International Journal of Mathematical Archive-9(2), 2018, 176-182
Available online through www.ijma.info ISSN 2229 – 5046

ADVECTION DIFFUSION EQUATION FOR NUTRIENT UPTAKE BY AQUATIC SPHERICAL BULB SURFACE WITH NONLINEAR BOUNDARY CONDITION

AVHALE P. S.
Department of Mathematics,
Shivaji Arts, Commerce and Science College Kannad Dist. Aurangabad (M.H.), India.

(Received On: 04-01-18; Revised & Accepted On: 31-01-18)

ABSTRACT

In this article, we derived mathematical expression for nutrient uptake by spherical bulb surface of aquatic plant, which is the advection diffusion equation in the spherical radial form. We assume Michal Menten boundary condition as boundary condition for advection diffusion equation, which is nonlinear surface boundary condition. We obtained the solution for advection diffusion equation reducing it into the Bessel's equation by the method of separation variable using the boundary and initial conditions by re-scaling the variables and using extreme parameters.

Key words and Phrases: *uptake of nutrient, complete solution of advection diffusion equation, spherical, bulb surface, re-scaling variable.*

1. INTRODUCTION

The primary physiological function of root is uptaking the water as well as nutrients and transport to leaves for photosynthesis. Investigations and observation of the uptake of water and nutrient in plant root and stem was traced back to many years ago, such bulb help to clean the dam or river. In recent years, a number of researchers from various fields, such as physics, applied mathematics and plant physiology, paid more attention to develop mathematical model for water and nutrient uptake. The outstanding work in this field is done by T.Roose and proposed a mathematical model for uptake of water and nutrient from groundwater. Roose work is development of Nye, Tinker and Barber model for water and nutrient uptake assuming that the root is an infinitely long cylinder [9][10], but some aquatic root are spherical bulb. To develop Mathematical model, we first derive advection diffusion equation of nutrient transport in the water and then try to solve the advection diffusion equation by transforming it into non-dimensional form with Michal Menten boundary condition as boundary condition.

2. NUTRIENT CONVECTION EQUATION IN WATER

The root surrounded by water and gas. We indicate $\emptyset_l$ volume fraction of system occupied by the liquid and $\emptyset_s$ volume fraction of system occupied by gas.

The conservation of system volume equation is written as [9][10]

$$\emptyset_g + \emptyset_l = 1. \tag{2.1}$$

Fraction of gas in water is negligible we may assume $\emptyset = \emptyset_l$.

Nutrient comes in contact with the bulb surface by flow of water in which diffusion of nutrient takes place. Then the equation for ions in the liquid phase is written as

$$\frac{\partial}{\partial t}(\emptyset_l c_l) + \nabla.(c_l u) = \nabla.(\emptyset_l D \nabla c_l), \tag{2.2}$$

where u is the Darcy flux of water , c_l is the nutrient concentration in the liquid, D is the diffusion coefficient in the liquid phase

Hence, the equation (2.2) in terms of c_l becomes,

$$\emptyset_l \frac{\partial c}{\partial t} + \nabla.(c_l u) = \nabla.(\emptyset_l D \nabla c_l), \tag{2.3}$$

Corresponding Author: Avhale P. S., Department of Mathematics,
Shivaji Arts, Commerce and Science College Kannad Dist. Aurangabad (M.H.), India.

Noting $c_l = c$ and writing equation (2.3) in spherical radial polar coordinates, we get [4]

$$\emptyset_l \frac{\partial c}{\partial t} - \frac{aV}{r}\frac{\partial C}{\partial r} = D\emptyset_l \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right), \tag{2.4}$$

where a is the radius of the bulb. The water flux is given by $u = -\frac{aV}{r}$, which derives from the law of mass conservation for water, i.e, $\nabla . u = 0$. The quantity V is the Darcy flux of water into the bulb. [1]

