

Certain Subclasses of Bi-Univalent Functions Associated with Salagean Operator

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ABSTRACT

The purpose of the present paper is to introduce new subclasses of the function class of bi-univalent functions by using Salagean operator. Furthermore, we obtain estimates on the coefficients and for functions of these subclasses. Relevant connections of well-known results are briefly indicated.

Keywords: Analytic functions, univalent functions, Bi-univalent functions, Salagean derivative.

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1. INTRODUCTION

Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the unit disc $U = \{z : |z| < 1\}$ and let S denote the subclass of A consisting function of the type (1.1) which are normalized and univalent in U .

It is well known that every $f \in S$ has an inverse f^{-1} defined by

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad z \in U \\ f(f^{-1}(w)) &= w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right), \end{aligned}$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U . Let Σ denote the class of bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). Examples of functions in the class Σ are $\frac{z}{1-z}$, $\log(1-z)$, $\frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$ and so on.

However, the familiar Koebe function is not a member of Σ . Other common examples of functions in S such as $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ .

Lewin [3] investigated the Bi-univalent function class Σ showed that $|a_2| < 1.5$.

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Subsequently by Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$.

The study of Bi-univalent functions is a recent topic of study in Geometric Function Theory. Several researchers for example see ([5], [7], [8]) introduced the various new subclass of the bi-univalent function class Σ and obtained non sharp bounds on the initial coefficients $|a_2|$ and $|a_3|$.

To prove our main results, we shall require the following lemma due to [2].

Lemma 1.1: If $h \in P$ then $|c_k| \leq 2$ for each k , where P is the family of all functions h analytic in U for which $\operatorname{Re}\{h(z)\} > 0$, $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$ for $z \in U$.

2. MAIN RESULT

Definition 2.1: A function $f(z)$ given by (1.1) is said to be in the class $\mathfrak{R}_{\Sigma,n}^\alpha$ if the following conditions are satisfied $f \in \Sigma$ and

$$\left| \arg \left(\frac{D^{n+1} f(z)}{z} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in U, 0 < \alpha \leq 1) \quad (2.1)$$

$$\left| \arg \left(\frac{D^{n+1} g(w)}{w} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in U, 0 < \alpha \leq 1) \quad (2.2)$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (2.3)$$

and D^n stands for the differential operator introduced by Salagean [6].

We first state and prove the following result.

Theorem 2.1: Let $f(z)$ given by (1.1) be in the function class $\mathfrak{R}_{\Sigma,n}^\alpha$. Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{\alpha 3^{n+1} + (1-\alpha) 2^{2n+1}}} \quad \text{and} \quad |a_3| \leq 2\alpha \frac{(3^{n+1} \alpha + 2^{2n+1})}{3^{n+1} \cdot 2^{2n+1}}. \quad (2.4)$$

Proof: We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$\frac{D^{n+1} f(z)}{z} = [Q(z)]^\alpha \quad \text{and} \quad \frac{D^{n+1} g(w)}{w} = [L(w)]^\alpha, \quad (2.5)$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities

$$\Re\{Q(z)\} > 0 \quad (z \in U) \quad \text{and} \quad \Re\{L(w)\} > 0 \quad (w \in U). \quad (2.6)$$

Furthermore, the functions $Q(z)$ and $L(w)$ have the forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (2.7)$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + \dots \quad (2.8)$$

respectively. Now, equating the coefficients of $\frac{D^{n+1} f(z)}{z}$ with $[Q(z)]^\alpha$ and the coefficient of $\frac{D^{n+1} g(w)}{w}$

with $[L(w)]^\alpha$

$$2^{n+1} a_2 = \alpha c_1 \quad (2.9)$$

$$3^{n+1} a_3 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2 \quad (2.10)$$

$$-2^{n+1} a_2 = \alpha l_1 \quad (2.11)$$

$$3^{n+1} (2a_2^2 - a_3) = \alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2 \quad (2.12)$$

From (2.9) and (2.11), we get

$$c_1 = -l_1$$

and

$$2^{2n+3} a_2^2 = \alpha^2 (l_1^2 + c_1^2) \quad (2.13)$$

Also, from (2.10), (2.12) and (2.13), we find that

$$2(3^{n+1}) a_2^2 = \alpha (l_2 + c_2) + \frac{\alpha(\alpha-1)}{2} (l_1^2 + c_1^2)$$

$$2(3^{n+1}) a_2^2 = \alpha (l_2 + c_2) + \frac{(\alpha-1)}{2\alpha} 2^{2n+3} a_2^2.$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2}{2(\alpha 3^{n+1} + (1-\alpha) 2^{2n+1})} (l_2 + c_2)$$

Which, in conjunction with the following well-known inequalities (see [2], page 41)

$$|c_2| \leq 2 \text{ and } |l_2| \leq 2,$$

gives us the desired estimate on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$ by subtracting (2.12) from (2.10), we get

$$2(3^{n+1} a_3 - 3^{n+1} a_2^2) = \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2 - \left(\alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2 \right).$$

Upon substituting the values of a_2^2 from (2.13) and observing that

$$c_1^2 = l_1^2.$$

It follows that

$$a_3 = \frac{1}{4^{n+1}} \alpha^2 c_1^2 + \frac{1}{2(3^{n+1})} \alpha (c_2 - l_2)$$

The familiar inequalities (cf. [2], page 41)

$$|c_2| \leq 2 \text{ and } |l_2| \leq 2$$

would now yield

$$|a_3| \leq \frac{1}{4^{n+1}} \alpha^2 \cdot 4 + \frac{1}{2(3^{n+1})} \alpha \cdot 4 = 2\alpha \frac{(3^{n+1} \alpha + 2^{2n+1})}{3^{n+1} \cdot 2^{2n+1}}.$$

This completes the proof of Theorem 2.1.

