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INTEGRAL EDGE SUM OF $\Gamma(Z_n)$

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ABSTRACT

The first simplification of Beck's [2] zero divisor graph was introduced by D.F.Anderson and P.S.Livingston[1]. Their motivation was to give a better illustration of the zero divisor structure of the ring. In this paper, we find the integral sum of zero divisor graph and study their properties.

Keywords: *Integral Edge Sum, Zero divisor graph.*

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1. INTRODUCTION

Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph $\Gamma(R)$ to R with vertices $\Gamma(R)^* = Z(R) - \{0\}$, the set of non-zero divisors of R and for distinct $u, v \in Z(R)^*$, the vertices u and v are adjacent if and only if $uv = 0$. The zero divisor graph is very useful to find the algebraic structures and properties of rings. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck in [2]. The first simplication of Beck's zero divisor graph was introduced by D.F.Anderson and P.S.Livingston [1]. Their motivation was to give a better illustration of the zero divisor structure of the ring. D.F.Anderson and P.S.Livinston, and others e.g., [4, 5, 6], investigate the interplay between the graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of R. Throught this paper, we consider the commutative ring R by Z_n and zero divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$. Let $\Gamma(Z_n)$ be a graph. A bijection $f : E(\Gamma(Z_n)) \to Z^+$, where Z^+ is a set of positive integers is called an edge mapping of the graph $\Gamma(Z_n)$. Now, we define, $F(v) = \sum\{ f(e) : e\, is\, incident\, on\, v\, on\, V(\Gamma(Z_n))\}$. Then, F is called the edge sum mapping of the edge mapping f. $f : E(\Gamma(Z_n)) \to N^+$ such that f and its corresponding edge sum mapping F on $V(\Gamma(Z_n))$ satisfy the following conditions: (i) F is into mapping to Z^+. That is, $F(v) = Z^+$, for every $v \in E(\Gamma(Z_n))$. (ii) If $e_1, e_2, ..., e_n \in E(\Gamma(Z_n))$ such that $f(e_1) + f(e_2) + ... + f(e_n) \in Z^+$, then $e_1, e_2, ..., e_n$ are incident on a vertex in $\Gamma(Z_n)$. The edge sum labeling was introduced by Paulraj Joseph et al., [3].

2. INTEGRAL EDGE SUM OF $\Gamma(Z_n)$

In this section, we evaluate the integral edge sum of $\Gamma(Z_n)$ and study their properties. The integral edge sum graph is defined as follows,

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Definition 2.1: Let $\Gamma(Z_n)$ be a graph. A bijection $f : E \to S$ where S is a set of integers is called as integral edge function of the graph $\Gamma(Z_n)$. Define $f(v) = \sum\{f(e) : e\, is\, incident\, on\, v\}$ on V. Then F is called the integral edge sum function of the integral edge function f.
$\Gamma(Z_n)$ is said to be an integral edge sum graph if there exists an integral edge function $f : E \to S$ such that f and its corresponding integral edge sum function F on V satisfy the following conditions:

i) F is into S. That is, $f(v) \in S$ for every $v \in V$.

ii) If $e_1, e_2, ..., e_n \in E$ such that $f(e_1) + f(e_2) + ... + f(e_n) \in S$, then $e_1, e_2, ..., e_n$ are incident on a vertex.

Definition 2.2: If e is an edge of $\Gamma(Z_n)$, then the graph obtained by subdividing 'e' exactly once is denoted as G(e).

Definition 2.3: Let $\Gamma(Z_n)$ be an integral edge sum graph. The integral edge function $f : E \to S$ and its corresponding integral edge sum function F which make G an integral edge sum graph are called optimal integral edge function and optimal integral edge sum function respectively.