3. BOUNDARY CONDITION AT AQUATIC BULB SURFACE

Bulb surface accept the nutrient up to a certain level even if the nutrient concentration in liquid increases indefinitely. It is also verified that the bulb surface accept nutrient up to a critical level (low) of nutrient in liquid phase near the bulb surface below which first it stop the uptake of nutrient and then start bleeding in the soil. The experimentally measured, heuristic Michaelis-Menten type nutrient uptake boundary condition is therefore given by, see [7]

$$\emptyset_l D \frac{\partial c}{\partial r} + Vc = \frac{F_m c}{K_m + c} - E \text{ at } r = a, \tag{3.1}$$

where c indicate the concentration of nutrient in the liquid phase of the soil, K_m indicate the Michaelis-Menten constant that is equal to the bulb surface nutrient concentration, when the flux of nutrient into the bulb is half of the maximum possible, F_m indicate the maximum flux of nutrient into the bulb, $\frac{F_m c_{min}}{K_m + c_{min}}$ where c_{min} indicate the minimum concentration when the nutrient uptake by the bulb stop, and a is the radius of the bulb.

4. INITIAL CONDITION AND BOUNDARY CONDITION AWAY FROM BULB SURFACE IN WATER

Initial condition can be write as for $t = 0$

$$c = c_0 \text{ at } t = 0 \text{ for } a < r < \infty, \tag{4.1}$$

for later time

$$c \to c_0 \text{ as } r \to \infty \text{ for } t > 0. \tag{4.2}$$

5. NON-DIMENSIONALISATION OF NUTRIENT CONVENTION EQUATION

Choosing time, space, and concentration-scale as follows and substitute in (2.4)

$$t = \frac{a^2}{D} t^*, \quad r = ar^*, c = K_m c^*, \tag{5.1}$$

where c^*, t^* and $r^*$are dimensionless nutrient concentration, time, and radial variables, respectively, we obtain (after dropping * s) the following dimensionless model

$$\frac{\partial c}{\partial t} - P_e \frac{1}{r}\frac{\partial c}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right), \tag{5.2}$$

with boundary conditions

$$\frac{\partial c}{\partial r} + P_e c = \lambda \frac{c}{1+c} - \epsilon \quad \text{at } r = 1, \tag{5.3}$$

$$c \to c_\infty \text{ as } r \to \infty \text{ for } t > 0, \tag{5.4}$$

the dimensionless initial condition is given by

$$c = c_\infty \text{ at } t = 0 \text{, for } 1 < r < \infty \tag{5.5}$$

The dimensionless parameters in above equations are defined as

$$P_e = \frac{aV}{D\emptyset_l}, \ \lambda = \frac{F_m a}{DK_m \phi_l}, \ \epsilon = \frac{Ea}{DK_m \phi_l}, \ c_\infty = \frac{c_0}{K_m}. \tag{5.6}$$

6. CONCENTRATION OF NUTRIENT ENTERING INTO BULB SURFACE AND TOTAL UPTAKE

The concentration of nutrient entering into bulb surface is obtain by solving equation (5.2) with boundary condition (5.3), (5.4) and (5.5) for extreme parameters. Rearranging the equation (5.2), we write as

$$\frac{\partial c}{\partial t} = \left(\frac{2+P_e}{r}\right)\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2} \tag{6.1}$$

Re-scaling independent variable as $r = (2 + P_e)R$, then $\partial r = (2 + P_e)\partial R$. Then equation (6.1) written as

$$(2 + P_e)\frac{\partial c}{\partial t} = \frac{1}{R}\frac{\partial c}{\partial R} + \frac{\partial^2 c}{\partial R^2} \tag{6.2}$$

Corresponding boundary condition changes into

$$\frac{\partial c}{\partial R} + (2 + P_e)P_e c = \lambda(2 + P_e)\left[\frac{c}{1+c} - \epsilon\right] \quad \text{at } R = \frac{1}{2+P_e}, \tag{6.3}$$

$$c \to c_\infty \text{ as } r \to \infty \text{ for } t > 0, \tag{6.4}$$

$$c = c_\infty \text{ at } t = 0 \text{ for } \frac{1}{2+P_e} < R < \infty. \tag{6.5}$$

We consider the extreme of parameter λ. For large value of ϕ_l with small radius of bulb a, i.e., for small value of λ we expect the region where c is less than order 1 to be larger. Mathematical this problem is similar to the Oseen type of problems [5]. Therefore $\lambda << 1$, i.e., $\lambda \approx 0$. Then the boundary condition space (6.3) reduces

$$\frac{\partial c}{\partial R} + (2 + P_e)P_e c = 0. \tag{6.6}$$

We separate the variable of equation (6.2) by substitution $c(R,t) = U(R)T(t)$, we have [6]

$$\frac{1}{T}(2 + P_e)\frac{\partial T}{\partial t} = \frac{1}{U}\left[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}\right], \tag{6.7}$$

