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On Hilbert Modules over Locally m-Convex H^*-Algebras

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ABSTRACT

In this paper a Hilbert $E-$module W is defined where $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ is a locally m-convex H^*-algebra. With each $\lambda\in\Lambda$, we associate a Hilbert module $\widehat{W}_\lambda$ over an H^*-algebra $\widehat{E}_\lambda$. We obtain relationship between these spaces and the initial space. Moreover the existence of orthonormal bases in a Hilbert $E-$module is proved. We topologized the space of bounded $E-$linear operators via suitable family of seminorms.

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Key Words: *Hilbert module, Locally m-convex H^*-algebra.*

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1. Introduction

A locally multiplicatively convex algebra (l.m.c.a in short) is a topological algebra (E,τ) whose topology τ is determined by a directed family $(|.|_\lambda)_{\lambda\in\Lambda}$ of submultiplicative seminorms. Such an algebra will usually denoted by $(E,(|.|_\lambda)_{\lambda\in\Lambda})$. If, in addition, E is endowed with an involution $x\mapsto x^*$ such that $|x|_\lambda=|x^*|_\lambda$, for any $x\in E, \lambda\in\Lambda$, then $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ is called an l.m.c.$*-$algebra. Let $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ be a complete l.m.c.a. It is known that $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ is the inverse limit of the normed algebras $(E_\lambda,(|.|'_\lambda)_{\lambda\in\Lambda})$, where $E_\lambda=E/N_\lambda$ with $N_\lambda=\{x\in E:|x|_\lambda=0\}$, and $|\bar{x}|'_\lambda=|x|_\lambda$. An element x of E is written $x=(x_\lambda)_\lambda=(\pi_\lambda(x))_\lambda$, where $\pi_\lambda:E\to E_\lambda$ is the canonical surjection. The algebra $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ is also the inverse limit of the Banach algebras $\widehat{E}_\lambda$, the completion of E_λ's. The norm in $\widehat{E}_\lambda$ will also be denoted by $|.|'_\lambda$.

In the following we define the locally m-convex H^*-algebra spaces. This notion was introduced in [5] as a natural extension of the classical H^*-algebras of W. Ambrose ([1]). Here we consider the case where the algebra is complete and it is endowed with a continuous involution.

Definition: 1.1 A locally m-convex H^*-algebra (l.m.c. H^*-algebra in short) is a complete l.m.c.$*$-algebra $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ on which is defined a family $(\langle.,.\rangle_\lambda)_{\lambda\in\Lambda}$ of positive semi-definite pseudo-inner products such that the following properties hold for all $x,y,z\in E$ and $\lambda\in\Lambda$:

$(i)\,|x|^2_\lambda=\langle x,x\rangle_\lambda$,

$(ii)\,\langle xy,z\rangle_\lambda=\langle y,x^*z\rangle_\lambda$,

$(iii)\,\langle yx,z\rangle_\lambda=\langle y,zx^*\rangle_\lambda$.

For every $\lambda\in\Lambda$, the quotient space $E_\lambda=E/N_\lambda$ is an inner product space under $\langle x_\lambda,y_\lambda\rangle_\lambda=\langle x,y\rangle_\lambda$. The

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underlying Banach space $\widehat{E}_\lambda$ is a Hilbert space. Moreover, the involutive Banach algebra $(\widehat{E}_\lambda, |.|'_\lambda)$ is an H^*-algebra. The algebra $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ is the inverse limit of the Banach H^*-algebras $(\widehat{E}_\lambda, |.|'_\lambda)$, ([5], Theorem 2.3). So there exists a unique homomorphism $\phi: E \to \lim_{\leftarrow\lambda} \widehat{E}_\lambda$ in which $\chi_\mu o\varphi = \pi_{\lambda,\mu} o\pi_\lambda$, where $|.|_\lambda \geq |.|_\mu$ and $\chi_\mu : \lim_{\leftarrow\lambda}\widehat{E}_\lambda \to \widehat{E}_\mu$ is the natural projection. One can see that φ is an isomorphism and $\lim_{\leftarrow\lambda}\widehat{E}_\lambda \cong E$ (See also the remarks following Satz 1.1 in [7]). One of the most useful consequences of this isomorphism is that every coherent sequence in $\{\widehat{E}_\lambda : \lambda\in\Lambda\}$ determines an element of E.