For example, Let $V = \{v_1, v_2, ..., v_{p-1}\}$ be the vertex set and $E = \{vv_i : 1 \le i \le p-1\}$ be the edge set of the graph $\Gamma(Z_{2p})$, where p is any prime number, greater than two. Let $S = \left\{0,1,2,...,(p-3), \frac{-(p-3)(p-2)}{2}\right\}$. The integral edge function $f : E \to S$ is defined as $f(vv_i) = i \; for 1 \le i \le (p-3)$ and $f(vv_2) = \frac{-(p-3)(p-2)}{2}$
.
The corresponding integral edge sum function is,

$$f(v) = 0$$
$$f(v_i) = i \;\; for \; 1 \le i \le (p-3)$$
$$f(v_{p-2}) = \frac{-(p-3)(p-2)}{2}$$
$$f(v_{p-1}) = 0$$

Therefore, F is into S. As all the edges are incident on the second condition becomes trivial. Hence $\Gamma(Z_{2p})$ is an integral edge graph for every natural number $p-1$.
For example, In $\Gamma(Z_{2p})$(e), $V = \{v_1, v_2, ..., v_{p-2}, u, v, w\}$ is the vertex set and $E = \{uv, uw\} \cup \{vv_i : 1 \le i \le (p-2)\}$ is the edge set of $\Gamma(Z_{2p})$.
$S = \left\{0,1,2,...,(p-3), \frac{(p-1)(p-2)}{2}\right\}$. The integral edge sum function $f : E \to S$ is defined as $f(uv) = 0$

$$f(uw) = \frac{(p-1)(p-2)}{2}$$
$$f(vv_i) = i, \;\; 1 \le i \le (p-2).$$

The corresponding integral edge sum function is as follows:

$$f(u) = f(v) = f(w) = \frac{(p-1)(p-2)}{2}$$
$$f(v_i) = i, \;\; 1 \le i \le (p-2).$$

Except uw all the other edges are incident on v. The value $f(uw) = \frac{(p-1)(p-2)}{2}$ can be got by summing either $f(vv_1)$, $f(vv_2)$, …, and $f(vv_n)$ (or) $f(uv)$, $f(vv_1)$, …, $f(vv_n)$. In both the cases the corresponding edges are incident on v. Hence, $\Gamma(Z_{2p})$(e) is an integral edge sum graph.

In both the examples 0 is one among the labels. But not all the integral edge sum graphs admit 0 as a label.

Definition 2.4: An integral edge sum graph $\Gamma(Z_n)$ is called a zero integral sum graph if there exists an optimal integral edge function $f : E \to S$ with $0 \in S$. We have proved that $\Gamma(Z_{2p})$ or $\Gamma(Z_{2p})(e)$, where p is any prime number, are zero integral edge sum graphs. In the following theorem, we prove that there is no other zero integral edge sum graph.

Theorem 2.5: Let G be an integral edge sum graph without isolated vertices. Then G is a zero integral edge sum graph $\Gamma(Z_{2p})$ or $\Gamma(Z_{2p})(e)$ for some prime number 'p'.

Proof: Let $f : E \to S$ be an optimal integral edge function and F be its corresponding optimal integral edge sum function. Let $e = uv$ be such that $f(e) = 0$. If e_1 is any other edge, then $f(e_1) = f(e_1) + f(e) \in S$ and therefore e and e_1 are incident on a vertex. That is, all the other edges are adjacent to e.

If w is any other then w and v, then $F(w) = F(w) + f(0) \in S$. Therefore if $w_1, w_2, ..., w_r$ are the vertices adjacent to w, then ww_1, ww_2, …, ww_r and e are adjacent on a vertex. The only possibility is that w is a pendent vertex adjacent on a vertex. The only possibility is that w is a pendant vertex adjacent to either u or v.

