

AN APPLICATION OF HYPERGEOMETRIC FUNCTION ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT

The purpose of the present paper is to investigate some interesting criteria for hypergeometric functions belonging to classes $T(\lambda, \alpha)$ and $C(\lambda, \alpha)$. We also studied an integral operator related to hypergeometric functions relevant connection of the results presented herewith various well known results are briefly indicated.

Keywords: Analytic, Univalent, Hypergeometric function, Integral Operator

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1. INTRODUCTION

Let T be the class consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \quad (1.1)$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition

$$\Re \left\{ \frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f(z)} \right\} > \alpha \quad (1.2)$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

Also let $C(\lambda, \alpha)$ denote the subclass of T consisting of function which satisfy the condition

$$\Re \left\{ \frac{f'(z) + zf''(z)}{f'(z) + \lambda zf''(z)} \right\} > \alpha \quad (1.3)$$

for some $\alpha(0 \leq \alpha < 1)$, $\lambda(0 \leq \lambda < 1)$ and for all $z \in U$ also from (1.2) and (1.3) we have

$$f(z) \in C(\lambda, \alpha) \Leftrightarrow zf'(z) \in T(\lambda, \alpha). \quad (1.4)$$

We note that $T(0, \alpha) = T^*(\alpha)$, the class of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and $C(0, \alpha) = C(\alpha)$, the class of convex functions of order $\alpha(0 \leq \alpha < 1)$, (see Mostafa [5], Silverman [7]).

Let $F(a, b; c; z)$ be the Gaussian Hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (1.5)$$

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where $c \neq 0, -1, -2, \dots$ and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1 & n = 0 \\ a(a+1)\dots(a+n-1), & n \in \mathbb{N} \end{cases}$$

We note that $F(a, b; c; 1)$ converges for $\Re(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Silverman [8] and Mosftafa [5] (see also [2]-[4], [6]) gives necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $T^*(\alpha)$ and $C(\alpha)$. Now presented here $I(a, b; c; z)$ by the $z\{(1-\mu)F(a, b; c; z) + \mu[zF(a, b; c; z)]'\}$ where $\mu(0 \leq \mu < 1)$.

2. MAIN RESULTS

To establish our main result, we need the following lemma due to Altinats and Owa [1].

Lemma 2.1: A function $f(z)$ defined by (1.1) is in the class $T(\lambda, \alpha)$ if and only if

$$\sum_{n=2}^{\infty} (n - \lambda \alpha n - \alpha + \lambda \alpha) a_n \leq 1 - \alpha. \quad (2.1)$$

Lemma 2.2: A function $f(z)$ defined by (1.1) is in $C(\lambda, \alpha)$, if and only if

$$\sum_{n=2}^{\infty} n(n - \lambda \alpha n - \alpha + \lambda \alpha) a_n \leq 1 - \alpha. \quad (2.2)$$

Theorem 2.1: If $a, b > -1$ and $ab < 0, c > a + b + 1$ then $zI(a, b; c; z)$ where

$I(a, b; c; z) = (1 - \mu)F(a, b; c; z) + \mu[zF(a, b; c; z)]'$ is in $T(\lambda, \alpha)$, if and only if

$$\begin{aligned} & \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\{(1-\lambda\alpha)(1-\mu) - \alpha\mu(1-\lambda)\} \left(\frac{a-bc-a-b-1}{ab} \right) \right. \\ & \left. + \mu(1-\lambda\alpha) \left\{ 5 + \frac{(a+1)(b+1)}{c-a-b-2} \right\} - \alpha(1-\lambda)(1-\mu)(c-a-b-1) \right] \leq (1-\alpha) \left| \frac{c}{ab} \right| + \alpha(1-\lambda)(1-\mu) \frac{c}{ab}. \end{aligned} \quad (2.3)$$

Proof: We have $zI(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} z^n$

$$= z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} z^n, \quad (2.4)$$

According to Lemma 2.1 we must show that

$$\sum_{n=2}^{\infty} (n - \lambda \alpha n - \alpha + \lambda \alpha) (1 - \mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} \leq \frac{c}{ab} (1 - \alpha). \quad (2.5)$$