Given an l.m.c. H^*-algebra $(E, (|.|_\lambda)_{\lambda\in\Lambda})$. Since $*$ is an involution, E is proper, namely $lan(E) = \{0\}$, where $lan(E) = \{x\in E : xE = \{0\}\}$ is the left annihilator of E and so each $\widehat{E}_\lambda$, for every $\lambda\in\Lambda$. The trace class $\tau(E)$ of E is defined as the set $\tau(E) = \{ab : a,b\in E\}$. Clearly, $\tau(E)$ is an ideal of E which is complete $*-$algebra in the topology τ determined by suitable submultiplicative seminorms $\tau_\lambda(.), \lambda\in\Lambda$ related to given seminorms on E by $\tau_\lambda(a^*a) = |a|_\lambda^2$, for all a in E. For every $\lambda\in\Lambda$, there exists a canonical continuous linear form on $\tau(E)$ called the trace$-\lambda$ of E and we denote it by tr_λ which is related with the semi-definite pseudo-inner product $\langle .,.\rangle_\lambda$ of E by $tr_\lambda(ab) = \langle a, b^*\rangle_\lambda$ for all $a,b\in E$. The trace class in the H^*-algebra $\widehat{E}_\lambda, \lambda\in\Lambda$ is defined as the set $\tau(\widehat{E}_\lambda) = \{[a+N_\lambda][b+N_\lambda] : a,b\in E\}$. It is known that $\tau(\widehat{E}_\lambda)$ is an ideal of $\widehat{E}_\lambda$, which is Banach $*$-algebra under a suitable norm $\hat{\tau}_\lambda(.)$. The norm $\hat{\tau}_\lambda$ is related to given norm $|.|'_\lambda$ on $\widehat{E}_\lambda$ by $\hat{\tau}_\lambda([a^*a+N_\lambda]) = |[a+N_\lambda]|'^2_\lambda$, for all $a\in E$. The trace class $\tau(E_\lambda)$ of $E_\lambda, \lambda\in\Lambda$ is defined similarly. Obviously $\tau(E_\lambda)$ is an ideal of E_λ which is norm $*-$algebra under a suitable norm $\tau_\lambda(.)$, in which $\tau_\lambda(a^*a+N_\lambda) = |a+N_\lambda|'^2_\lambda$ for all $a\in E$. For $\lambda\in\Lambda$, there exists a continuous linear form $\widehat{tr}_\lambda$ on $\tau(\widehat{E}_\lambda)$ satisfying $\widehat{tr}_\lambda([a+N_\lambda][b+N_\lambda]) = \widehat{tr}_\lambda([b+N_\lambda][a+N_\lambda]) = \langle a, b^*\rangle_\lambda$. Similarly there exists a continuous linear form tr_λ on the $\tau(E_\lambda), \lambda\in\Lambda$, satisfying $tr_\lambda(ab+N_\lambda) = tr_\lambda(ba+N_\lambda) = \langle a, b^*\rangle_\lambda$.

Now suppose that $\chi_\lambda : \tau(E) \to \tau(E_\lambda)$ defined by $\chi_\lambda(ab) = ab + N_\lambda$ and $\pi_{\lambda,\mu} : \tau(E_\lambda) \to \tau(E_\mu)$ defined by $\pi_{\lambda,\mu}(a+N_\lambda)(b+N_\lambda) = (a+N_\mu)(b+N_\mu)$, where $|.|_\lambda \geq |.|_\mu$. Then $\{\tau(E), \tau; \chi_\lambda\}$ is the inverse limit of the inverse system $\{\tau(E_\lambda), \tau_\lambda; \pi_{\lambda,\mu}, \lambda,\mu\in\Lambda, |.|_\lambda \geq |.|_\mu\}$ and it is also inverse limit of the inverse system $\{\tau(\widehat{E}_\lambda), \hat{\tau}_\lambda; \pi_{\lambda,\mu}, \lambda,\mu\in\Lambda, |.|_\lambda \geq |.|_\mu\}$.

In this paper we will intoduce a Hilbert module W over an l.m.c. H^*-algebra $(E, (|.|_\lambda)_{\lambda\in\Lambda})$ and for each $\lambda\in\Lambda$ we will associate a Hilbert module $\widehat{W}_\lambda$ over an H^*-algebra $\widehat{E}_\lambda$. We shall see that $W \cong \lim_{\leftarrow\lambda}\widehat{W}_\lambda$. Then we will discuss about orthonormal bases in these spaces. Also we will topologize $L_E(V,W)$ and $B_E(V,W)$, the set of adjointable $E-$linear operators and the set of bounded $E-$linear operators from Hilbert $E-$module V into Hilbert $E-$module W, respectively, via suitable families of seminorms. Throughout this paper E is an l.m.c. H^*-algebra and W is a Hilbert $E-$module except some results about unitary operators in Hilbert H^*-modules at the end of the paper. The paper is organized as follows.

In section 2 we will introduce Hilbert modules over l.m.c. H^*-algebras and their properties are studied. The existence of orthonormal bases in these spaces is proved.

In section 3 we topologize the space of bounded $E-$linear operators and the space of all adjointable $E-$linear operators. Also, more properties of these spaces are detected.

2. Hilbert modules over l.m.c. H^*-algebras and orthonormal bases

Hilbert modules over l.m.c. H^*-algebras generalize the notion of Hilbert H^*-modules by allowing the $\tau(E)-$valued product in an l.m.c. H^*-algebra.

Definition: 2.1 Let $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ be a Hausdorff l.m.c. H^*-algebra. A pre-Hilbert E-module is a left module W over E provided with a mapping

$[.|.]:W\times W\to\tau(E)$ (called $\tau(E)-$valued product) where $\tau(E)=\{ab:a,b\in E\}$ which satisfies the following conditions:

$(i)\,[\alpha x\mid y]=\alpha[x\mid y]\,\forall\alpha\in C,\forall x,y\in W,$
$(ii)\,[x+y\mid z]=[x\mid z]+[y\mid z]\,\forall x,y,z\in W,$
$(iii)\,[ax\mid y]=a[x\mid y],\forall a\in E,\forall x,y\in W,$
$(iv)\,[x\mid y]^*=[y\mid x]\,\forall x,y\in W,$
$(v)\,\forall x\in W,x\neq 0,\exists a\in E,a\neq 0,$ such that $[x\mid x]=a^*a,$
(vi) for each $\lambda\in\Lambda,W$ is a semi-definite pseudo-inner product space with $(x,y)_\lambda=\langle a,b^*\rangle_\lambda$ (or $tr_\lambda(ab)$) where $[x\mid y]=ab\in\tau(E)$.

We say that W is a Hilbert $E-$module if it is complete with respect to the topology determined by the family of seminorms $\|x\|_\lambda=\sqrt{(x,x)_\lambda}, x\in W,\lambda\in\Lambda$.
Given a Hilbert $E-$module W, then for $\lambda\in\Lambda$, $\xi_\lambda=\{x\in W:\|x\|_\lambda=0\}$ is a closed submodule of W. Indeed, for non zero x in ξ_λ and $a\in E$, if $[x\mid x]=b^*b$ for some non zero $b\in E$, then $\|ax\|_\lambda=|ab^*|_\lambda\leq|a|_\lambda|b|_\lambda=0$. For $\lambda\in\Lambda, W_\lambda=W/\xi_\lambda$ is an inner product space with $(x,y)_\lambda=tr_\lambda[x+\xi_\lambda\mid y+\xi_\lambda]$ (or $\langle a,b^*\rangle_\lambda$ where $[x\mid y]=ab$) and its completion $\widehat{W}_\lambda$ is a Hilbert space.