Suppose $\deg u > 1$. Let $u_1, u_2, ..., u_m$ be the pendent vertices adjacent to u apart from v and $v_1, v_2, ..., v_{p-1}$ be the pendent vertices adjacent to v apart from u. Then $F(u) = f(uu_1) + f(uu_2) + ... + f(uu_m) + f(uv) \in S$. Let $F(u) = f(e_1)$. If $e_1 = uu_i$ for some i (say uu_1), then $F(u) = f(uu_1) = f(uu_1) + f(uu_2) + ... + f(uu_m)$. That is, $f(uu_2) + ... + f(uu_m) = 0$. Hence $F(v) = F(v) + f(uu_2) + ... + f(uu_m) \in S$. That is, $vv_1, vv_2, ..., vv_{p-1}, uu_1, uu_2, ..., uu_m$ and uv are incident on a vertex. That is positive only if $m = 0$.

If $e_1 = e = uv$, then $F(u) = 0$ and $F(u) + F(v) \in S$. Again this is a possibility only if v is not adjacent to any other than u. Here again $m = 0$.

Let us consider the last case. Let $e_1 = vv_i$ for some i. assuming $e_1 = vv_1$, we get the following: $F(v) = f(vv_1) + f(vv_2) + ... + f(vv_{p-1}) = f(uu_1) + f(uu_2) + ... + f(uu_m) + f(vv_2) + f(vv_3) + ... + f(vv_{p-1})$ is an element of S. This is not possible if $p - 1 \geq 2$, where p is any prime greater than 2.
Hence we have the following:

(i) e is adjacent to all other edges of $\Gamma(Z_n)$.

(ii) Any vertex other than u and v is a pendent vertex adjacent to either u or v.

(iii) If $\deg u > 1$, then $\deg v = 1$. Thus G is either $\Gamma(Z_{2p})$ or $\Gamma(Z_{2p})(e)$ for some $p - 1$.

In all the other integral edge sum graphs zero is not among the edge labels. In a similar way it is easily seen that zero sum of edge labels is also not possible in the nonzero integral edge sum graphs. That is, if f is an optimal integral edge sum graph $\Gamma(Z_{2n})$, then $f(e_1) + f(e_2) + ... + f(e_n)$ is not zero for any sub collection $\{e_1, e_2, ..., e_n\}$ of the edge set $\Gamma(Z_{2n})$.

The optimal edge function and its corresponding optimal edge sum function of an edge sum graph and the optimal integral edge function and its corresponding optimal integral edge sum of a nonzero integral edge sum graph act almost similarly. We state the following observations concerning non zero integral edge sum of $\Gamma(Z_n)$ without proof, as the result and its proof are similar to that of edge sum graphs in each case.

Observation 1: Let $\Gamma(Z_n)$ be a nonzero integral edge sum graph with integral edge function $f : E \to S$ and the integral edge sum function F of f. Let $l_1, l_2, ..., l_m$ where $m > l$ be a collection of edges incident on a vertex w (say).

Let $ww_i = l_i$ for $l \le i \le m$. If there exists an edge $e = uv$ such that $f(l_1) + f(l_2) + ... + f(l_m) = f(e)$, then one of the following holds:

(i) $\{u, v\}$ forms a $\Gamma(Z_9)$ component in $\Gamma(Z_n)$.

(ii) $\langle\{u, v, w\}\rangle$ is either $\Gamma(Z_4)$ or $\Gamma(Z_9)$ with one of u, v as a pendent vertex in $\Gamma(Z_n)$.

Observation 2: $\Gamma(Z_n)$ be a nonzero integral edge sum graph with integral edge function $f : E \to S$ and the integral edge sum function F of f. let w be a non pendent vertex and $e = uv \in E$ be such that $F(w) = f(e)$. Then one of the following holds:

(i) $\{u, v\}$ forms a $\Gamma(Z_9)$ component in $\Gamma(Z_n)$.

(ii) $\langle\{u, v, w\}\rangle$ is either $\Gamma(Z_9)$ with one of u, v as a pendent vertex in $\Gamma(Z_n)$.