Corresponding boundary condition (6.3) reduces

$$\frac{\partial U}{\partial R} + (2 + P_e)P_e U = 0. \tag{6.8}$$

We equate equation (6.7) with $-\beta^2$

$$\frac{1}{T}(2 + P_e)\frac{\partial T}{\partial t} = \frac{1}{U}\left[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}\right] = -\beta^2, \tag{6.9}$$

Then the Bessel's equation with boundary condition at bulb surface is,

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \beta^2 U = 0, \tag{6.10}$$

$$\frac{\partial U}{\partial R} + (2 + P_e)P_e U = 0, \tag{6.11}$$

and

$$\frac{\partial T}{\partial t} = -\frac{\beta^2 T}{(2+P_e)}, \tag{6.12}$$

with boundary conditions

$$c = c_\infty \text{ at } t = 0 \text{ for } \frac{1}{(2+P_e)} < R < \infty. \tag{6.13}$$

Then the complete solution is given by

$$c(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{1}{(2+P_e)}\beta^2 t} U(\beta,R)\, d\beta \int_{R=\frac{1}{(2+P_e)}}^{\infty} R' U(\beta,R') c_\infty\, dR', \tag{6.14}$$

where $U(\beta,R)$ and $N(\beta)$ is given by

$$U(\beta,R) = J_0(\beta R)\left[\beta Y_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)Y_0\left(\beta\frac{1}{(2+P_e)}\right)\right] \\ -Y_0(\beta R)\left[\beta J_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)J_0\left(\beta\frac{1}{(2+P_e)}\right)\right], \tag{6.15}$$

Also

$$N(\beta) = \left[\beta J_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)J_0\left(\beta\frac{1}{(2+P_e)}\right)\right]^2 \\ + \left[\beta Y_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)Y_0\left(\beta\frac{1}{(2+P_e)}\right)\right]^2, \tag{6.16}$$

Replacing R by $R = \frac{r}{(2+\mathrm{Pe})}$ in equation (6.14) and (6.15), we get concentration entering through bulb surface

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{1}{(2+P_e)}\beta^2 t} U\left(\beta,\frac{r}{(2+\mathrm{Pe})}\right) d\beta \int_{r=1}^{\infty} \frac{r'}{(2+\mathrm{Pe})} U\left(\beta,\frac{r'}{(2+\mathrm{Pe})}\right) c_\infty\, dr', \tag{6.17}$$

$$U(\beta,r) = J_0\left(\beta\frac{r}{(2+\mathrm{Pe})}\right)\left[\beta Y_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)Y_0\left(\beta\frac{1}{(2+P_e)}\right)\right] \\ -Y_0\left(\beta\frac{r}{(2+\mathrm{Pe})}\right)\left[\beta J_1\left(\beta\frac{1}{(2+P_e)}\right) - P_e(2+P_e)J_0\left(\beta\frac{1}{(2+P_e)}\right)\right]. \tag{6.18}$$

Amount of nutrient absorbed by bulb surface of radius r in time is given as, [2][4]

$$M = 4\pi r t\frac{\partial c}{\partial r}. \tag{6.19}$$

7. NUTRIENT CONVENTION EQUATION WITH LIMIT $c_\infty << 1$ and $\epsilon < P_e << 1$

In this section we consider P_e, ϵ and c_∞ are negligible. If Michaelis-Menten coefficient K_∞ much larger than the far field concentration c_0, i.e, $c_\infty << 1$ then the equation (5.2) reduces to the form

$$\frac{\partial c}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right). \tag{7.1}$$

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r}. \tag{7.2}$$

Corresponding boundary condition reduces to the form

$$\frac{\partial c}{\partial r} = \lambda \frac{c}{1+c} \text{ at } r = 1,$$
$$c \to c_\infty \text{ at } t > 0 \text{ for } r \to \infty,$$
$$c = c_\infty \text{ at } t = 0 \text{ for } 1 < r < \infty \quad (7.3)$$

Re-scaling $c = c_\infty C$, then the model in scaled concentration is written as

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} + \frac{2}{r}\frac{\partial C}{\partial r}, \quad (7.4)$$