Example: 2.2 Let $(E,(|.|_\lambda)_{\lambda\in\Lambda})$ be an l.m.c. H^*-algebra. Then E is a Hilbert module over itself with $\tau(E)-$valued product defined by $[a\mid b]=ab^*$. Also it is easy to verify that a closed submodule of a Hilbert $E-$module is again a Hilbert $E-$module. Note that analogue of Lemma 2.2 of $[1]$ holds for l.m.c. H^*-algebras. More precisely if x is a non zero element in an l.m.c. H^*-algebra E, then x^*x, xx^*, x^* are also non zero. The proof of the following proposition can be based on the direct application of previous comments about Hilbert $E-$module W.

Proposition: 2.3 Let W be a Hilbert module over an l.m.c. H^*-algebra $(E,(|.|_\lambda)_{\lambda\in\Lambda})$. For each $\lambda\in\Lambda,\widehat{W}_\lambda$ is a Hilbert module over the proper H^*-algebra $\widehat{E}_\lambda$ with $\pi_\lambda(a)(x+\xi_\lambda)=ax+\xi_\lambda$ and $[x+\xi_\lambda\mid y+\xi_\lambda]=\pi_\lambda([x\mid y])$ for every $a\in E$ and for every $x,y\in W$. Let σ_λ^W be the canonical map from W onto $\widehat{W}_\lambda,\lambda\in\Lambda$. For $\lambda_1,\lambda_2\in\Lambda,|.|_{\lambda_1}\geq|.|_{\lambda_2}$, there is a canonical surjective linear map $\sigma_{\lambda_1\lambda_2}^W:\widehat{W}_{\lambda_1}\to\widehat{W}_{\lambda_2}$ such that $\sigma_{\lambda_1\lambda_2}^W(\sigma_{\lambda_1}^W(x))=\sigma_{\lambda_2}^W(x)$. Also $\{\widehat{W}_\lambda,\widehat{E}_\lambda;\sigma_{\lambda_1\lambda_2}^W,|.|_{\lambda_1}\geq|.|_{\lambda_2},\lambda_1,\lambda_2\in\Lambda\}$ is an inverse system of Hilbert H^*-modules in the following sense:

$$\sigma_{\lambda_1\lambda_2}^W(\pi_{\lambda_1}(a)\sigma_{\lambda_1}^W(x))=\pi_{\lambda_1\lambda_2}(\pi_{\lambda_1}(a))\sigma_{\lambda_1\lambda_2}^W((\sigma_{\lambda_1}^W(x))$$

and

$$(\sigma_{\lambda_1\lambda_2}^W((\sigma_{\lambda_1}^W(x)),\sigma_{\lambda_1\lambda_2}^W((\sigma_{\lambda_1}^W(y)))_{\lambda_2}=\langle\pi_{\lambda_1\lambda_2}(\pi_{\lambda_1}(a)),\pi_{\lambda_1\lambda_2}(\pi_{\lambda_1}(b^*))\rangle_{\lambda_2}$$

Where $[x\mid y]=ab,$ for every $x,y\in W$ and for every $a\in E$;

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$\sigma^W_{\lambda_2\lambda_3}\sigma^W_{\lambda_1\lambda_2} = \sigma^W_{\lambda_1\lambda_3}, |.|_{\lambda_1} \geq |.|_{\lambda_2} \geq |.|_{\lambda_3}; \sigma^W_{\lambda\lambda} = id_{W_\lambda}$, $\lim_{\leftarrow\lambda}\widehat{W}_\lambda$ is a Hilbert $E-$module with

$$(\pi_\lambda(a))_\lambda(\sigma^W_\lambda(x))_\lambda = (\sigma^W_\lambda(ax))_\lambda,$$

and

$$\langle(\sigma^W_\lambda(x))_{\lambda\in\Lambda}, (\sigma^W_\lambda(y))_{\lambda\in\Lambda}\rangle = (\langle\pi_\lambda(a), \pi_\lambda(b^*)\rangle_\lambda)_{\lambda\in\Lambda}$$

Where $[x\,|\,y] = ab$. Indeed, $\lim_{\leftarrow\lambda}\widehat{W}_\lambda$ maybe identified with W. So every coherent sequence in $\{\widehat{W}_\lambda : \lambda\in\Lambda\}$ determines an element of W.

For the basic facts about Hilbert H^*-modules we refer to [2] and [4]. In particular with the assumption of the previous proposition, we have the following three relations in Hilbert $\widehat{E}_\lambda-$module $\widehat{W}_\lambda$ ($\lambda\in\Lambda$),

$$\| x+\xi_\lambda \|^2_\lambda = \widehat{tr}_\lambda([x+\xi_\lambda \,|\, x+\xi_\lambda]) = \hat{\tau}_\lambda([x+\xi_\lambda \,|\, x+\xi_\lambda]).$$

$$|[x+\xi_\lambda \,|\, y+\xi_\lambda]|'_\lambda \leq \hat{\tau}_\lambda([x+\xi_\lambda \,|\, y+\xi_\lambda]) \leq \| x+\xi_\lambda \|_\lambda \| y+\xi_\lambda \|_\lambda .$$

$$\| (a+N_\lambda)(x+\xi_\lambda) \|_\lambda \leq | a+N_\lambda |'_\lambda \| x+\xi_\lambda \|_\lambda .$$

