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STABILITY ANALYSIS FOR RECURRENT NEURAL NETWORKS WITH TIME-VARYING DELAYS

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ABSTRACT

In this paper global asymptotic stability analysis of static recurrent neural networks with time-varying delay is studied by the LMI approach. Firstly, a novel Lyapunov functional is introduced, which involves the integral terms of the neuron state. Furthermore, a new technique is applied when estimating the upper bound of the derivative of the Lyapunov functional. Based on this, some less conservative criteria are obtained for the concerned static neural networks.Throughout this paper, R^n and $R^{n\times n}$ denote the n-dimension Euclidean space and set of all $n \times n$ real matrices, respectively. A real symmetric matrix $P > 0$ (≥ 0) denotes P being a positive definite matrix. I is used to denote an identity matrix with proper dimensions. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The symmetric term in a symmetric matrix are denoted by $*$.

Keywords: *Stability Analysis, Recurrent Neural Networks, Time-Varying Delays.*

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INTRODUCTION:

In the past few decades, recurrent neural networks (RNNs) have found many successful applications in signal processing, image processing, pattern classification, realizing associative memories, solving certain optimization problems and so on. Because the integration and communication delays are unavoidably encountered in RNNs, often constituting a source of instability and oscillations, considerable attention has been focused on the stability problem of neural networks with time delays. The stabilities can be classified into two types: delay-independent stability and delay dependent stability. Since delay-dependent criteria make use of information on the size of delay, they are generally less conservative than delay-independent ones especially when the delay is small in size.

Neural networks can be classified as static neural networks or local field neural networks. Nowadays, many results have been obtained for the local field neural networks. For example, several criteria are proposed to deal with the exponential stability analysis. The linear matrix inequality (LMI) technique is developed to derive the criteria, which can guarantee the globally asymptotic stability of the static neural networks with time-varying delays. Less conservative results have been established based on a Lyapunov functional. However, there is some conservatism in these analysis results, and it is necessary to make further investigation into static neural networks.

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SOME RESULTS FROM THE THEORY OF ABSOLUTE STABILITY:

In this section we describe some approaches and results known from the theory of absolute stability. They form the basis of our approach for stability analysis of RNN. One of the most efficient methods of stability theory of dynamical systems is the method of absolute stability theory. A system to be analyzed for stability with this method is written in the form

$$x^{k+1} = Ax^k + B\varepsilon^k, \quad \sigma^k = \Theta x^k + b = \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_n \end{pmatrix} \tag{1}$$

$$\varepsilon^k = \begin{pmatrix} \varepsilon_1 \\ \cdots \\ \varepsilon_n \end{pmatrix}, \quad \varepsilon_i = \psi_i(\sigma_i), \quad i = 1,2,\ldots\ldots,n$$

where A, B and Θ are some constant matrices and b is a constant vector. It should be emphasized that the nonlinearities ψ_i do not include explicit compositions of functions of different variables σ_i.

Let F be a quadratic form with arguments x, ε for the system (1). Assume that the pair (A, B) is stabilizable.

Theorem: Assume that the following conditions hold.

1) For any solution of the system (1) there exist a sequence of positive numbers $N_j \to \infty$ such that the inequality $\sum_{k=0}^{N_j} F(x^k \varepsilon^k) \geq 0$ is fulfilled for all j.
2) There exist a positive number $\in$ such that the inequality $Re\{F(z,\eta)\} \leq -\epsilon(|z^2| + |\eta^2|)$ holds for all pairs (z,η) such that $Az + B\eta = e^{i\omega}z$ for some $\omega \epsilon [0,\pi]$.
3) There exist a matrix D such that D such that $F(y, D_y) \geq 0$ for all vectors y and the matrix $A + BD$ is stable.Then the equilibrium point of the system (1) is globally exponentially stable.If the condition (1) holds, then the system (1) is said to satisfy an integral quadratic constraint with the form F. If $F(x,\varepsilon) \geq 0$ for all x, then the system (1) is said to satisfy a local quadratic constraint with the form F.Satisfaction of the local quadratic constraint evidently implies satisfaction of the integral constraint.The condition (3) is usually called the condition of minimal stability. For almost all cases it is checked easily with $d = diag\{\mu_j\}\Theta$ and some appropriate choice of the numbers μ_j.The main condition (2) is usually called the frequency domain condition. The problem of checking this condition is reduced to checking for positive definiteness of some parameter-dependent matrix. This complicated problem is equivalent to the problem of existence of a hermitian solution to

