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ON SOME SUMMATION-DIFFERENCE INEQUALITIES

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ABSTRACT

In this paper we shall consider the equation

$$f(t, \Delta x(t), x(t), Fx(t)) = 0; \ x(0)) = x_0$$

where $f : J \times R^3 \to R$ and F be an operator from $J \to R$ into $J \to R$. We also discuss about over and under function of above equation and its δ- approximate solution.

Keywords: *Difference Equation, Summation equation, Summation inequality, Under and over Function.*

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1. INTRODUCTION:

Agarwal [1], Kelley and Peterson [9] developed the theory of difference equations and difference inequalities. Some difference inequalities and comparison results are obtained by K. L. Bondar [2, 3]. Some summation and difference inequalities are obtained in K. L. Bondar [4, 5]. K. L. Bondar, V. C. Borkar, S. T. Patil [6, 7] and Dang H., Oppenheimer S.F.[8] obtained the existence and uniqueness results for difference equations. Some differential and integral inequalities are given in [10]. In this paper we shall consider the equation

$$f(t, \Delta x(t), x(t), Fx(t)) = 0, \ x(0) = x_0 \tag{1}$$

where $f : J \times R^3 \to R$ and F be an operator from $J \to R$ into $J \to R$. We also discuss about over and under function of above equation and its δ- approximate solution.

2. PRELIMINARY NOTES

Let $J = \{t_0, t_0 + 1\ldots t_0 + a\}$, $t_0 \geq 0$, $t_0 \in R$, and E be an open subset of R. Consider the difference equations with an initial condition,

$$\Delta u(t) = g(t, u(t)), u(t_0) = u_0 \tag{2}$$

where $u_0 \in E$, $u: J \to E$, $g : J \times E \to R$.

The function $\phi : J \to R$ is said to be a solution of initial value problem (2), if it satisfies

$$\Delta \phi(t) = g(t, \phi(t)); \quad \phi(t_0) = u_0.$$

The initial value problem is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convention $\sum_{s=t_0}^{t_0-1} g(s, u(s)) = 0$ and so u (t) given above is the solution of (2).

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3. MAIN RESULTS:

Theorem: 3.1 *Assume that*
(i) $f: J \times R^3 \to R$ *and* $f(t, x, y, z)$ *is nondecreasing in x for fixed (t, y, z) and nonincreasing in z for fixed (t, x, y);*

(ii) the operator F maps from $J \to R$ *into* $J \to R$*, and for any two functions* $u_1, u_2 : J \to R$*, the inequality*

$$u_1(t) \leq u_2(t), \quad 0 \leq t \leq t^*, \; t^* > 0, \; t^* \in J$$

implies

$$Fu \leq Fv, \text{ for } t = t^*;$$

(iii) $v, w : J \to R$ *and the inequalities*

$$f(t, \Delta v(t), v(t), Fv(t)) \leq 0$$

$$f(t, \Delta w(t), w(t), Fw(t)) \geq 0$$

hold for $t > 0$, $t \in J$, *one of them being strict.*

Then, $v(0) < w(0)$ *implies*

$$(t) < w(t), \; t \geq 0. \tag{3}$$

Proof: Assume that the set

$$Z = [t \in J: v(t) \geq w(t)]$$

is nonempty. Let $t^* = \inf Z$. Then $t^* > 0$, because $v(0) < w(0)$. Furthermore, we have

$$v(t^*) = w(t^*), \tag{4}$$

$$v(t) \leq w(t), \; 0 \leq t \leq t^*, \tag{5}$$

and

$$\Delta v(t^*) \geq \Delta w(t^*). \tag{6}$$

It then follows from assumption (ii) that

$$v(t) \leq Fw(t), \quad \text{for } t = t^*. \tag{7}$$

The monotonicity of the function f now yields

$$f(t^*, \Delta v(t^*), v(t^*), Fv) \geq F(t^*, \Delta w(t^*), w(t^*), Fw)$$

because of the relations (4), (5), (6) and (7). This implies a contradiction in view of the strictness of one of the inequalities assumed in (iii). Consequently, the set Z is empty, and (3) is true. The proof is complete.