As an immediate consequence of the above relations and the previous comments we obtain:

$$\| x \|^2_\lambda = tr_\lambda([x\,|\,x]) = \tau_\lambda([x\,|\,x]).$$

$$|[x\,|\,y]|'_\lambda \leq \tau_\lambda([x\,|\,y]) \leq \| x \|_\lambda \| y \|_\lambda .$$

$$\| ax \|_\lambda \leq | a |'_\lambda \| x \|_\lambda .$$

Proposition: 2.4. Let W be a Hilbert module over an l.m.c. H^*-algebra E and $b(E) = \{a\in E : \| a \|_\infty = sup_\lambda | a |_\lambda < \infty\}$ and $b(W) = \{x\in W : \| x \|_\infty = sup_\lambda \| x \|_\lambda < \infty\}$. Then $b(E)$ is an H^*-algebra and $b(W)$ is a $b(E)-$Hilbert module.

Proof: Clearly, the sets $b(E)$ and $b(W)$ are complex vector spaces and $b(W)$ is a left $b(E)-$module. Because, when W is identified with $\lim_{\leftarrow\lambda}\widehat{W}_\lambda$, we see that $b(W)$ corresponds to the set of bounded coherent sequences. The Cauchy-Schwarz inequality, applied to Hilbert $\widehat{E}_\lambda-$module $\widehat{W}_\lambda$, yields for $x, y\in b(W)$, the inequality $\| (x,y) \|^2_\infty \leq \| (x,x) \|_\infty \| (y,y) \|_\infty$, so that the restriction of $b(W)$ of the $\tau(E)-$valued product on W is a $\tau(b(E))-$valued product on $b(W)$. Obviously, $(b(E), (\langle.,.\rangle_\lambda |_{b(E)\times b(E)})_{\lambda\in\Lambda})$ and $(b(W), [.,.] |_{b(W)\times b(W)})$ take more properties being as a subset of E and W respectively. To proof of completeness in [7], Satz 3.1, also applied here and show that $b(E)$ and $b(W)$ are complete for norm $\| a \|_\infty = \| \langle a,a\rangle \|^{\frac{1}{2}}_\infty$ and $\| x \|_\infty = \| (x,x) \|^{\frac{1}{2}}_\infty$, respectively, for every $a\in E, x\in W$. Q.E.D.

Definition: 2.5 A non zero projection e in an l.m.c. H^*-algebra E is called minimal, if $eEe = Ce$. Also element u in a Hilbert $E-$module W is said to be a basic element if there exists a minimal projection $e\in E$ such that $[u\,|\,u] = e$. An orthonormal system in W is a family of basic elements $\{u_\alpha\}_\alpha, \alpha\in I$ satisfying $[u_\alpha \,|\, u_\beta] = 0$ for all $\alpha, \beta\in I, \alpha\neq\beta$. An orthonormal basis in W is an orthonormal system generating a dense submodule of W.

If V is a subset of a Hilbert $E-$module W, we define $V^\perp = \{w\in W \,|\, [w\,|\,v] = 0\ \forall v\in V\}$. Clearly $V^\perp$ is a closed submodule of W. If V is a submodule of W then $V^\perp = \{x\in W \,|\, (x,v)_\lambda = 0, \forall v\in V, \forall\lambda\in\Lambda\}$.

Our next result is a generalization of Lemma 1.3 in [4].

Lemma 2.6. Let W be a Hilbert $E-$module and let u in W be such that $e=[u\mid u]$ is an idempotent in E. If the closed submodule generated by u is complemented or e does not belong to N_λ, for each $\lambda\in\Lambda$ then $[w\mid u]=[w\mid u]e$ for all $w\in W$.

We can also generalized Corollary 1.4 of [4] to Hilbert modules over l.m.c. H^*-algebras.

Corollary: 2.7 If $\{u_\alpha\}_{\alpha\in I}$ is an orthonormal system in a Hilbert $E-$module W in which for every $\alpha\in I$ the closed submodule generated by u_α is complemented or u_α does not belong to ξ_λ, for each $\lambda\in\Lambda$, then

$$[w-\sum_{\alpha\in J}[w\mid u_\alpha]u_\alpha \mid w-\sum_{\alpha\in J}[w\mid u_\alpha]u_\alpha]=[w\mid w]-\sum_{\alpha\in J}[w\mid u_\alpha][w\mid u_\alpha]^*$$

for every finite subset J of I and for all w in W.

Our next result is a generalization of Proposition 1.5 of [4].

Proposition: 2.8 Let W be a Hilbert $E-$module, let u be a basic element in W, and let M denote the closed submodule of W generated by u. If M is orthogonally complemented in W then the mapping $w\to[w\mid u]u$ is the orthogonal projection from W onto M. As a consequence we have $M=Eu$.

Our next result is a generalization of Theorem 1.6 of [4] which provides a very useful characterization of orthonormal bases in a Hilbert $E-$module W.

Theorem: 2.9 Let $\{u_\alpha\}_{\alpha\in I}$ be an orthonormal system in a Hilbert $E-$module W in which for every $\alpha\in I$, the closed submodule generated by u_α is complemented, then the following statement are equivalent.

(i) For all w_1,w_2 in W the family $\{[w_1\mid u_\alpha][w_2\mid u_\alpha]^*\}_{\alpha\in I}$ is summable in the space $(\tau(E),(\tau_\lambda)_{\lambda\in\Lambda})$, with sum equale to $[w_1\mid w_2]$.

(ii) For every w in W, we have $[w\mid w]=\sum_{\alpha\in I}[w_1\mid u_\alpha][w_2\mid u_\alpha]^*$ (Parseval's identity) in the space $(\tau(E),(\tau_\lambda)_{\lambda\in\Lambda})$.

