.

Lemma 2: Let $x(t)$ be a positive solution of equation (1) and the function $z(t)$ satisfies case (II) of Lemma (1). If
$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{v}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s) ds du dv = \infty, \tag{3}$$
then $lim_{t\to\infty} x(t) = 0$.

Proof: Let $x(t)$ be a positive solution of equation (1) and the function $z(t)$ satisfies the case (II) of Lemma 1. Therefore $\lim_{t\to\infty} z(t)$ exists and finite. Assume that $\lim_{t\to\infty} z(t) = l > 0$. From the definition of $z(t)$, we have $x(t) \leq z(t)$ and hence
$$\lim_{t\to\infty} x(t) \leq \lim_{t\to\infty} z(t) = l.$$

Therefore there exists a $0 \leq l_1 \leq l$ such that $\lim_{t\to\infty} x(t) = l_1$. Next we claim that $l_1 = 0$. Let us assume that $l_1 > 0$, then $l_1 < x(t) < l_1 + \epsilon$ for $\epsilon > 0$
and t large enough. From equation (1), we have
$$\left(a(t)\left(b(t)z'(t)\right)'\right)' \leq -l_1 q(t), t \geq t_1.$$
The remaining part of the proof is similar to that of Lemma 2 of [3], and hence it is omitted.

Lemma 3: If $a > 0, b > 0$ and $0 < \alpha \leq 1$, then
$$a^{\alpha} b^{1-\alpha} \leq \alpha\, a + (1-\alpha) b.$$

Proof: The proof can be found in [9].

For convenience, we use that following notations without further mention:
$$A(t) = \int_{t_0}^{t} \frac{1}{a(s)} ds$$
$$B(t) = \int_{t_0}^{t} \frac{A(s)}{b(s)} ds$$
$$Q(t) = q(t)\left(1 - \alpha\, p\left(\tau(t)\right) - \frac{1-\alpha}{M} p\left(\tau(t)\right)\right) > 0,$$
for all constants $M > 0$.

Theorem 4: Let conditions $(H_1) - (H_3)$, (2) and (3) hold. If $\rho(t)$ is a positive, nondecreasing and differentiable function such that
$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left[\frac{\rho(s)Q(s)B\left(\tau(s)\right)}{A(s)} - \frac{a(s)\left(\rho'(s)\right)^2}{4\rho(s)}\right] ds = \infty \tag{4}$$
for all $t \geq t_1$, then equation (1) is almost oscillatory.

Proof: Let $x(t)$ be a positive solution of equation (1) such that $x(t) > 0, x(\sigma(t)) > 0$ and $x(\tau(t)) > 0$ for all $t \geq t_1 \geq t_0$, where t_1 is chosen so that $z(t)$ satisfies both the cases of Lemma 1.

Case-(I): From the property of the $z(t)$, we have
$$x(t) = z(t) - p(t)x^{\alpha}\left(\sigma(t)\right) \geq z(t) - p(t)z^{\alpha}(t).$$
By using Lemma 3 with $b = 1$, we obtain
$$\begin{aligned} x(t) &\geq z(t) - p(t)\left(\alpha\, z(t) + (1-\alpha)\right) \\ &\geq \left(1 - \alpha\, p(t) - \frac{(1-\alpha)p(t)}{M}\right) z(t) \end{aligned} \tag{5}$$
where we have used $z(t) \geq M > 0$ for all $t \geq t_1$.

Using (5) in equation (1), we get

$$\left(a(t)\left(b(t)z'(t)\right)'\right)' + Q(t)z\left(\tau(t)\right) \le 0, t \ge t_1. \tag{6}$$

Define

$$w(t) = \rho(t)\frac{a(t)(b(t)z'(t))'}{b(t)z'(t)}, t \ge t_1. \tag{7}$$

Then $w(t) > 0$ and

$$w'(t) = \rho'(t)\frac{a(t)\left(b(t)z'(t)\right)'}{b(t)z'(t)} + \rho(t)\frac{\left(a(t)\left(b(t)z'(t)\right)'\right)'}{b(t)z'(t)} - \rho(t)\frac{a(t)\left(b(t)z'(t)\right)'}{\left(b(t)z'(t)\right)^2}\left(b(t)z'(t)\right)'$$

$$\le \frac{\rho'^{(t)}}{\rho(t)}w(t) - \frac{w^2(t)}{\rho(t)a(t)} - \frac{\rho(t)Q(t)z\left(\tau(t)\right)}{b(t)z'(t)}, \tag{8}$$

where we have used (6) and (7).

Now

$$b(t)z'(t) \ge \int_{t_1}^{t} \frac{a(s)\left(b(s)z'(s)\right)'}{a(s)}\, ds$$

$$\ge a(t)(b(t)z'(t))'A(t)$$

and hence

$$\left(\frac{b(t)z'(t)}{A(t)}\right)' \le 0.$$

That is $\frac{b(t)z'(t)}{A(t)}$ is decreasing for all $t \ge t_1$.

Further

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{b(s)z'(s)A(s)}{A(s)b(s)} ds$$

$$\ge \frac{b(t)z'(t)}{A(t)}B(t)$$

or

$$z\left(\tau(t)\right) \ge B\left(\tau(t)\right)\frac{b\left(\tau(t)\right)z'\left(\tau(t)\right)}{A\left(\tau(t)\right)}$$

$$\ge B\left(\tau(t)\right)\frac{b(t)z'(t)}{A(t)} \tag{9}$$

where we have used $\frac{b(t)z'(t)}{A(t)}$ is decreasing. Using (9) in (8), we obtain

$$w'(t) \le -\frac{\rho(t)Q(t)}{A(t)}B\left(\tau(t)\right) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{w^2(t)}{\rho(t)a(t)}.$$

Using the completing the square in the above inequality we obtain

$$w'(t) \le -\frac{\rho(t)Q(t)}{A(t)}B\left(\tau(t)\right) + \frac{a(t)\left(\rho'(t)\right)^2}{4\rho(t)}, t \ge t_1.$$

Integrating the last inequality from t_1 to t yields

$$\int_{t_1}^{t}\left[\frac{\rho(s)Q(s)}{A(s)}B\left(\tau(s)\right) - \frac{a(s)\left(\rho'(s)\right)^2}{4\rho(s)}\right]ds \le w(t_1) < \infty.$$

Taking limit supremum as $t \to \infty$ in the above inequality, we get a contradiction with (4).

Case-(II): For this case, using Lemma 2, we see that by condition (3) that $\lim_{t\to\infty} x(t) = 0$.

This completes the proof.

3. EXAMPLE

In this section, we present an example to illustrate the main result.

Example 1: Consider the third order neutral differential equation

$$\left(t\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right)\right)''\right)'+\frac{\lambda}{t^2}x\left(\frac{t}{3}\right)=0, t\geq 1 \tag{10}$$

where $\lambda>0$. Here $a(t)=t, b(t)=1, q(t)=\frac{\lambda}{t^2}, p(t)=\frac{1}{t}, \alpha=\frac{1}{3}, \sigma(t)=\frac{t}{2}$ and $\tau(t)=\frac{t}{3}$. A simple calculation shows that $A(t)=log\ t$ and $B(t)=t(log\ t-1)$. By taking $\rho(t)=t$, we see that all conditions of Theorem 4 are satisfied if $\lambda>\frac{1}{4}$. Hence equation (10) is almost oscillatory if $\lambda>\frac{1}{4}$.

Remark 1: If $\alpha=1$, then Theorem (4) reduces to Theorem 2.1 of (10) and so our result generalize and complement to that of in (10).

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Source of support: Nil, Conflict of interest: None Declared.