Definition: 3.2 *A function* $v : J \to R$ *is said to be an under function with respect to equation (1), if it satisfies the inequality*

$$f(t, \Delta v(t), v(t), Fv(t)) < 0.$$

On the other hand if v *satisfies the inequality*

$$f(t, \Delta v(t), v(t), Fv(t)) < 0,$$

then a function $v(t)$ *is said to be an over function with respect to equation (1).*

As a consequence of Theorem 3.1, we have the following result.

Theorem: 3.3 *Let $u(t), w(t) : J \to R$ be under and over functions respectively with respect to (1) and $v(t)$ be a solution of (1) existing on J. Then,*

$$u(0) < v(0) < w(0)$$

implies

$$u(t) < v(t) < w(t), \quad t \geq 0.$$

Proof: As u (t) is an under function and $v(t)$ is a solution of (1) respectively, we have

$$f(t, \Delta u(t), u(t), Fu(t)) < 0 \text{ and}$$

$$f(t, \Delta v(t), v(t), Fv(t)) = 0, \quad v(0) = 0.$$

Thus if $u(0) < v(0)$, then by Theorem 3.1, we have

$$u(t) < v(t), \ t \geq 0.$$

Similarly using definition of solution, an over function of (1) and by Theorem 3.1 again we obtain

$$v(t) < w(t), t \geq 0.$$

Hence, $u(t) < v(t) < w(t), \quad t \geq 0$.

Definition: 3.4 *Let $v : J \to R$. Then $v(t)$ is said to be a δ-approximate solution of the equation (1), if $v(t)$ satisfies the inequality*

$$| f(t, \Delta v(t), v(t), v(t), Fv(t)) | \leq \delta(t), \ t \in J, t \geq 0, \quad \text{where } \delta : J \to R_+.$$

A result that gives an error estimation of the δ-approximate solution is the following.

Theorem: *Let $v(t)$ be a δ-approximate solution of (1). Suppose further that*

$$f(t, x_1, y_1, Fy_1) - f(t, x_2, y_2, Fy_2) \geq g(t, x_1 - x_2, y_1 - y_2, G(y_1 - y_2)),$$

$x_1 \geq x_2$, $y_1 \geq y_2$, where $g : J \times R^3 \to R$, and G is an operator that maps $J \to R$ into $J \to R$. Assume that the function $g(t, x, y, z)$ is nondecreasing in x for fixed (t, y, z) and nonincreasing in z for (t, x, y), and for any two function $u, v : J \to R$, the inequality

$$u(t) \leq v(t), \ 0 \leq t \leq t^*, \ t^* \in J, t^* > 0,$$

implies

$$Gu \leq Gv \text{ for } t = t^*.$$

Then, if $u(t)$ is any solution of (1) such that $u(0) = x_0$ and $| v(0) - x_0 | \leq \rho_0$, we have

$$| v(t) - u(t) | < \rho(t), \ t \geq 0, \text{ where } \rho(t) > 0 \text{ is increasing and satisfying}$$

$$g(t, \Delta\rho(t), \rho(t), G\rho) > \delta(t), \ t \in J.$$

Proof: We shall first show that $v(t) - u(t) < \rho(t)$, $t \geq 0$. Setting $z(t) = v(t) - u(t)$ and proceeding as in Theorem 3.1, we arrive at $t^* > 0$ with the properties,

$$z(t^*) = \rho(t^*)$$

$$\Delta z(t^*) \geq \Delta\rho(t^*),$$

and

$$Gz \leq G\rho, \quad t = t^*.$$

Since $\rho(t^*) > 0$ and increasing we have, $\Delta\rho(t^*) > 0$ and so that $v(t^*) \geq u(t^*)$, $\Delta v(t^*) \geq \Delta u(t^*)$. Hence

$$\delta(t^*) \geq f(t^*, \Delta v(t^*), v(t^*), Fv) - f(t^*, \Delta u(t^*), u(t^*), Fu)$$

$$\geq g(t^*, \Delta z(t^*), z(t^*), Gz).$$

Now, using monotonicity property of g, it follows that

$$g(t^*, \Delta z(t^*), z(t^*), Gz) \leq g(t^*, \Delta\rho(t^*), \rho(t^*), G\rho)$$

$$< \delta(t^*),$$

which implies $\delta(t^*) < \delta(t^*)$. This absurdity proves

$$v(t) - u(t) < \rho(t), \; t \geq 0.$$

A similar argument shows that $u(t) - v(t) < \rho(t)$, $t \geq 0$. The theorem is therefore proved.

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