$$(Ax + B\varepsilon)^* H(Ax + B\varepsilon) - x^* Hx + F(x,\varepsilon) < 0 \text{ for all } (x,\varepsilon) \neq 0 \tag{2}$$

Inequality (2) is a linear matrix inequality (LMI) with respect to the components of the matrix H. Furthermore, the quadratic form F is usually of the following type.

$$F = \sum_{j=1}^{r} \tau_j F_j \tag{3}$$

Where τ_j are arbitrary positive numbers and F_j are the quadratic forms with fixed coefficients, each of the forms describing some property of a nonlinearity. In this case the inequality (2) becomes an LMI also with respect to the parameter τ_j. These parameters and the matrix H may be found by an efficient interior point algorithm of the convex optimization. The corresponding quadratic form is

$$F_j = (\varepsilon_j - \gamma_j\sigma_j)(\mu_j\sigma_j - \varepsilon_j)$$

Note that a function for which $F_j \geq 0$ may be nonmonotone and time varying. Thus, if the conditions of the theorem (1) holds with the form F_j then the system (1) would be stable even if the function φ_j is monotone and time varying. It is required only that its plot lie in the sector $[\gamma_j\mu_j]$. The frequency domain condition (1) with a quadratic form (3) that guarantees stability of the system (1) with nonlinear functions φ_j satisfying the local quadratic constraint $F_j \geq 0$ for all $j = 1,\ldots\ldots,r$ is called the circle criterion. It is not specific in terms of the amount of information used about nonlinear functions; therefore the circle criterion is necessarily conservative when used for stability analysis of systems with nonlinear functions of a particular kind. Here we use two kinds of quadratic forms

$$F = \varepsilon^*\Gamma(M\Theta x - \varepsilon) \text{ and}$$
$$F = (\varepsilon - N\Theta x)^*\Gamma(M\Theta x - \varepsilon)$$

Here diagonal matrices M and N represent sector bounds for nonlinearities;

$$M = diag\{\mu_j\} \text{ and } N = diag\{\gamma_j\}.$$

Γ is a matrix satisfying certain properties and is the another argument of the LMI (2) which is to be solved with respect to both H and Γ.

SYSTEM TRANSFORMATION

To apply the method of absolute stability theory to stability analysis of RNN, it is necessary to transform the system

$$\begin{aligned} x_1^{k+1} &= tanh(W_1 x_1^k + V_n x_n^k + b_1) \\ x_2^{k+1} &= tanh(W_2 x_2^k + V_1 x_1^k + b_2) \\ x_3^{k+1} &= tanh(W_3 x_3^k + V_2 x_2^k + b_3) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{k+1} &= tanh(W_n x_n^k + V_{n-1} x_{n-1}^{k+1} + b_n) \end{aligned} \tag{i}$$

To the form (1). Hence the first step of our approach is a transformation. In (1), Θ becomes a matrix of blocks W_j and V_j and b becomes a vector of biases. Without loss of generality we henceforth assume that all nonlinearities are hyperbolic tangents or tanh, although the minimum requirement is to have all nonlinearities of a sector type. A recurrent network of the above system containing just one layer is already cast in the form (1). An RMLP with n layers without global feedback ($V_n = 0$) can be analyzed for stability using (1) layer by layer. However, the above system with ($V_n \neq 0$) must be modified to fit the form (1). To transform these equations to (1) we propose to use a special state-space extension method. We consider a two layer Recurrent Multilayer Perception (RMLP) with global feedback described by

$$\begin{aligned} x_1^{k+1} &= tanh(W_1 x_1^k + V_n x_n^k + b_1) \\ x_2^{k+1} &= tanh(W_2 x_2^k + V_1 x_1^k + b_2) \end{aligned} \tag{4}$$