Scaled boundary condition are as follows

$$\frac{\partial C}{\partial r} = \lambda \frac{C}{1+c_\infty C} \text{ at } r = 1,$$
$$C \to 1 \text{ at } t > 0 \text{ for } r \to \infty,$$
$$C = 1 \text{ at } t = 0 \text{ for } 1 < r < \infty, \quad (7.5)$$

for $c_\infty << 1$, we can approximate the root surface boundary condition, using the binomial expansion, at the leading order given by

$$\frac{\partial C}{\partial r} \approx \lambda C \text{ at } r = 1. \quad (7.6)$$

Re-scaling equation (7.4) and its boundary condition by independent variable as $r = 2R$, we can write

$$2\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial R^2} + \frac{1}{R}\frac{\partial C}{\partial R} \quad (7.7)$$

Scaled boundary condition are written as

$$\frac{\partial C}{\partial R} = 2\lambda C \quad \text{at } R = \frac{1}{2}$$
$$C \to 1 \text{ at } t > 0 \text{ as } R \to \infty,$$
$$C = 1 \text{ at } t = 0 \text{ for } \frac{1}{2} < R < \infty \quad (7.8)$$

We solve the above boundary value problem by separation of the variables. Substituting the substitution $C(R,t) = T(t)U(R)$ the value in equation (7.7), separating the variables, we write as

$$\frac{1}{U}\left[\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}\right] = \frac{2}{T}\left[\frac{\partial T}{\partial t}\right] = -\beta^2 \quad (7.9)$$

Now consider the boundary value problem

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \beta^2 U = 0. \quad (7.10)$$

With the boundary condition

$$\frac{dU}{dR} - 2\lambda U = 0 \quad (7.11)$$

Then the complete solution is given by, see [10],

$$C(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{\beta^2}{2}t} U(\beta,R)\, d\beta \int_{R=\frac{1}{2}}^{\infty} R' U(\beta,R')\, dR', \quad (7.12)$$

where $U(\beta,R)$ is eigen value function.

$$U(\beta,R) = J_0(\beta R)[\beta Y_1(\beta) + 2\lambda Y_0(\beta)] - Y_0(\beta R)[\beta J_1(\beta) + 2\lambda J_0(\beta)]. \quad (7.13)$$
$$N(\beta) = [\beta J_1(\beta) + 2\lambda J_0(\beta)]^2 + [\beta Y_1(\beta) + 2\lambda Y_0(\beta)]^2. \quad (7.14)$$

Substituting $R = \frac{r}{2}$ in equation (7.12) to (7.14), we get concentration entering into bulb surface

$$C(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{\beta^2}{2}t} U\left(\beta,\frac{r}{2}\right) d\beta \int_{R=\frac{1}{2}}^{\infty} R' U\left(\beta,\frac{r'}{2}\right) dr', \quad (7.15)$$

where $U(\beta,r)$ is eigen value function.

$$U(\beta,r) = J_0\left(\beta\frac{r}{2}\right)[\beta Y_1(\beta) + 2\lambda Y_0(\beta)] - Y_0\left(\beta\frac{r}{2}\right)[\beta J_1(\beta) + 2\lambda J_0(\beta)] \quad (7.16)$$
$$N(\beta) = [\beta J_1(\beta) + 2\lambda J_0(\beta)]^2 + [\beta Y_1(\beta) + 2\lambda Y_0(\beta)]^2 \quad (7.17)$$

So the general solution of equation (7.1) is given by

$$c(r,t) = c_\infty \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{\beta^2}{2}t} U\left(\beta,\frac{r}{2}\right) d\beta \int_{r=1}^{\infty} R' U\left(\beta,\frac{r'}{2}\right) dr'. \quad (7.18)$$

The model with this condition is known as the "root absorbing power" model.