(iii) For every w in W, we have $w=\sum_{\alpha\in I}[w\mid u_\alpha]u_\alpha$ (Fourier expansion).

(iv) $\{u_\alpha\}_{\alpha\in I}$ is an orthonormal basis in W.

Remark: 2.10 Parseval's identity leads to the equality $\sum_{\alpha\in I}|[w\mid u_\alpha]|_\lambda^2=\|w\|_\lambda^2$ for every $\lambda\in\Lambda$. Indeed,

$$\sum_{\alpha\in I}|[w\mid u_\alpha]|_\lambda^2=\sum_{\alpha\in I}tr_\lambda([w\mid u_\alpha][w\mid u_\alpha]^*)=tr_\lambda(\sum_{\alpha\in I}([w\mid u_\alpha][w\mid u_\alpha]^*))=tr_\lambda([w\mid w])=\|w\|_\lambda^2.$$

The existence of basic elements in a Hilbert module over an l.m.c. H^*-algebra can be guaranteed by an argument similar to Proposition 1.7 of [4]. A slightly modification of Theorem 1.9 of [4] gives the following theorem in a Hilbert $E-$module.

Theorem: 2.11 Let S be a subset of a Hilbert $E-$module W. If S is a maximal orthonormal system then it is an orthonormal basis in W and converse is true when each closed submodule generated by every element of S is complemented in W.

Proof: Let S be a maximal orthonormal system in W and let M be the closed submodule generated by S. If $M\neq W$, then there exists $x\in W-M$. It implies that $\|x\|_{\lambda_0}\neq 0$, for some $\lambda_0\in\Lambda$. Now if $M^\perp=\{0\}$ then $(M+\xi_{\lambda_0})^\perp\subseteq M^\perp=\{0\}$. Hence in the Hilbert $\widehat{E}_{\lambda_0}-$module $\widehat{W}_{\lambda_0}$, we have $M+\xi_{\lambda_0}=W+\xi_{\lambda_0}$ which is a contradiction. So $M^{\perp} \neq \{0\}$ and therefore there exists a basic element u in $M^{\perp}$. Obviously $S \cup \{u\}$ is an orthonormal system strictly containing S, which is a contradiction. The converse is obvious by Fourier expansion. Q.E.D.

Corollary: 2.12 Every non zero Hilbert module over an l.m.c. H^*-algebra has an orthonormal basis.

Theorem: 2.13. In a Hilbert $E-$module $W, \{v_\alpha\}_{\alpha \in I}$ is an orthonormal system if and only if $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in the Hilbert $\widehat{E}_\lambda-$module $\widehat{W}_\lambda$ when v_α does not belong to ξ_λ for each $\alpha \in I$ and for each $\lambda \in \Lambda$. Also if $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W and v_α does not belong to ξ_λ for each $\alpha \in I$ then $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal basis in the Hilbert $\widehat{E}_\lambda-$module $\widehat{W}_\lambda$. So we have an analogue of Fourier expansion and Parseval's identity associated to this orthonormal basis in Hilbert $\widehat{E}_\lambda-$module $\widehat{W}_\lambda$.

Proof: Suppose that v is a basic element in W. If for some $\lambda_0 \in \Lambda, v \in \xi_{\lambda_0}$, then $v + \xi_{\lambda_0}$ is not a basic element in $\widehat{W}_{\lambda_0}$. It is clear that for $\mu \in \Lambda$ in which, $|.|_{\lambda_0} \geq |.|_\mu, v \in \xi_\mu$ and $v + \xi_\mu$ is not a basic element in $\widehat{W}_\mu$. Now if v does not belong to $\xi_\lambda, \lambda \in \Lambda$ then we have

$$[v+\xi_\lambda \mid v+\xi_\lambda]\widehat{E}_\lambda[v+\xi_\lambda \mid v+\xi_\lambda] = \pi_\lambda([v \mid v])\widehat{E}_\lambda\pi_\lambda([v \mid v])$$

$$= \pi_\lambda([v \mid v]E[v \mid v])$$

$$= \pi_\lambda(C[v \mid v])$$

$$= C[v+\xi_\lambda \mid v+\xi_\lambda].$$

Hence $v + \xi_\lambda$ is a basic element in $\widehat{W}_\lambda$. Conversely, if for each $\lambda \in \Lambda, v + \xi_\lambda$ is a basic element in $\widehat{W}_\lambda$ then we have

$$[v+\xi_\lambda \mid v+\xi_\lambda]\widehat{E}_\lambda[v+\xi_\lambda \mid v+\xi_\lambda] = C[v+\xi_\lambda \mid v+\xi_\lambda]$$

and this implies that

$$\pi_\lambda([v \mid v]E[v \mid v] - C[v \mid v]) = 0,$$

for every $\lambda \in \Lambda$. So $[v \mid v]E[v \mid v] - C[v \mid v] \subseteq \widehat{N}_\lambda$ for every $\lambda \in \Lambda$ and therefore $[v \mid v]E[v \mid v] - C[v \mid v] = 0$. It is easy to verify that, $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal system in W if and only if $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in $\widehat{W}_\lambda$, when v_α does not belong to ξ_λ for each $\alpha \in I$ and for each $\lambda \in \Lambda$. Now suppose that $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W and for each $\alpha \in I, v_\alpha$ does not belonge to ξ_λ then as we mentioned before, $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal system in $\widehat{W}_\lambda$. We are going to show that it generates a dense submodule of $\widehat{W}_\lambda$. For this, let $x + \xi_\lambda \in \widehat{W}_\lambda$. Since $\{v_\alpha\}_{\alpha \in I}$ is an orthonormal basis in W, So we have

$$\exists \gamma_{i_k} \in C, \exists v_{\alpha_{i_k}} \in \{v_\alpha\}_{\alpha \in I}; k \in IN, \sum_{k=1}^{n} \gamma_{i_k} v_{\alpha_{i_k}} \to x,$$

as n tends to ∞. It implies that

$$\sum_{k=1}^{n} \gamma_{i_k} (v_{\alpha_{i_k}} + \xi_\lambda) \to x + \xi_\lambda,$$

when n tends to ∞. Thus if for all $\alpha \in I, v_\alpha$ does not belong to ξ_λ then $\{v_\alpha + \xi_\lambda\}_{\alpha \in I}$ is an orthonormal basis in $\widehat{W}_\lambda$. Q.E.D.