3. COEFFICIENT BOUNDS

Coefficient Bounds for the function class $\mathfrak{R}_{\Sigma,n}(\beta)$

Definition 3.1: A function $f(z)$ given by (1.1) is said to be in the class $\Re_{\Sigma,n}(\beta)$ ($0 \leq \beta < 1$) if the following conditions are satisfied $f \in \Sigma$ and

$$R\left(\frac{D^{n+1}f(z)}{z}\right) > \beta \quad (w \in U, 0 \leq \beta < 1), \quad (3.1)$$

$$R\left(\frac{D^{n+1}g(w)}{z}\right) > \beta \quad (w \in U, 0 \leq \beta < 1), \quad (3.2)$$

where the function g is given by (2.3).

Theorem 3.1: Let $f(z)$ given by (1.1) be in the function class $\Re_{\Sigma,n}(\beta)$ ($0 \leq \beta < 1$). Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3^{n+1}}}$$

and

$$|a_3| \leq (1-\beta) \left(\frac{1}{2^{2n}} + \frac{2}{3^{n+1}} - \beta \right).$$

Proof: First of all, the argument inequalities in (3.1) and (3.2) can easily be rewritten in their equivalent forms:

$$\frac{D^{n+1}f(z)}{z} = \beta + (1-\beta)Q(z) \quad \text{and} \quad \frac{D^{n+1}g(w)}{z} = \beta + (1-\beta)L(w), \quad (3.3)$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities

$$\Re\{Q(z)\} > 0 \quad (z \in U) \quad \text{and} \quad \Re\{L(w)\} > 0 \quad (w \in U) \quad (3.4)$$

Moreover, the functions $Q(z)$ and $L(w)$ have the following forms:

$$Q(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (3.5)$$

and

$$L(w) = 1 + l_1 w + l_2 w^2 + \dots, \quad (3.6)$$

As in the proof of Theorem 2.1, by suitably comparing coefficient, we get

$$2^{n+1}a_2 = (1-\beta)c_1 \quad (3.7)$$

$$3^{n+1}a_3 = (1-\beta)c_2 \quad (3.8)$$

$$-2^{n+1}a_2 = (1-\beta)l_1 \quad (3.9)$$

$$3^{n+1}(2a_2^2 - a_3) = (1-\beta)l_2 \quad (3.10)$$

From (3.7) and (3.9), we get

$$c_1 = -l_1$$

and

$$2^{2n+3}a_2^2 = (1-\beta)^2(l_1^2 + c_1^2) \quad (3.11)$$

Also, from (3.8) and (3.10), we find that

$$2(3^{n+1})a_2^2 = 3^{n+1}a_3 + (1-\beta)l_2 = (1-\beta)c_2 + (1-\beta)l_2 = (1-\beta)(c_2 + l_2). \quad (3.12)$$

Therefore, we have

$$|a_2^2| \leq \frac{(1-\beta)}{2(3^{n+1})}(|l_2| + |c_2|) = \frac{(1-\beta)}{2(3^{n+1})}.4 = \frac{2(1-\beta)}{(3^{n+1})}.$$

This gives the bound on $|a_2|$, as asserted in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2(3^{n+1}a_3 - 3^{n+1}a_2^2) = (1 - \beta)(c_2 - l_2),$$

which, upon substituting the values of a_2^2 from (3.11), yields

$$2(3^{n+1})a_3 = \frac{3^{n+1}}{2^{2n+2}}(1 - \beta)^2(l_1^2 + c_1^2) + (1 - \beta)(c_2 - l_2),$$

This last equation, together with the well-known estimates:

$$|c_1| \leq 2, |l_1| \leq 2, |c_2| \leq 2 \text{ and } |l_2| \leq 2$$

would lead us to the following inequality:

$$2(3^{n+1})a_3 = \frac{3^{n+1}}{2^{2n+2}}(1 - \beta)^2.8 + (1 - \beta).4.$$

Therefore, we have

$$|a_3| \leq (1 - \beta) \left(\frac{1}{2^{2n}} + \frac{2}{3^{n+1}} - \beta \right).$$

This completes the proof of theorem 3.1.

Remarks 1: If we put $n = 0$ then we obtain the corresponding results of Srivastava *et.al* [7].

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REFERENCES

1. D.A. Brannan and J.G. Clunie (Eds) Aspects of Contemporary Complex Analysis, Proceedings of the NATO Advanced Study Institute held at the university of Durham, Durham; July 1-20, 1979), Academic Press, New York and London (1980).
2. P.L. Duren, Univalent Functions, in: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
3. M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63-68.
4. E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal., 32 (1969), 100-112.
5. Saurabh Porwal and M. Darus, On a new subclass of bi-univalent functions, J. Egypt. Math. Soc., 21 (2013), 190-193.
6. G.S. Salagean, Subclasses of univalent functions, Complex analysis-Fifth Romanian Finish Seminar, Bucharest 1 (1983) 362-372.
7. H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188-1192.
8. Xu, Y.C. Gui and H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett., 25 (6) (2012), 990-994.

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