Note that the left side of (2.5) diverges if $c < a + b + 1$. Now

$$\begin{aligned} & \sum_{n=2}^{\infty} (n - \lambda \alpha n - \alpha + \lambda \alpha) (1 - \mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} \\ &= [(1-\lambda\alpha)(1-\mu) - \alpha\mu(1-\lambda)] \sum_{n=2}^{\infty} n \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} \\ &+ \mu(1-\lambda\alpha) \sum_{n=2}^{\infty} n^2 \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} - \alpha(1-\mu)(1-\lambda) \sum_{n=2}^{\infty} \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_{n-1}} \\ &= \{(1-\lambda\alpha)(1-\mu) - \alpha\mu(1-\lambda)\} \left\{ \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} + \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} \right\} \\ &+ \mu(1-\lambda\alpha) \sum_{n=0}^{\infty} (n+2)^2 \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} - \alpha(1-\mu)(1-\lambda) \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} \end{aligned}$$

$$\begin{aligned}
 &= \{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \left\{ F(a+1, b+1; c+1; 1) + \frac{c}{ab} F(a, b; c; 1) \right\} \\
 &+ \mu(1-\lambda\alpha) \left\{ 4F(a+1, b+1; c+1; 1) + \frac{c}{a} \left(\frac{(a)_2(b)_2}{b(c-a-b-2)_2} + \frac{ab}{(c-a-b-1)} \right) F(a, b; c; 1) \right\} \\
 &- \alpha(1-\mu)(1-\lambda) \frac{c}{ab} \{F(a, b; c; 1) - 1\} \\
 &= \{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \left\{ \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{c}{a} \frac{\Gamma(c)\Gamma(c-a-b)}{b\Gamma(c-a)\Gamma(c-b)} \right\} \\
 &+ \mu(1-\lambda\alpha) \left\{ \frac{4\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{c(a+1)(b+1)}{(c-a-b-2)_2} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \right. \\
 &\left. + \frac{c}{a} \frac{a}{(c-a-b-1)} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \right\} - \alpha(1-\mu)(1-\lambda) \frac{c}{a} \left\{ \frac{\Gamma(c)\Gamma(c-a-b)}{b\Gamma(c-a)\Gamma(c-b)} - 1 \right\} \\
 &= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \\
 &\left[\{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \left(\frac{ab+c-a-b-1}{ab} \right) + \mu(1-\lambda\alpha) \left\{ 5 + \frac{(a+1)(b+1)}{(c-a-b-2)} \right\} \right] \\
 &- \alpha(1-\mu)(1-\lambda)(c-a-b-1) \leq (1-\alpha) \left| \frac{c}{ab} \right| + \alpha(1-\mu)(1-\lambda) \frac{c}{ab}. \tag{2.6}
 \end{aligned}$$

Remark 1: If we put $\mu = 0$ in (2.6) of Theorem 2.1 we obtain the result which is correct to the Mostafa [5].

Corollary 2.1: If $a, b > -1, ab < 0$ and $c > a + b + 1$ then $zI(a, b; c; z)$ is in $T(\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} [ab + (c-a-1)(1-ab)] \geq c.$$

Theorem 2.2: If $a, b > -1, ab < 0$ and $c > a + b + 1$, then $zI(a, b; c; z)$ where

$I(a, b; c; z) = (1-\mu)F(a, b; c; z) + \mu(zF(a, b; c; z))'$ is in $C(\lambda, \alpha)$ if and only if

$$\begin{aligned}
 &\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{\mu(1-\lambda\alpha)}{abc} \{(a)_3(b)_3(c+3) + (a)_2(b)_2(c+2)(c-a-b-2) + 7a(b-a-b-2)_2(c+1) \right. \\
 &\left. + (c+1)(c-a-b)_3\} + \{1-\lambda\alpha\}(1-\mu) - \alpha\mu(1-\lambda)\} n \left(5 + \frac{(a+1)(b+1)}{(c-a-b-2)} \right) - \alpha(1-\mu)(1-\lambda)(c-a-b-1) \right] \\
 &\leq (1-\alpha) \left| \frac{c}{ab} \right| + \{(\alpha-\mu)(1-\alpha) - \lambda\alpha\} \frac{c}{ab}.
 \end{aligned}$$

Proof: We have

$$\begin{aligned}
 zI(a, b; c; z) &= z + \frac{ab}{c} \sum_{n=2}^{\infty} (1-\mu + \mu n) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n \\
 &= z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} (1-\mu + \mu n) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n
 \end{aligned}$$