3. Space of bounded operators

Definition: 3.1 Let V and W be two Hilbert modules over an l.m.c. H^*-algebra $(E,(|.|_\lambda)_{\lambda\in\Lambda})$. An operator $T:V\to W$ is called $E-$linear if it is linear and satisfies $T(ax)=aT(x)$ for all $a\in E$ and for all $x\in V$. We say that $E-$linear operator T is bounded if for each $\lambda\in\Lambda,$ and for each $x\in V$ there exists $K_\lambda>0$ in which $\|T(x)\|_\lambda\le K_\lambda\|x\|_\lambda$.

Put $\overline{P}_\lambda^V(x)=(x,x)_\lambda^{\frac{1}{2}}$ and $\overline{P}_\lambda^W(Tx)=(Tx,Tx)_\lambda^{\frac{1}{2}}$, where $(x,x)_\lambda$ and $(Tx,Tx)_\lambda$ are denoted positive semi-definite pseudo-inner products in V and W respectively. So the $E-$linear operator T is bounded if for each $\lambda\in\Lambda,$ and for each $x\in V$ there exists $K_\lambda>0$ in which $\overline{P}_\lambda^W(Tx)\le K_\lambda\overline{P}_\lambda^V(x)$.

The set of all bounded $E-$linear operators from Hilbert module V into W is denoted by $B_E(V,W)$ and when $V=W$ is denoted by $B_E(V)$. It is easy to see that the map $\widetilde{P}_\lambda,\lambda\in\Lambda,$ defined by $\widetilde{P}_\lambda(T)=sup\{\overline{P}_\lambda^W(Tx):x\in V,\overline{P}_\lambda^V(x)\le 1\}$ is a seminorm on $B_E(V,W)$.

Theorem: 3.2 Let V and W be Hilbert modules over an l.m.c. H^*-algebra $(E,(|.|_{\lambda\in\Lambda}))$. Then

(i) $B_E(V,W)$ is a complete locally convex space with topology determined by the family of seminorms $\{\widetilde{P}_\lambda\}_{\lambda\in\Lambda}$.

(ii) $B_E(V)$ is a locally C^*- algebra with the topology determined by the family of seminorms $\{\widetilde{P}_\lambda\}_{\lambda\in\Lambda}$.

Proof: Suppose that $\lambda_1,\lambda_2\in\Lambda,|.|_{\lambda_1}\ge|.|_{\lambda_2}, S\in B_{\widehat{E}\lambda_1}(\widehat{V}_{\lambda_1},\widehat{W}_{\lambda_1}),$ set of bounded $\widehat{E}_{\lambda_1}-$linear operators. We have

$$(\sigma^{\widehat{W}}_{\lambda_1\lambda_2}(S(\sigma^{\widehat{V}}_{\lambda_1}(x))),\sigma^{\widehat{W}}_{\lambda_1\lambda_2}(S(\sigma^{\widehat{V}}_{\lambda_1}(x)))_{\lambda_2}\le\|S\|^2_{\lambda_1}(\sigma^{\widehat{V}}_{\lambda_2}(x),\sigma^{\widehat{V}}_{\lambda_2}(x))_{\lambda_2}.$$

Therefore the following map is a bounded $\widehat{E}_{\lambda_2}$-bounded.

$$(\pi_{\lambda_1\lambda_2})_*(S):\widehat{V}_{\lambda_2}\to\widehat{W}_{\lambda_2}$$

$$\sigma^{\widehat{V}}_{\lambda_2}(x)\mapsto\sigma^{\widehat{W}}_{\lambda_1\lambda_2}(S(\sigma^{\widehat{V}}_{\lambda_1}(x))).$$

So we yield a bounded operator $(\pi_{\lambda_1\lambda_2})_*$ from $B_{\widehat{E}\lambda_1}(\widehat{V}_{\lambda_1},\widehat{W}_{\lambda_1})$ into $B_{\widehat{E}\lambda_2}(\widehat{V}_{\lambda_2},\widehat{W}_{\lambda_2})$. Also $\{B_{\widehat{E}\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda),(\pi_{\lambda_1\lambda_2})_*;|.|_{\lambda_1}\ge|.|_{\lambda_2},\lambda_1,\lambda_2\in\Lambda\}$ is an inverse system of Banach spaces. We are going to show that $B_E(V,W)$ and $\lim_{\leftarrow\lambda}B_{\widehat{E}\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$ are isomorphic. Suppose that $\lambda\in\Lambda,T\in B_E(V,W)$. One can see that, $T(\xi^V_\lambda)\subseteq\xi^W_\lambda$ and so there exists a unique operator $T_\lambda:V_\lambda\to W_\lambda$ in which $\sigma^W_\lambda oT=T_\lambda o\sigma^V_\lambda$. Moreover T_λ is a bounded $E_\lambda-$linear operator. It has a continuous extension $\widehat{T}_\lambda:\widehat{V}_\lambda\to\widehat{W}_\lambda$. Thus we can define the following continuous linear operator

$$(\pi_\lambda)_*:B_E(V,W)\to B_{\widehat{E}\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$$