The original system above can be transformed into

$$\begin{aligned} x_{11}^{k+1} &= tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1) \\ x_{12}^{k+1} &= x_{11}^k \\ x_{21}^{k+1} &= tan(W_2 x_{22}^k + V_1 x_{11}^k + b_2) \\ x_{22}^{k+1} &= x_{21}^k. \end{aligned} \tag{5}$$

The system (5) can now be written in the form suitable for our stability analysis

$$\begin{aligned} & x^{k+1} = Ax^k + B\varepsilon^k\ , \qquad \varepsilon^k = col(\varepsilon_1^k, \varepsilon_2^k) \\ & \varepsilon_j^k = tanh(\sigma_j^k),\ j = 1,2, \ldots \ldots \ldots \\ & \sigma^k = \Theta x^k + b,\ b = col(b_{1,} b_2) \\ & \Theta_1 = [0, W_{1,} V_{2,} 0], \quad \Theta_2 = [V_{1,} 0,0, W_2] \end{aligned} \tag{6}$$

Where, $x = col(x_{11} x_{12} x_{21} x_{22})$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix},\ B = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}$$

We would like to demonstrate that the system (5) may be converted to two independent systems which are counterparts of the original system (5). We can write

$$\begin{aligned} x_{12}^{k+2} &= \varphi_1(x_{12}^k, x_{21}^k) \\ x_{21}^{k+2} &= \varphi_2\left(x_{21}^k, \varphi_1(x_{21}^k, x_{21}^k)\right) \end{aligned} \tag{7}$$

We note that this system is identical to the original system

$$\begin{aligned} x_1^{k+1} &= tanh(W_1 x_1^k + V_n x_n^k + b_1) \\ x_2^{k+1} &= tanh(W_2 x_2^k + V_1 x_1^k + b_2) \end{aligned}$$

Since one time step of the system

$$\begin{aligned} x_1^{k+1} &= tanh(W_1 x_1^k + V_n x_n^k + b_1) \\ x_2^{k+1} &= tanh(W_2 x_2^k + V_1 x_1^k + b_2) \end{aligned}$$

Corresponds to two time steps of the system

$$\begin{aligned} x_{11}^{k+1} &= tanh(W_1 x_{12}^k + V_2 x_{21}^k + b_1) \\ x_{12}^{k+1} &= x_{11}^k \\ x_{21}^{k+1} &= tan(W_2 x_{22}^k + V_1 x_{11}^k + b_2) \\ x_{22}^{k+1} &= x_{21}^k. \end{aligned}$$

Indeed let us choose $x_{12}^0 = x_1^0, x_{21}^0 = x_2^0$, then $x_1^1 = x_{12}^2, x_2^1 = x_{21}^2$, etc. for the second counterpart, we have

$$\begin{aligned} x_{22}^{k+2} &= \varphi_2(x_{11}^k, x_{22}^k) \\ x_{11}^{k+2} &= \varphi_1\left(x_{11}^k, \varphi_1(x_{11}^k, x_{22}^k)\right) \end{aligned} \tag{8}$$

The system,

$$x_{22}^{k+2} = \varphi_2(x_{11}^k, x_{22}^k)$$
$$x_{11}^{k+2} = \varphi_1\left(x_{11}^k, \varphi_1(x_{11}^k, x_{22}^k)\right)$$

Corresponds to the following system:

$$x_2^{k+1} = \tanh(W_2 x_2^k + V_1 x_1^k + b_2)$$
$$x_1^{k+1} = \tanh(W_1 x_1^k + V_2 x_2^{k+1} + b_1) \quad (9)$$

It is easy to see that the system,

$$x_2^{k+1} = \tanh(W_2 x_2^k + V_1 x_1^k + b_2)$$
$$x_1^{k+1} = \tanh(W_1 x_1^k + V_2 x_2^{k+1} + b_1)$$

Is like the original system

$$x_1^{k+1} = (W_1 x_1^k + V_n x_n^k + b_1)$$
$$x_2^{k+1} = tanh(W_2 x_2^k + V_1 x_1^k + b_2)$$