Remark: Let $f : E \to S$ be an optimal edge function and F be its corresponding optimal edge sum graph $\Gamma(Z_n)$.If u is a non pendent vertex of $\Gamma(Z_n)$, then there exist vertices v, w such that $F(u) = f(vw)$. If vw is not a $\Gamma(Z_9)$ component then either one of them is adjacent to u. Let v be the adjacent to u. Then $F(v) \ge f(u\ v) + f(vw) > F(u) > f(e)$ for every edge incident on u. Hence if $F(v) = f(e_1)$, then e_1 is incident on v and $F(u) = f(e_2)$, then e_2 is not incident on u. But this is not the case in integral edge sum graphs. We will see it in proving complete graph in $\Gamma(Z_n)$ is integral edge sum graph.

Definition 2.6: Let $V = \{v_1, v_2, ..., v_{p-1}\}$ be the vertex set of $\Gamma(Z_{p^2})$ for $p \ge 5$ is any prime number. The integral edge function matrix $A = (a_{i,j})$ of order $p-1$ of the integral edge function f of $\Gamma(Z_{p^2})$ is defined as $a_{i,j} = f(v_i v_j)$ if $i \ne j$ and is 0 if $i = j$. The integral edge function matrix is a symmetric matrix.

Theorem 2.7: $\Gamma(Z_{p^2})$ for $p \ge 5$ is an integral edge sum graph, where p is a prime number.

Proof: Let $V = \{v_1, v_2, ..., v_{p-1}\}$ be the vertex set of $\Gamma(Z_{p^2})$ for $p \ge 5$ is any prime number and $E = \{v_i v_j : 1 \le i \le p-2\ and\ i+1 \le j \le p-1\}$ be the edge set of $\Gamma(Z_{p^2})$. For defining the integral edge function matrix we need the following definitions.

Let $B = (b_{i,j})$ be a $(p-5)\times(p-1)$ matrix defined as $b_0 = 20$, $b_{i,j} = b_{j,i}$ for $1 \le i, j \le p-5$, $b_{i,i+1} = b_{i-1} = \sum\{b_{i-1,j} : 1 \le i \le p-5\}$ and $b_{i,j} = \sum\{2^{j-i} b_{i-1} : 1 \le i \le p-5\ and\ (i+2) \le j \le p-1\}$.

Let $d_{p-1} = \lfloor (b_{1,p-1} - b_{1,2}) + (b_{2,p-1} - b_{2,3}) + ... + (b_{p-5,p-1} - b_{p-5,p-4}) \rfloor$.

$$d_{p-2} = \lfloor (b_{1,p-2} - b_{1,2}) + (b_{2,p-2} - b_{2,3}) + ... + (b_{p-5,p-2} - b_{p-5,p-4}) \rfloor.$$
$$d_{p-3} = \lfloor (b_{1,p-3} - b_{1,2}) + (b_{2,p-3} - b_{2,3}) + ... + (b_{p-5,p-3} - b_{p-5,p-4}) \rfloor.$$
$$d_{p-4} = \lfloor (b_{1,p-4} - b_{1,2}) + (b_{2,p-4} - b_{2,3}) + ... + (b_{p-5,p-4} - b_{p-5,p-4}) \rfloor.$$

Since $b_{i,i+1} = b_{i-1}$ for $1 \le i \le p-5$, we get that

$$\begin{aligned} d_j &= (b_{1,j} + b_{2,j} + ... + b_{p-5,j}) - (b_{1,2}) + (b_{2,3} + ...b_{p-5,p-4}). \\ &= \sum\{b_{i,j} : 1 \le i \le p-5\} - \sum\{b_{i-1} : 1 \le i \le p-5\} \\ &= \sum\{b_{i,j} : 1 \le i \le p-5\} - B_{p-6} \end{aligned}$$

Where, $B_j = \sum\{b_i : 1 \le i \le j\}$ for $p-4 \le j \le p-1$.