$$T\mapsto\widehat{T}_\lambda$$

where $\sigma^{\widehat{W}}_\lambda oT=\widehat{T}_\lambda o\sigma^{\widehat{V}}_\lambda$. Also, for $\lambda_1,\lambda_2\in\Lambda,|.|_{\lambda_1}\ge|.|_{\lambda_2}$ we have $(\pi_{\lambda_1\lambda_2})_*o(\pi_{\lambda_1})_*=(\pi_{\lambda_2})_*$. Now we can define the following isomorphism operator

$$\phi:B_E(V,W)\to\lim_{\leftarrow\lambda}B_{\widehat{E}\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$$

$$\phi(T)=((\pi_\lambda)_*(T))_\lambda.$$

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For each $T\in B_E(V,W), \|\phi(T)\|_\lambda=\widetilde{P}_\lambda(T)$. Linear operator ϕ is surjective. Indeed, let $([T_\lambda])_{\lambda\in\Lambda}\in \lim_{\leftarrow\lambda} B_{\widehat{E}_\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$. We define linear operator T as follows

$$T: V\to W$$

$$x\mapsto (\widehat{T}_\lambda(\sigma_\lambda^{\widehat{V}}(x)))_\lambda.$$

For $\lambda_1,\lambda_2\in\Lambda$ that $|.|_{\lambda_1}\geq |.|_{\lambda_2}$ we have

$$\sigma_{\lambda_1\lambda_2}^{\widehat{V}}(\widehat{T}_{\lambda_1}(\sigma_{\lambda_1}^{\widehat{V}}(x)))=(\pi_{\lambda_1\lambda_2})_*(\widehat{T}_{\lambda_1})(\sigma_{\lambda_2}^{\widehat{V}}(x))=\widehat{T}_{\lambda_2}(\sigma_{\lambda_2}^{\widehat{V}}(x)).$$

So T is well-defined. Also it is a bounded $E-$module map and $\phi(T)=((\pi_\lambda)_*(T))_{\lambda\in\Lambda}$. Completeness of $\lim_{\leftarrow\lambda} B_{\widehat{E}_\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$ implies that $B_E(V,W)$ is complete.

For $\lambda\in\Lambda, \widetilde{P}_\lambda$ is a submultiplicative seminorm on $B_E(V)$ and $\{B_{\widehat{E_\lambda}}(\widehat{V_\lambda}),(\pi_{\lambda_1\lambda_2})_*; |.|_{\lambda_1}\geq |.|_{\lambda_2}, \lambda_1,\lambda_2\in\Lambda\}$ is an inverse system of C^*-algebras and linear operator

$$\widetilde{\phi}: B_E(V)\to \lim_{\leftarrow\lambda} B_{\widehat{E}_\lambda}(\widehat{V}_\lambda)$$

$$T\mapsto ((\pi_\lambda)_*(T))_\lambda$$

is an isomorphism of topological algebras. Also, $\|\widetilde{\phi}(T)\|_\lambda=\widetilde{P}_\lambda(T)$ and since $B_{\widehat{E}_\lambda}(\widehat{V_\lambda})$'s are C^*-algebras, so $B_E(V)$ is a locally C^*-algebra. Q.E.D.

Definition: 3.3 We say that $E-$linear operator T has an adjoint if there exists $E-$linear operator $T^*: W\to V$ in which $[Tx\mid y]=[x\mid T^*y]$ for each $x\in V$ and $y\in W$. The set of adjointable $E-$linear operators from Hilbert $E-$module V into Hilbert $E-$module W is denoted by $L_E(V,W)$ and for each $\lambda\in\Lambda,$ the set of adjointable operators from V_λ into W_λ is denoted by $L_{E_\lambda}(V_\lambda,W_\lambda)$. Let $T\in L_E(V,W)$. For each $\lambda\in\Lambda,$ since $T(\xi_\lambda^v)\subseteq \xi_\lambda^W$, we can define

$$(\pi_\lambda)_*: L_E(V,W)\to L_{E_\lambda}(V_\lambda,W_\lambda)$$
$$(\pi_\lambda)_*(T)(x+\xi_\lambda^V)=T(x)+\xi_\lambda^W,$$

Obviously $(\pi_\lambda)_*(T)\in L_{E_\lambda}(V_\lambda,W_\lambda)$ and $\widetilde{|T|_\lambda}=\|(\pi_\lambda)_*(T)\|_{L_{E_\lambda}(V_\lambda,W_\lambda)}$ defines a seminorm on $L_E(V,W)$, where $\|.\|_{L_{E_\lambda}(V_\lambda,W_\lambda)}$ is the operator norm in $L_{E_\lambda}(V_\lambda,W_\lambda)$.

We topologize $L_E(V,W)$ via these seminorms. By similar argument just like previous theorem $L_E(V,W)$ may be identified with $\lim_{\leftarrow\lambda} L_{\widehat{E}_\lambda}(\widehat{V}_\lambda,\widehat{W}_\lambda)$. In particular $L_E(V)\cong \lim_{\leftarrow\lambda} L_{\widehat{E}_\lambda}(\widehat{V}_\lambda)$ and we conclude that $L_E(V)$ is a locally C^*-algebra. The connecting maps of the inverse system $\{L_{E_\lambda}(V_\lambda,W_\lambda)\}_{\lambda\in\Lambda}$ will be denoted by $(\pi_{\lambda_1\lambda_2})_*, \lambda_1,\lambda_2\in\Lambda, |.|_{\lambda_1}\geq |.|_{\lambda_2}$, where

$$(\pi_{\lambda_1\lambda_2})_*: L_{E_{\lambda_1}}(V_{\lambda_1},W_{\lambda_1})\to L_{E_{\lambda_2}}(V_{\lambda_2},W_{\lambda_2})$$
$$(\pi_{\lambda_1\lambda_2})_*(T)(x+\xi_{\lambda_2}^V)=\sigma_{\lambda_1\lambda_2}^W(T(x+\xi_{\lambda_1}^V)).$$