For which the forward propagation begins with the second layer and ends with the first layer it is the other way around for the system

$$x_1^{k+1} = tanh(W_1 x_1^k + V_n x_n^k + b_1)$$
$$x_2^{k+1} = tanh(W_2 x_2^k + V_1 x_1^k + b_2).$$

For each state vector x_l of a layer l of an n −th RMLP we introduce n copies

$$x_{11}^{k+1} = tanh\big(W_1 x_{ln}^k + V_{l-1} x_{l-1,1}^k + b_l\big)\ l = 1,2, \dots .n(mod\ n)$$
$$x_{12}^{k+1} = x_{11}^k$$
$$x_{13}^{k+1} = x_{12}^k$$
$$\dots\dots\dots\dots..$$
$$x_{1n}^{k+1} = x_{l,n-1}^k \quad (10)$$

It can be seen that $x_{1n}^{k+1} = x_{l,n-1}^k$ has the form

$$x^{k+1} = Ax^k + B\varepsilon^k, \quad \sigma^k = \Theta x^k + \ b = \begin{pmatrix} \sigma_1 \\ \cdots \\ \sigma_n \end{pmatrix}$$

Equation (10) can be interpreted as describing n independent process in the RMLP

$$x_1^{k+1} = \tanh(W_1 x_1^k + V_n x_n^k + b_1)$$
$$x_2^{k+1} = \tanh(W_2 x_2^k + V_1 x_1^{k+1} + b_2)$$
$$x_3^{k+1} = \tanh(W_3 x_3^k + V_2 x_2^{k+1} + b_3)$$
$$x_n^{k+1} = \tanh(W_n x_n^k + V_{n-1} x_{n-1}^{k+1} + b_n)$$

And it is the general form illustrated in the example above for $n = 2$. Each process represents a system that is a counterpart of the original system,

$$x_1^{k+1} = tanh(W_1 x_1^k + V_n x_n^k + b_1)$$
$$x_2^{k+1} = tanh(W_2 x_2^k + V_1 x_1^k + b_2)$$
$$x_3^{k+1} = tanh(W_3 x_3^k + V_2 x_2^k + b_3)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_n^{k+1} = tanh(W_n x_n^k + V_{n-1} x_{n-1}^{k+1} + b_n)$$

It differs from systems represented by other process. For instance, a system with the state vector $col(x_{1n}, x_{2,n-1}, x_{3,n-2} \dots \dots \dots . x_{n1})$ can be written as,

$$x_{1n}^{k+n} = \phi_1(x_{1n}^k, x_{2,n-1}^k, x_{3,n-2}^k, \dots \dots .. x_{n1}^k)$$
$$x_{2,n-1}^{k+n} = \phi_2(x_{1n}^k, x_{2,n-1}^k, x_{3,n-2}^k, \dots \dots .. x_{n1}^k)$$
$$..$$
$$x_{1n}^{k+n} = \phi_n(x_{1n}^k, x_{2,n-1}^k, x_{3,n-2}^k, \dots \dots .. x_{n1}^k) \quad (11)$$

Where vector functions φ_i may be compositions of the functions $tanh$ with appropriate arguments. However, there are no compositions of nonlinear functions in (10), which is important for stability analysis. Setting $k + n$ instead of $k + 1$ as the time index of the left-hand side of (11), we arrive at a system which is an exact equivalent of the system (i).

Since

$$x_{1n} = x_{1,}$$
$$x_{2,n-1} = x_2$$
$$x_{3,n-2} = x_3$$
$$\ldots\ldots\ldots\ldots$$
$$x_{n1} = x_n$$

In this case, it is also natural to count the forward propagation of signals sequentially from the first layer of

$$x_1^{k+1} = tanh(W_1 x_1^k + V_n x_n^k + b_1)$$
$$x_2^{k+1} = tanh(W_2 x_2^k + V_1 x_1^k + b_2)$$
$$x_3^{k+1} = tanh(W_3 x_3^k + V_2 x_2^k + b_3)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n^{k+1} = tanh(W_n x_n^k + V_{n-1} x_{n-1}^{k+1} + b_n).$$