Let h_0 be a content satisfying $h_0 > (d_{p-1} - d_{p-2} + 2 - b_0)$; $h_0 \equiv 0(\mathrm{mod}\,20)$ and $h_0 > B_{p-6}$.

Let $H=(h_{i,j})$ be a $(p-5)\times(p-1)$ matrix defined as $h_{i,j}=h_0$ for $2\le j\le p-1$ and $h_{i,j}=h_{i-1}=\sum\{h_{i-1,k}:1\le k\le p-1\}$ for $2\le i\le p-5$ and $i+1\le j\le p-1$ where $h_{j,i}=h_{i,j}$ for $1\le i\le p-5$ and $1\le j\le p-5$. Let $x_i=b_i+h_i$ for $0\le i\le p-6$.

The integral edge function matrix $A=(a_{i,j})$ of order $p-1$ is defined as,

$$a_{i,j}=b_{i,j}+h_{i,j} \text{ for } 1\le i\le p-5 \text{ and } i+1\le j\le p-1.$$
$$a_{p-4,p-3}=x_{p-5}=\sum\{a_{p-5,j}:1\le j\le p-1\}$$
$$a_{p-4,p-2}=-y$$
$$a_{p-4,p-1}=x_{p-5}+a$$
$$a_{p-3,p-2}=x_{p-4}=\sum\{a_{p-4,j}:1\le j\le p-1\}$$
$$a_{p-3,p-1}=-z$$
$$a_{p-2,p-1}=x_{p-3}=\sum\{a_{p-3,j}:1\le j\le p-1\}$$

Where a, z, y are defined as follows.

Let $X_i=\sum\{x_j:0\le j\le i\}$

$$y=3X_{p-5}-2x_{p-4}+d_{p-1}+d_{p-3}+d_{p-4}+4$$
$$z=2y-X_{p-5}+d_{p-1}-d_{p-2}+d_{p-4}+4 \text{ and}$$
$$2a=5(y-X_{p-5})+d_{p-1}-2d_{p-2}-d_{p-3}-3d_{p-4}$$

We have, $d_j=\sum\{b_{i,j}:1\le i\le p-5\}-B_{p-6}$ for $p-4\le j\le p-1$

$$=\sum\{(b_{i,j}+h_{i,j}):1\le i\le p-5\}-\sum\{(b_{i-1}+h_{i,j}):1\le i\le p-5\}$$
$$=\sum\{a_{i,j}:1\le i\le p-5\}-\sum\{(b_{i-1}+h_{i,j}):1\le i\le p-5\}$$
$$=\sum\{a_{i,j}:1\le i\le p-5\}-\sum\{x_{i-1}:1\le i\le p-5\}$$
$$=\sum\{a_{i,j}:1\le i\le p-5\}-X_{p-6}$$

Let $S=\{a_{,j}:1\le i\le p-2\, and\, i+1\le j\le p-1\}$.

The edge function $f:E\to S$ is defined $i+1\le j\le p-1$. The corresponding integral edge sum function F is as follows:

$$F(v_i)=\sum\{f(v_iv_j):1\le j\le p-1 and\ j\ne i\}\, for 1\le i\le p-3.$$
$$=\sum\{a_{i,j}:1\le j\le p-1 and\ \ j\ne i\}$$
$$=\sum\{(b_{i-1}+h_{i,j}):1\le j\le p-1\}$$
$$=b_i+h_i$$
$$=x_i$$

$$F(v_{p-2})=\sum\{f(v_{p-2}v_j):1\le j\le p-1 and\ j\ne p-2\}$$
$$=\sum\{a_{p-2,j}:1\le j\le p-1\ and\ \ j\ne p-2\}$$
$$=\sum\{a_{p-2,j}:1\le j\le p-5\}+a_{p-2,p-4}+X_{p-6}-y+x_{p-4}+x_{p-3}$$
$$=x_{p-5}$$
$$=a_{p-4,p-3}$$
$$=f(v_{p-4}v_{p-3}) \text{ and}$$