8. SITUATION WHEN WILL BE HIGH NUTRIENT UPTAKE FOR $\lambda >> 1$ AND $Pe << 1$

If the gradient of nutrient concentration near root surface is high, i.e., $\frac{\partial c}{\partial r}|_{r=1} = \lambda \gg 1$ for $c \sim O(1)$. Then re-scaling the independent variables r and t to stretched variables R and T, i.e., $r = 1 + \frac{R}{\lambda}$ and $t = \frac{T}{2\lambda^2}$, the equation (7.1) reduces to [2][8][9]

$$\frac{\partial c}{\partial T} = \frac{1}{2}\frac{\partial^2 c}{\partial R^2} + \frac{1}{R+\lambda}\frac{\partial c}{\partial R}. \tag{8.1}$$

Which at the leading order simples to

$$\frac{\partial c}{\partial T} = \frac{1}{2}\frac{\partial^2 c}{\partial R^2}, \tag{8.2}$$

Since $\frac{1}{\lambda+R} << 1$ for $\lambda >> 1$. The re-scaled boundary condition is

$$\frac{\partial c}{\partial R} = c \text{ at } R = 0, \text{ and } c \to 1 \text{ as } R \to \infty, \tag{8.3}$$

and the initial condition is c = 1 at T = 0 for 0 < R < 1. Then the general solution to this leading order problem is given by

$$c(R,T) = erf\left(\frac{R}{\sqrt{2T}}\right) + e^{R+\frac{T}{2}} erfc\left(\frac{R}{\sqrt{2T}} + \sqrt{\frac{T}{2}}\right), \tag{8.4}$$

with the flux $F(T) = \frac{\partial c}{\partial R}\frac{\partial R}{\partial r}|_{R=0}$ of nutrient into the root given by

$$F(T) = \lambda e^{T} erfc\left(\sqrt{\frac{T}{2}}\right). \tag{8.5}$$

As $T \to \infty$, the concentration of nutrient at the surface $c \to 0$ and $F \to 0$, since $e^{\frac{T}{2}} erfc\left(\sqrt{\frac{T}{2}}\right) \to 0$ as $T \to \infty$.

9. ZERO-SINK MODEL

For $t > tc\_\sim\frac{1}{\lambda^2}$ 1the root surface nutrient concentration has dropped to a very low level, then we take the boundary condition at the root surface at the leading order to be c = 0 at r = 1, i.e., the mathematical model reduces with following boundary condition

$$\frac{\partial c}{\partial t} + \frac{(-P_e)}{r}\frac{\partial c}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right), \tag{9.1}$$

$$c = 0 \text{ at } r = 1 \text{ and } c \to 1 \text{ as } r \to \infty. \tag{9.2}$$

Rearranging the terms of equation (9.1), we can write the equation

$$\frac{\partial c}{\partial t} = \left(\frac{2+P_e}{r}\right)\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial r^2}. \tag{9.3}$$

Re-scaling r in the equation (9.3) as $r = (2 + Pe)R$, we have

$$(2 + P_e)\frac{\partial c}{\partial t} = \frac{1}{R}\frac{\partial c}{\partial R} + \frac{\partial^2 C}{\partial R^2}. \tag{9.4}$$

Re-scaling the boundary condition

$$c = 0 \text{ at } R = \frac{1}{2+P_e}, \text{ and } c \to 1 \text{ as } R \to \infty. \tag{9.5}$$

By separating the variable, it can be shown that the time-variable function is given by $e^{\left(-\frac{\beta^2}{(2+P_e)}\right)t}$ and space variable function $U(\beta, R)$ is the solution of the following boundary value problem

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \beta^2 U = 0, \text{ in } \frac{1}{2+P_e} < R < \infty, \tag{9.6}$$

with the boundary condition

$$C = 0 \text{ at } R = \frac{1}{2+P_e}, \tag{9.7}$$

Then the complete solution for $c(R,t)$ is constructed as

$$c(R,t) = \int_{\beta=0}^{\infty} C(\beta) e^{-\frac{\beta^2}{2+P_e}t} U(\beta, R) d\beta. \tag{9.8}$$

The application of the initial condition yields

$$1 = \int_{\beta=0}^{\infty} C(\beta) U(\beta, R) d\beta, \text{ as } R \to \infty \tag{9.9}$$

where

$$C(\beta) \equiv \frac{1}{N(\beta)}\beta \int_{R=\frac{1}{2+P_e}}^{\infty} R' U(\beta, R') dR'. \tag{9.10}$$