For other $n-1$ process with state vectors

$$col(x_{11}, x_{2n}, x_{3.n-1}, \ldots \ldots, x_{n2})$$
$$col(x_{12}, x_{21}, x_{3,n}, \ldots \ldots \ldots .. x_{n3})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$col(x_{1,n-1}, x_{2,n-2}, x_{3,n-3}, \ldots \ldots \ldots \ldots .. x_{nn})$$

Their system differ (i) only by the order in which signals propagate through the layers. Stability is established at once for all n processes(11), counterparts of the original system (i). If this is possible, then the global stability of the original system (i) and the system (10) follows immediately. Based on theorem (1) there is a special case which is being discussed below. Consider the form

$$p_{k+1} = P_1 Q_1 P_2 Q_2 \ldots \ldots .. P_q Q_q p_k \tag{12}$$

Where $p_k \in R^n$ is a state vector, Q_j are known matrices and P_j are diagonal matrices. The problem is to check Lyapunov stability of the system $p_{k+1} = P_1 Q_1 P_2 Q_2 \ldots \ldots .. P_q Q_q p_k$ for all matrices P_j satisfying the condition $0 \le P_j \le I$. The system (i) and (12) have different forms but the system (i) can be transformed into the form of (12)

$$x^{k+1} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & f_q \end{pmatrix} \cdot \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & V_{q-1} & W_q \end{pmatrix} \ldots\ldots$$
$$\begin{pmatrix} f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} W_1 & 0 & \ldots & 0 & 0 & V_q \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} x^k$$

Where $f_j(s) = tanh(s)$, $x^k = col(x_1^k \ldots \ldots \ldots x_q^k)$ and all the off diagonal blocks are set to zero. The system has the form (12) with

$$P_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & f_q \end{pmatrix} \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$P_q = \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..$$

$$Q_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & V_{q-1} & W_q \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q_2 = \begin{pmatrix} W_1 & 0 & \ldots & 0 & 0 & V_q \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

If there exist diagonal positive – definite matrices D_j such that $\left\|D_j Q_j D_{j+1}^{-1}\right\| < 1$ for all $j = 1, \ldots\ldots, q(mod\ q)$, then the system (12) is stable.

PROBLEM FORMULATION

Consider the following recurrent neural networks with time-varying delay: $\dot{x} = -Ax(t) + f\big(Wx\big(t - \tau(t)\big) + J\big)$

$$x(t) = \varphi(t), -\tau \leq t \leq 0 \tag{13}$$

Where, $x(.) = [x_1(.), x_2(.) \ldots\ldots, x_n(.)]^T \in R^n$ is the neuron state vector $f\big(x(.)\big) = \big[f_1\big(x_1(.)\big), f_2\big(x_2(.)\big), \ldots\ldots, f_n\big(x_n(.)\big)\big]^T \in R^n$ denote the neuron activation function $J = \big[j_1, j_{2,} \ldots\ldots j_n\big]^T \in R^n$ is a constant input vector $A = diag\{a_1, a_2, \ldots\ldots, a_n\}$ is a positive diagonal matrix W is the delayed connection weight matrix $\tau(t)$ is a time-varying delay satisfying $0 \leq \tau(t) \leq \tau$ and $\dot{\tau}(t) \leq \mu$ and $\varphi(\mathrm{t})$ $(-\tau \leq t \leq 0)$ is the initial condition. In addition, each neuron activation function in system (1) is assumed to satisfy

$$0 \leq \frac{f_i(x) - f_i(y)}{x - y} \leq l_{i,} \forall\, x, y \in R, i = 1,2, \ldots\ldots, n \tag{14}$$