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So $\{L_{E_\lambda}(V_\lambda, W_\lambda); (\pi_{\lambda_1\lambda_2})_*\}_{|.|_{\lambda_1} \geq |.|_{\lambda_2}}$ is an inverse system of normed spaces and $\{L_{\widehat{E_\lambda}}(\widehat{V_\lambda}, \widehat{W_\lambda}); (\pi_{\lambda_1\lambda_2})_*\}_{|.|_{\lambda_1} \geq |.|_{\lambda_2}}$ is an inverse system of Banach spaces. Also,

$$L_E(V,W) \cong \lim_{\leftarrow\lambda} L_{\widehat{E_\lambda}}(\widehat{V_\lambda}, \widehat{W_\lambda}).$$

So $L_E(V,W)$ is a complete locally convex space. On the other hand by [2] each $T \in B_{\widehat{E_\lambda}}(\widehat{V_\lambda})$ belongs to $L_{\widehat{E_\lambda}}(\widehat{V_\lambda})$. From this we obtain that each $T \in B_E(V)$ belongs to $L_E(V)$.

Definition: 3.4 Let W be a Hilbert module over an l.m.c. H^* −algebra, E. Let $v, w \in W$ be basic vectors and let the operator $F_{v,w}: W \to W$ be defined with $F_{v,w}(x) = [x \mid w]v$. The linear span of the set $\{F_{v,w} : v, w \in W\}$ is denoted by $F_E(W)$ and an operator T belonging to $F_E(W)$ is called a generalized finite rank operator. Observe that $F_E(W) \subseteq B_E(W)$ and $F_{v,w}^* = F_{w,v}, TF_{v,w} = F_{Tv,w}, F_{v,w}T = F_{v,T^*w}$, for each $v, w \in W$, for each $T \in B_E(W)$. Therefore $F_E(W)$ is a selfadjoint two-sided ideal in $B_E(W)$.

Definition: 3.5 An operator $T \in B_E(W)$ is said to be a generalized compact operator if there exists a sequence of generalized finite rank operators $\{F_n\}$ such that $\lim_n F_n = T$. The set of all generalized compact operators is denoted by $K_E(W)$. By definition $K_E(W) = \overline{F_E(W)}$ is a closed two-sided ideal in $B_E(W)$. Moreover, $K_E(W)$ may be identified with $\lim_{\leftarrow\lambda} K_{\hat{E}\lambda}(\widehat{W}\lambda)$.

We terminate with a result about unitary operators in Hilbert H^* −modules.

Definition: 3.6 Let E be a proper H^* −algebra. We say that Hilbert E− modules V and W are unitary equivalent if there is a unitary element U in $L_E(V,W)$, namely, $UU^* = id_W$ and $U^*U = id_V$.

If $U \in L_E(V,W)$ is unitary then it is clear that U is a surjective E− linear map and also that U is isometric,

since $\| U(x) \|^2 = tr[U(x) \mid U(x)] = tr[U^*U(x) \mid x] = tr[x \mid x] = \| x \|^2$. Our next result will be the converse assertion, that if $U : E \to F$ is an isometric, surjective E− linear map then U is unitary. For this we need the following lemma.

Lemma: 3.7 Let E be a proper H^* −algebra and $a \in E$. If $\| ac \| = \| bc \|$ for each $c \in E$ then $a^*a = b^*b$.

Proof: We have $\| ac \|^2 = \| bc \|^2$, so that $\langle ac, ac \rangle = \langle bc, bc \rangle$ and $\langle a^*ac, c \rangle = \langle b^*bc, c \rangle$.

Hence $\langle (a^*a - b^*b)c, c \rangle = 0$ for each $c \in E$. From this and by Lemma 3.1 of [1] we have

$$\sup_{\|c\|=1} \| a^*a - b^*b)c \| = \sup_{\|c\|=1} | \langle (a^*a - b^*b)c, c \rangle | = 0.$$

Thus $\| (a^*a - b^*b)c \| = 0$, where $\| c \| = 1$, so that $(a^*a - b^*b)c = 0$ for arbitrary $c \in E$. Therefore

$(a^*a - b^*b)E = 0,$ so that $a^*a - b^*b = 0$. Q.E.D.

Proposition: 3.8 With E, V, W as before, let U be an E− linear map from V to W. The following conditions are equivalent:

(i) U is an isometric surjective E− linear map;

(ii) U is a unitary element of $L_E(V,W)$.

Proof. Suppose that (i) holds. For x in V, $[U(x)|U(x)]=b^*b$ and $[x|x]=c^*c$ for some b,c in E. For each a in E, we have

$$\begin{aligned}\|ab^*\|^2 &= tr(a[U(x)|U(x)]a^*)\\ &= tr([U(ax)|U(ax)])\\ &= \|U(ax)\|^2\\ &= \|ax\|^2\\ &= tr([ax|ax])\\ &= tr(a[x|x]a^*)\\ &= tr(a(c^*c)a))\\ &= \|ac^*\|^2.\end{aligned}$$

Thus $\|ba^*\|=\|ca^*\|$ for each a in E. By previous lemma $b^*b=c^*c$. This implies that $[U(x)|U(x)]=[x|x]$ for each x in E and by polarization identity $[U(x)|U(y)]=[x|y]$ for each x,y in E.

Now let $x\in V$ and $z\in W$. Since U is surjective, there is a $y\in V$ such that

$U(y)=z$. We have $[U(x)|z]=[U(x)|U(y)]=[x|y]=[x|U^{-1}(z)]$.

Hence $U^*=U^{-1}$. This implies that U satisfies (ii); and the implication $(ii)\Rightarrow(i)$ is obvious as already discussed. Q.E.D.

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