Now according to the Lemma 2.2, we have

$$\begin{aligned}
 \sum_{n=2}^{\infty} n(n-\lambda\alpha n - \alpha + \lambda\alpha)(1-\mu + \mu n) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} &\leq \left| \frac{c}{ab} \right| (1-\alpha) \\
 \sum_{n=2}^{\infty} n(n-\lambda\alpha n - \alpha + \lambda\alpha)(1-\mu + \mu n) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} &
 \end{aligned}$$

$$= \mu(1-\lambda\alpha) \sum_{n=0}^{\infty} (n+2)^3 \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} + \{(1-\lambda\alpha)(1-\mu) - \alpha\mu(1-\lambda)\} \\ \sum_{n=0}^{\infty} (n+2)^2 \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} - \alpha(1-\mu)(1-\lambda) \sum_{n=0}^{\infty} \frac{(n+2)(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}}$$

By solving the above calculation we get the result.

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{\mu(1-\lambda\alpha)}{abc} \{(a)_3(b)_3(c+3) + (a)_2(b)_2(c+2)(c-a-b-2) \right. \\ \left. + 7a(b-a-b-2)_2(c+1) + (c+1)(c-a-b)_3\} + \{(1-\lambda\alpha)(1-\mu) - \alpha\mu(1-\lambda)\} \right. \\ \left. \left(5 + \frac{(1+1)(b+1)}{(c-a-b-2)} \right) - \alpha(1-\mu)(1-\lambda)(c-a-b-1) \right] \\ \leq (1-\alpha) \left| \frac{c}{ab} \right| + (\alpha-\mu)(1-\alpha) - \lambda\alpha \frac{c}{ab}.$$

3. AN INTEGRAL OPERATOR

In the theorem below, we obtain similar results in connection with a particular integral operator $k(a, b; c; z)$ acting on $I(a, b; c; z)$ as follows

$$k(a, b; c; z) = \int_0^z I(a, b; c; t) dt. \quad (3.1)$$

Theorem 3.1: Let $a, b > -1$, $ab < 0$ and $c > a+b+1$. Then $k(a, b; c; z)$ defined in (3.1) is in $T(\lambda, \alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left\{ \mu(1-\lambda\alpha) \frac{(c-a-b+1)}{ab} + \{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \left(\frac{c-a-b}{ab} \right) \right. \\ \left. + \alpha(1-\mu)(1-\lambda) \frac{(a)_2(b)_2}{(c)_2} \left(\frac{c-a-b}{c} \right) \right\} - (1-\alpha\lambda + \mu - \lambda\mu) \frac{c}{ab} - \left(1 + \frac{a}{c} \right) \frac{b}{\alpha} (1-\mu)(1-\lambda) \frac{(a)_2(b)_2}{(c)_2} \leq (1-\alpha) \left| \frac{c}{ab} \right|. \quad (3.2)$$

Proof: Since

$$k(a, b; c; z) = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} (1+\mu-\mu n) \frac{(a+1)_{n-2} (b+1)_{n-2} z^n}{(c+1)_{n-2} (1)_n}. \quad (3.3)$$

By Lemma 2.1, we have

$$\sum_{n=2}^{\infty} (n-\lambda\alpha n - \alpha + \lambda\alpha)(1-\mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_n} \leq \left| \frac{c}{ab} \right| (1-\alpha).$$

Now

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n - \alpha + \lambda\alpha)(1-\mu + \mu n) \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_n} \\ = \{\mu(1-\lambda\alpha)\} \sum_{n=2}^{\infty} \frac{n^2 (a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_n} + \{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \\ \sum_{n=2}^{\infty} \frac{n^2 (a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_n} + \alpha(1-\mu) \sum_{n=2}^{\infty} n \frac{(a+1)_{n-2} (b+1)_{n-2}}{(c+1)_{n-2} (1)_n} \\ = \mu(1-\lambda\alpha) \{F(a+1, b+1; c+1; 1) + \frac{c}{ab} F(a, b; c; 1) - 1\} \\ + \{(1-\lambda\alpha)(1-\mu) - \mu\alpha(1-\lambda)\} \frac{c}{ab} \{F(a, b; c; 1) - 1\} + \alpha(1-\mu)(1-\lambda) \frac{(a)_2(b)_2}{(c)_2} \left\{ F(a, b; c; 1) - 1 - \frac{ab}{c} \right\} \\ = \mu(1-\lambda\alpha) \left\{ \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c+1)\Gamma c \Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right\}$$