Where l_i are some constants.The neural networks of system (13) are so called static neural networks. Under the assumption that W is invertible and $WA = AW$ holds and the equation $y(t) = Wx(t) + J$, system (13) can be easily rewritten as

$$\dot{y}(t) = -Ay(t) + Wf\Big(y\big(t - \tau(t)\big)\Big) + AJ \tag{15}$$

However, in many applications, static neural networks do not satisfy the transform condition. That is, system

$$\dot{x} = -Ax(t) + f\big(Wx\big(t - \tau(t)\big) + J\big)$$
$$x(t) = \varphi(t), -\tau \leq t \leq 0 \;\&$$
$$\dot{y}(t) = -Ay(t) + Wf\Big(y\big(t - \tau(t)\big)\Big) + AJ$$

are not always equivalent. Therefore, it is necessary to study the neural networks of system (13)

Under assumption $\dot{x} = -Ax(t) + f\big(Wx\big(t - \tau(t)\big) + J\big)$

$$x(t) = \varphi(t), -\tau \leq t \leq 0$$

there is an equilibrium x^* of (13). For simplicity, make the transformation $(.) = x(.) - x^*$. Then the system (13) can be transformed into

$$\dot{z}(t) = -Az(t) + g\Big(Wz\big(t - \tau(t)\big)\Big)$$
$$x(t) = \psi(t), -\tau \leq t \leq 0 \tag{16}$$

Where $z(.) = [z_1(.), z_2(.), \ldots\ldots\ldots\ldots z_n(.)]^T$ is the state vector of transformed system (14)$\psi(t) = \varphi(t) - x^*$ is the initial condition and the transformed neuron activation functions is

$$g\big(z(.)\big) = \big[g_1\big(z_1(.)\big), g_2\big(z_2(.)\big), \ldots\ldots\ldots\ldots g_n\big(z_n(.)\big)\big]^T$$
$$= f(Wz(.) + Wx^* + J) - f(Wx^* + J)$$

From (14) it is easy to derive that

$$0 \leq \frac{g_i(x) - g_i(y)}{x - y} \leq l_{i,} \forall\, x, y \in R, i = 1,2, \ldots\ldots, n \tag{17}$$

Based on the analysis above, we know that the problem of how to analyze the stability of system (13) at equilibrium is changed into a problem of how to analyze the zero stability of system (16).

MAIN RESULTS:

In this section, we will present asymptotical criteria for the considered neural network.

THEOREM:
For given diagonal matrix $L = diag\{l_1, l_2, \dots \dots . l_n\}$ and scalars $\tau \geq 0. \mu \geq 0$ System
$\dot{z}(t) = -Az(t) + g\left(Wz\big(t - \tau(t)\big)\right)$
$x(t) = \psi(t)$, $-\tau \leq t \leq 0\dots$ with $0 \leq \frac{g_i(x)-g_i(y)}{x-y} \leq l_{i,} \forall\, x, y \in R, i = 1,2, \dots \dots, n$
is globally asymptotically stable, if there exist matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0, Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0, Q_i > 0, (i = 1,2,3)$, Diagonal matrix $R \geq 0, U_1 \geq 0, U_2 \geq 0$ and appropriately dimensional matrices $M_j = \begin{bmatrix} M_{j^1} \\ M_{j^2} \end{bmatrix} (j = 1,2)$ such that the following inequalities hold

$$q\begin{bmatrix} \Omega + \varphi + \varphi^T & \tau M_1^T \\ * & -\tau Z \end{bmatrix} < 0 \tag{18}$$

Where $l = 1,2$ and $\Omega = \begin{bmatrix} \Omega_{11} & 0 & -P_2 & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} \\ * & \Omega_{22} & 0 & 0 & 0 & 0 & W^T L U_2 \\ * & * & -Q_2 & -P_3^T & -P_3^T & 0 & 0 \\ * & * & * & 0 & 0 & 0 & P_2^T \\ * & * & * & * & 0 & 0 & P_2^T \\ * & * & * & * & * & \Omega_{66} & RW \\ * & * & * & * & * & * & \Omega_{77} \end{bmatrix}$