Substituting value (9.10) into the equation (9.8)

$$c(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{\beta^2}{2+P_e}t} U(\beta,R)d\beta \int_{R=\frac{1}{2+P_e}}^{\infty} R' U(\beta,R)dR'. \quad (9.11)$$

where $U(\beta,R)$ and $N(\beta)$

$$U(\beta,R) = J_0(\beta R)Y_0\left(\frac{\beta}{2+P_e}\right) - Y_0(\beta R)J_0\left(\frac{\beta}{2+P_e}\right), \quad (9.12)$$

$$N(\beta) = J_0^2\left(\frac{\beta}{2+P_e}\right) + Y_0^2\left(\frac{\beta}{2+P_e}\right). \quad (9.13)$$

Substituting the value of $U(\beta,R)$ and $N(\beta)$

$$c(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2\left(\frac{\beta}{2+P_e}\right)+Y_0^2\left(\frac{\beta}{2+P_e}\right)} e^{-\frac{\beta^2}{2+P_e}t} \left[J_0(\beta R)Y_0\left(\frac{\beta}{2+P_e}\right) - Y_0(\beta R)J_0\left(\frac{\beta}{2+P_e}\right)\right] d\beta$$
$$\int_{R=\frac{1}{2+P_e}}^{\infty} R' \left[J_0(\beta R')Y_0\left(\frac{\beta}{2+P_e}\right) - Y_0(\beta R')J_0\left(\frac{\beta}{2+P_e}\right)\right] dR'. \quad (9.14)$$

Re-substitute $R = \frac{r}{2+P_e}$ in equation (9.14), we get general solution.

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2\left(\frac{\beta}{2+P_e}\right)+Y_0^2\left(\frac{\beta}{2+P_e}\right)} e^{\frac{\beta^2}{2+P_e}t} \left[J_0\left(\beta\frac{r}{2+P_e}\right)Y_0\left(\frac{\beta}{2+P_e}\right) - Y_0\left(\beta\frac{r}{2+P_e}\right)J_0\left(\frac{\beta}{2+P_e}\right)\right] d\beta$$
$$\int_{r=1}^{\infty} \frac{r'}{2+P_e} \left[J_0\left(\beta\frac{r'}{2+P_e}\right)Y_0\left(\frac{\beta}{2+P_e}\right) - Y_0\left(\beta\frac{r'}{2+P_e}\right)J_0\left(\frac{\beta}{2+P_e}\right)\right] dr'. \quad (9.15)$$

10. Zero-sink model with $P_e << 1$

If $P_e << 1$, then the equation (9.1) is reduced into the form

$$\frac{\partial c}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) \quad (10.1)$$

Re-scaling by $r = 2R$, the equation (10.1) is written as

$$2\frac{\partial c}{\partial t} = \frac{1}{R}\frac{\partial c}{\partial R} + \frac{\partial^2 C}{\partial R^2}, \quad (10.2)$$

boundary condition are re-scaled as

$$c = 0 \text{ at } R = \frac{1}{2}, \text{ and } c \to 1 \text{ as } R \to \infty \quad (10.3)$$

Separating the variables by substituting c = U(R)T(t)

$$\frac{2}{T}\frac{\partial T}{\partial t} = \frac{1}{U}\left[\frac{1}{U}\frac{\partial U}{\partial R} + \frac{\partial^2 U}{\partial R^2}\right] = -\beta^2, \quad (10.4)$$

solution for time-variable function is given by $e^{\frac{-\beta^2}{2}t}$ and space variable function $U(\beta,r)$ is the solution of the following problem

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{dU}{dR} + \beta^2 U = 0 \quad \text{for } \frac{1}{2} < R < \infty \quad (10.5)$$

with boundary conditions

$$c = 0 \text{ at } R = \frac{1}{2} \text{ and } c \to 1 \text{ as } R \to \infty \quad (10.6)$$

Then the complete solution for $c(R,t)$ is constructed as

$$c(R,t) = \int_{\beta=0}^{\infty} C(\beta)e^{-\frac{\beta^2}{2}t}U(\beta,R)d\beta, \quad (10.7)$$

with the application of initial condition we get

$$1 = \int_{\beta=0}^{\infty} C(\beta)U(\beta,R)d\beta \quad \text{in } \frac{1}{2} < R < \infty, \quad (10.8)$$

Using the orthogonality of eigen value functions we have

$$C(\beta) \equiv \frac{1}{N(\beta)}\beta\int_{R=\frac{1}{2}}^{\infty} R' U(\beta,R')dR'. \quad (10.9)$$