$$\begin{aligned}
 & +\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1-\frac{ab}{c}\right\} \\
 & =\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(\frac{c-a-b}{ab}\right)\right. \\
 & \left.+\mu(1-\lambda\alpha)\left(\frac{c-a-b+1}{ab}\right)+\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left(\frac{c-a-b}{c}\right)\right\} \\
 & -\left\{1-\alpha\lambda+\mu-\lambda\mu\right\}\frac{c}{ab}-\left(1+\frac{ab}{c}\right)\alpha(1-\mu)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\leq(1-\alpha)\left|\frac{c}{ab}\right|.
 \end{aligned}$$

Corollary 3.2: Let $a, b > -1$, $ab < 0$ and $c > a+b+1$ with constant $\mu = 0$ then $k(a, b; c; z)$ is in $T(\lambda, \alpha)$, if and only if

$$\begin{aligned}
 & \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{(1-\lambda\alpha)\frac{(c-a-b)}{ab}+\alpha(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\left(\frac{c-a-b}{c}\right)\right\} \\
 & -\alpha\left(a+\frac{ab}{c}\right)(1-\lambda)\frac{(a)_2(b)_2}{(c)_2}\leq(2-\alpha-\lambda\alpha)\left|\frac{c}{ab}\right|.
 \end{aligned}$$

Proof: By the (3.2) of the above Theorem 3.1 if we put $\mu = 0$ then the above result can be find easily.

Corollary 3.3: If $a, b > -1$, $ab < 0$ and $c > a+b+1$, then $k(a, b; c; z)$ is in $T(\alpha)$, if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\alpha\frac{(a)_2(b)_2}{(c)_2}\left(\frac{ab}{c}\right)\right\}-\alpha\left(1+\frac{ab}{c}\right)\frac{(a)_2(b)_2}{(c)_2}ab\geq(2-\alpha)|c|$$

Theorem 3.4: If $a, b > -1$, $ab < 0$ and $c > a+b+1$, then $k(a, b; c; z)$ defined by (3.1) is in $C(\lambda, \alpha)$ iff

$$\begin{aligned}
 & \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\mu(1-\lambda\alpha)\left\{\frac{(ab)}{c^2}(c-a-b)+3+\left(\frac{c-a-b}{c}\right)\right\}\right. \\
 & \left.+\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)\}\left(1+\frac{c-a-b}{ab}\right)+\alpha(1-\mu)(1-\lambda)\left(\frac{c-a-b}{ab}\right)\right] \\
 & \leq\left|\frac{c}{ab}\right|(1-\alpha)+\frac{c}{ab}\{(1-\lambda\alpha)(1-\mu)-\mu\alpha(1-\lambda)+\alpha(1-\mu)(1-\lambda)\}.
 \end{aligned}$$

Proof: The proof of above theorem is much akin to that of Theorem 3.1, therefore we omit the details involved.

REFERENCES

1. O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients, Pusan Kyongnum Math. J., 4 (1988), 41-56.
2. B.C. Carlson and D.B. Shafffr, Starlike and prestarlike hypergeometric functions, J. Math. Anal. Appl., 15 (1984), 737-745.
3. N.E. Cho, S.Y. Woo and S. Owa, Uniform convexity properties for hypergeometric functions, Fract. Cal. Appl. Anal., 5(3) (2002), 303-313.
4. E. Merkes and B.T. Scott, Starlike hypergeometric functions, Proc. Amer. Math. Soc., 12 (1961), 885-888.
5. A.O. Mostafa, Study on starlike and convex properties for hypergeometric functions. Journal of inequalities in pure and applied Math., Vol. 10, Iss. 3, Art. 87, (2009).
6. St. Ruscheweyh and V. Singh, On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl., 113 (1986), 1-11.
7. H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975), 109-116.
8. H. Silverman, Starlike and Convexity properties for hypergeometric functions, J. Math., Anal. Appl., 172 (1993), 574-581.

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