$\Phi = [\ M_{12}^T\ \ M_{22}^T\ \ -M_{12}^T\ \ -M_{22}^T\ \ M_{11}^T\ \ M_{21}^T\ \ 0\ \ 0\ ]$ with
$\Omega_{11} = -P_1 A - A P_1^t + P_2 + P_2^T + Q_1 + Q_2 + \tau Z_{11} - \tau Z_{12} A - \tau A^T Z_{12}^T + \tau A^T Z_{22} A$ $\Omega_{14} = -AP_2 + P_3^T$
$\Omega_{15} = -AP_2 + P_3^T$
$\Omega_{16} = -AW^T R + W^T L U_1$
$\Omega_{17} = P_1 + \tau Z_{12} - \tau A^T Z_{22}$
$\Omega_{22} = -(1 - \mu)Q_1$
$\Omega_{66} = Q_3 - 2U_1$
$\Omega_{77} = -(1 - \mu)Q_3 - 2U_2 + \tau Z_{22}$.

Proof:
Firstly, we introduce the following Lyapunov Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{19}$$

Where, $V_1(t) = \chi(t) P \chi(t)$
$V_2(t) = 2\sum_{i=1}^{n} r_i \int_0^{W_i z(t)} g_i(s) ds$ $V_3(t) = \int_{t-\tau(t)}^{(t)} [z(s) Q_1 z(s) + g(Wz(s))$
$Q_3 g(Wz(s))] ds + \int_{t-\tau}^{(t)} [z(s) Q_2 z(s)$
$V_4(t) = \int_{-\tau}^{0} \int_{t+\theta}^{t} \eta(s) Z \eta(s) ds d\theta$

With $\chi(t) = [z(t)\ (\int_{t-\tau}^{t} z(s)ds)], \eta(s) = [z(s)\ \dot{z}(s)]$, W_i denoting the i-th row of matrix W $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} > 0$, $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0,\ Q_i > 0, (i = 1, 2, 3), R = diag\{r_1, r_2, \cdots, r_n\} \geq 0$.

Calculating the derivative of $V(t)$ along the solution of system (16) yields

$$\dot{V}_1(t) = 2\chi(t) P \begin{bmatrix} \dot{z}(t) \\ z(t) - z(t - \tau) \end{bmatrix} = 2\begin{bmatrix} z(t) \\ \int_{t-\tau(t)}^{t} z(s)ds + \int_{t-\tau}^{t-\tau(t)} z(s)ds \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \begin{bmatrix} -Az(t) + g(Wz(t - \tau(t))) \\ z(t) - z(t - \tau) \end{bmatrix} \tag{20}$$

& $$\dot{V}_2(t) = 2g(Wz(t) RW[-Az(t) + g(Wz(t - \tau(t)))] \tag{21}$$

Moreover, it can be deduced that

$$\dot{V}_3(t) \leq z(t)[Q_1 + Q\,2\,] z(t) + g\,T\left(Wz(t)\right) Q\,3\,g\left(Wz(t)\right)$$
$$-(1 - \mu)\begin{bmatrix} z\left(t - \tau(t)\right) Q_1 z\left(t - \tau(t)\right) \\ +g\left(Wz\left(t - \tau(t)\right)\right) Q_3 g\left(Wz\left(t - \tau(t)\right)\right) \end{bmatrix} - z(t - \tau) Q_2 z(t - \tau) \tag{22}$$

and it follows that

$$\dot{V}_4(t) \le \tau\eta(t)Z\eta(t) - \int_{t-\tau}^{t}\eta(s)Z\eta(s)ds$$
$$\le \tau z(t)\left[Z_{11} - Z_{12}A - A^T Z_{12}^T + AZ_{22}\right]z(t)$$
$$+2\tau z(t)\,[Z_{12} - AZ_{22}]g\left(Wz\big(t-\tau(t)\big)\right) + \tau g\left(Wz\big(t\,sw-\tau(t)\big)\right)Z_{22}g\left(Wz\big(t-\tau(t)\big)\right)$$
$$-\int_{t-\tau(t)}^{t}\eta(s)Z\eta(s)ds + \int_{t-\tau}^{t-\tau(t)}\eta(s)Z\eta(s)ds \qquad (23)$$

According to lemma. For any symmetric positive-definite matrix M > 0, scalar γ > 0, and vector function ω: [0, r]→ R m such that the integrations concerned are well defined, the following inequality holds:
$\gamma\int_0^1\omega(s)W\omega(s)ds \ge (\int_0^\gamma\omega(s)dsM(\int_0^\gamma\omega(s)ds)$.