Substituting equation (10.9) into equation (10.7) gives

$$c(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{N(\beta)} e^{-\frac{\beta^2}{2}t} U(\beta,R)d\beta \int_{R=\frac{1}{2}}^{\infty} R' U(\beta,R')dR'. \quad (10.10)$$

Where

$$U(\beta,R) = J_0(\beta R)Y_0\left(\frac{\beta}{2}\right) - Y_0(\beta R)J_0\left(\frac{\beta}{2}\right) \quad (10.11)$$

and $$N(\beta) = \left[J_0^2\left(\frac{\beta}{2}\right) + Y_0^2\left(\frac{\beta}{2}\right)\right]. \quad (10.12)$$

Then complete integral is given by

$$c(R,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2\left(\frac{\beta}{2}\right) + Y_0^2\left(\frac{\beta}{2}\right)} e^{-\frac{\beta^2}{2}t} \left[J_0(\beta R)Y_0\left(\frac{\beta}{2}\right) - Y_0(\beta R)J_0\left(\frac{\beta}{2}\right)\right] d\beta$$

$$\int_{R=\frac{1}{2}}^{\infty} R' \left[J_0(\beta R')Y_0\left(\frac{\beta}{2}\right) - Y_0(\beta R')J_0\left(\frac{\beta}{2}\right)\right] dR' \qquad (10.13)$$

Re-substitution of $R = \frac{r}{2}$, we get

$$c(r,t) = \int_{\beta=0}^{\infty} \frac{\beta}{J_0^2\left(\frac{\beta}{2}\right) + Y_0^2\left(\frac{\beta}{2}\right)} e^{-\frac{\beta^2}{2}t} \left[J_0\left(\beta \frac{r}{2}\right)Y_0\left(\frac{\beta}{2}\right) - Y_0\left(\beta \frac{r}{2}\right)J_0\left(\frac{\beta}{2}\right)\right] d\beta$$

$$\cdot \int_{r=1}^{\infty} \frac{r'}{2}\left[J_0\left(\beta \frac{r'}{2}\right)Y_0\left(\frac{\beta}{2}\right) - Y_0\left(\beta \frac{r'}{2}\right)J_0\left(\frac{\beta}{2}\right)\right] dr' \qquad (10.14)$$

11. CONCLUSIONS

We developed the mathematical model for nutrient uptake by aquatic spherical surface of bulb which is advection diffusion equation in the spherical radial coordinates and obtain solution by re-scaling and reducing into Bessel's equation. So the solutions are in the form of Bessel's function. The model is studied by using initial and using different surface boundary conditions.

REFERENCES

1. Avhale P. S. *On some realistic Models in Differential equations*, Dr Babasaheb Ambedkar Marathawada University, Aurangabad, 2016
2. Crank. J, *The Mathematical Of Diffusion*, Oxoford University Press, 1975.
3. Gangurde AB and Boraste SS, *Perliminary evalution of bauhinia racemosa lam caesalpinaceae seed mucilage as tablet binder*, Int. J. Pharm 2(1), 80-83, 2012.
4. H.S.Carslaw and J.C.Jaeger, *Conduction of heat in solid* , Oxford clarendon press 1959.
5. J. Kevorkian and J.D.Cole, *Perturbation methods in applied mathematics,* Springer-Verlag, 1981.
6. M.Necati Ozisik, *Heat Conduction*, A Wiley-Interscience Publication,1993
7. Michael M.Cox and David L.Nelson, *Principles of Biochemistry*, W.H.Freeman And Company New York.
8. R.N.Singh, *Advection diffusion equation models in near-surface geo-physical and environmental sciences*, J. Ind. Geophys Union, Vol.17 (2), (2013), 117-127.
9. T.Roose, *Mathematical model of plant nutrient uptake*, Doctor Dissertation, Oxford Linacre College, University of Oxford, 2000.
10. T.Roose, A.C.Fowler. and P.R.Darrah *A mathematical model of plant nutrient uptake*, J. Math. Biol Vol.42, (2001), 347-360.
11. Vineet Kumar, Sudesh Kumar Yadav, *Synthesis of variable of shaped goil nanoparticles in one solution using leaf extract of Bauhinia variegata l*, Digest Journal of Nanomaterials and Biostructures, 6(4), 2011, 1685-1693.

.

Source of support: Nil, Conflict of interest: None Declared.