$$\begin{aligned} F(v_{p-1}) &= \sum\left\{f(v_{p-1}v_j) : 1 \le j \le p-2\right\} \\ &= \sum\left\{a_{p-1,j} : 1 \le j \le p-2\right\} \\ &= \sum\left\{a_{p-1,j} : 1 \le j \le p-5\right\} + a_{p-1,p-4} + a_{p-1,p-3} + a_{p-1,p-2} \\ &= d_{p-1} + X_{p-6} + x_{p-5} + a - z + x_{p-3} = x_{p-5} \\ &= a_{p-4,p-3} \\ &= f(v_{p-4}v_{p-3}) \end{aligned}$$

Hence, F is into S. Let $S_1 = \{a_{i,j} : 1 \le i \le p-5 \text{ and } i+1 \le j \le p-1\}$ and $S_2 = \{a_{i,j} : p-4 \le i \le p-2 \text{ and } i+1 \le j \le p-1\}$. The elements of S_1 satisfy the following properties:

(i) $a_{i,j} = b_{i,j} + h_{i,j}$

(ii) $b_{i,i+1} = \sum\left\{b_{i-1,j} : 1 \le i \le p-5\right\}$

(iii) $b_{i,j} > (b_{1,2} + b_{1,3} + \ldots + b_{1,p-1}) + (b_{2,3} + b_{2,4} + \ldots + b_{2,p-1}) + \ldots.$
$+ (b_{i-1,i} + b_{i-1,i+1} + \ldots + b_{i-1,p-1}) + (b_{i,i+1} + b_{i,i+2} + \ldots + b_{i,p-1})$

(iv) $h_{i,j} = h_0$ for $2 \le j \le p-1$ and $h_{i,j} = h_{i-1} = \sum\left\{h_{i-1,k} : 1 \le k \le p-1\right\}$ for $2 \le j \le p-5$ and $i+1 \le j \le p-1$.

(v) All the elements of S_1 are congruent to $0(\text{mod}\,20)$. Hence, no element of S_1 except x_i for $1 \le j \le p-6$ is a sum of two or more elements of S.

The elements of S_2 satisfy the following properties:

(i) $a_{p-4,p-3} = x_{p-5} = \sum\left\{a_{p-5,j} : 1 \le i \le p-1\right\}$

(ii) $y = 3X_{p-5} - 2x_{p-5} + d_{p-1} + d_{p-3} + d_{p-4} + 4 \equiv 4(\text{mod}\,20)$.

Hence, $a_{p-3,p-2} = -y \equiv 10(\text{mod}\,20)$.

(iii) $a \equiv 10(\text{mod}\,20)$ and hence $x_{p-4,p-1} = x_{p-5} + a \equiv 10(\text{mod}\,20)$.

(iv) $a_{p-3,p-2} = x_{p-4} \equiv 6(\text{mod}\,20)$.

(v) $z \equiv 8(\text{mod}\,20)$ and hence $a_{p-3,p-1} = -z \equiv 12(\text{mod}\,20)$.

(vi) $x_{p-3} \equiv 8(\text{mod}\,20)$.

(vii) $x_{p-5} + a - z = 2$.

(viii) $x_{p-4} - y = d_{p-1} - d_{p-2} + 2$.

(ix) $$\begin{aligned} x_0 &= b_0 + h_0 \\ &> \left\{d_{p-1} - d_{p-2} + 2\right\} \\ &= x_{p-4} - y. \end{aligned}$$

Hence, no element of S_2 except x_{p-3}, x_{p-4} and x_{p-5} are a sum of two or more elements of S. Thus, we get that if $f(e_1) + f(e_2) + \ldots + f(e_n) \in S$, then $e_1, e_2, \ldots, e_n$ from the set of all edges incident on a vertex. Hence, $\Gamma(Z_{p^2})$ is an integral edge sum graph for all prime $p \ge 5$.

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