For any real vectors a, b, and any matrix, Q > 0 with appropriate dimensions, we have $2ab \le aXa + bX^{-1}b$. For any appropriately dimensional matrices $M_i = \begin{bmatrix} M_{i1} \\ M_{i2}\end{bmatrix}$ $(i = 1{,}2)$, the enlargement of the following integral terms can be conducted

$$-\int_{t-\tau(t)}^{t}\eta(s)Z\eta(s)ds \le 2\int_{t-\tau(t)}^{t}\eta(s)M_1\xi(t)ds + \int_{t-\tau(t)}^{t}\xi(t)M_1^TZ^{-1}M_1\xi(t)ds$$
$$= 2\left(\int_{t-\tau(t)}^{t}z(s)ds\right)^T M_{11}\xi(t) + 2[z(t) - z(t-\tau(t)]M_{12}\xi(t)$$
$$+\tau(t)\xi^T(t)M_1^TZ^{-1}M_1\xi(t) \qquad (24)$$

$$-\int_{t-\tau}^{t-\tau(t)}\eta(s)Z\eta(s)ds \le 2\int_{t-\tau}^{t-\tau(t)}\eta(s)M_1\xi(t)ds + \int_{t-\tau}^{t-\tau(t)}\xi(t)M_2^TZ^{-1}M_2\xi(t)ds$$
$$=2\left(\int_{t-\tau}^{t-\tau(t)}z(s)ds\right)^T M_{21}\xi(t) + \left[z\left(t-\tau(t)\right) - z(t-\tau)\right].$$
$$M_{22}\xi(t)\,\tau(t)\xi^T(t)M_2^TZ^{-1}M_2\xi(t) \qquad (25)$$

Where $\xi(t) = [z(t)z\big(t-\tau(t)\big)z(t-\tau)\left(\int_{t-\tau(t)}^{t}z(s)ds\right)^T\left(\int_{t-\tau}^{t-\tau(t)}z(s)ds\right)^T g(Wz(t))\,g(Wz(t-\tau(t)))]$.

From (17) it is well known that there exist diagonally matrices $U_1 \ge 0\, and\, U_2 \ge 0$ such that the following inequalities

$$2g(Wz(t))U_1[LWz(t) - g(Wz(t))] \ge 0 \qquad (26)$$
$$2g(Wz(t-\tau(t)))U_2[LWz(t) - g(Wz(t-\tau(t)))] \ge 0 \qquad (27)$$

Where $L = diag\{l_1, l_2, \cdots, l_n\}$.

Adding the terms on the left hand side of (26) & (27) to $\dot{V}(t)$ yield

$$\dot{V}(t) \le \xi(t)\{\Omega + \Phi + \Phi + \tau(t)M_1^TZ^{-1}M_1 + [\tau - \tau(t)]M_2^TZ^{-1}M_2\}\xi(t) \qquad (28)$$

It is clear that the inequalities $\Omega + \Phi + \Phi + \tau M_1^TZ^{-1}M_1 < 0$
$\Omega + \Phi + \Phi + \tau M_2^TZ^{-1}M_2 < 0$
can guarantee $\dot{V}(t) < 0$, which indicates that $V(t) \le V(0)$.

From the schur complement the inequalities
$\Omega + \Phi + \Phi + \tau M_1^TZ^{-1}M_1 < 0$
$\Omega + \Phi + \Phi + \tau M_2^TZ^{-1}M_2 < 0$
are equivalent to the condition q$\begin{bmatrix}\Omega + \varphi + \varphi^T & \tau M_1^T \\ * & -\tau Z\end{bmatrix} < 0$ in the proceeding theorem. Then the global asymptotic stability for neural network $\dot{z}(t) = -Az(t) + g\left(Wz\big(t-\tau(t)\big)\right)$ and $x(t) = \psi(t), -\tau \le t \le 0$ is achieved.

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