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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN COMPLETE METRIC SPACE

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ABSTRACT

In this paper, we persent fixed point results for generalization on spaces with two metrics. The focus in on continuation results for such type of mappings.

Key words: *Metric Space, Complete Metric Space, Self Mapping, Fixed Point, Commute Mapping.*

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INTRODUCTION

The study of common fixed point of mapping contractive type condition has been a very active field of research activity during the last three decades. The most general of the common fixed point pertain to two or three mapping of a metric space (X, d) and use either a Banach type contractive condition or other contractive condition. Many, Hardy [1], Rajput [2], Yadav [3], Sengupta [4] and so many authors work in this field and prove more interesting result. Throughout this section (X, d') denotes a complete metric space and d be an metric on X. if $x_0 \in X$ and $r > 0$ denote by $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and by $clos.B(x_0, r)^{d'}$ the d'-closer of $B(x_0, r)$.

Fixed point results for Banach Generalized contractions

Theorem: 1 Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and T be the mapping from, $clos.B(x_0, r)^{d'}$ into X, satisfying the following conditions;

$$d(Tx, Ty) \le \alpha . d(x, y) \tag{1.1}$$

Where non negative α, such that, $0 \le \alpha < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < (1 - \alpha)\, r \tag{1.2}$$

If $d \ngeq d'$ then T is uniformaly continuous from $(B(x_0, r), d)$ into (X, d') (1.3)

if $d \neq d'$ then T is continuous from $\left(clos.B(x_0, r)^{d'}, d'\right)$ into (X, d') (1.4)

then T has fixed point, that is there exists $x \in clos.B(x_0, r)^{d'}$ with $Tx = x$.

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Proof: Let $x_1 = Tx_0$ then from (1.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < (1-\alpha)\, r \le r$$

So, that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (1.1)

$$d(Tx_0, Tx_1) \le \alpha\, d(x_0, x_1)$$

$$d(Tx_0, Tx_1) \le \alpha\,(1-\alpha)\, r$$

Now

$$\begin{aligned}
d(x_0, x_2) &\le d(x_0, x_1) + d(x_1, x_2)\\
d(x_0, x_2) &\le (1-\alpha)\, r + \alpha\,(1-\alpha)\, r\\
d(x_0, x_2) &\le (1-\alpha)\, r\, (1+\alpha)\\
d(x_0, x_2) &< (1-\alpha)\, r\, (1+\alpha+\alpha^2+\alpha^3+\cdots\ldots\ldots..)\\
d(x_0, x_2) &< (1-\alpha)\, r\, (1-\alpha)^{-1}\\
d(x_0, x_2) &< r
\end{aligned}$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$\begin{aligned}
d(x_{n+1}, x_n) &\le \alpha^n\, d(x_0, x_1)\\
d(x_0, x_{n+1}) &< (1-\alpha)^n\, r\, (1-\alpha)^{-1}
\end{aligned}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d , which converges to x.

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'.

If $d \ge d'$ then this is trivial.

Next we suppose that, $d \ngeq d'$

Let $\varepsilon > 0$ be given. Now from (1.3) that there exists $\delta > 0$ such that,

$d'(Tx, Ty) < \varepsilon$ Whenever $x, y \in B(x_0, r)$ and $d(x, y) < \delta$ (1.5)

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with

$$d(x_n, x_m) < \delta \text{ for all } n, m \ge N \tag{1.6}$$

Now from (1.5) and (1.6) implies

$$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon \text{ whenever } n, m \ge N$$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'

Now since (X, d') is complete there exists $x \in \text{clos}.B(x_0, r)^{d'}$ with

$d'(x_n, x) \to 0$ and $n \to \infty$.

We claim that, $x = Tx$ (1.7)

First consider the case, when $d \ne d'$.

$$d'(x, Tx) \le d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$$

Let $n \to \infty$ and using (1.4), we obtain

$d'(x, Tx) \le d(x, x) + d(Tx, Tx)$

$d'(x, Tx) = 0$

And thus (8.7) is true,

Next we suppose that $d = d'$ then

$d'(x, Tx) \le d(x, x_n) + d(Tx_{n-1}, Tx)$

From (1.1),

$$d'(x, Tx) \le d(x, x_n) + \alpha d(x_{n-1}, Tx)$$

As $\to \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$(1-\alpha) d(x, Tx) \le 0$$

So that, $d(x, Tx) = 0$ and (1.7) holds.

This the proof of the theorem.

Theorem: 2 Let (X, d') be a complete metric space, d another metric on X, $x_0 \in X$, $r > 0$ and T be the mapping from, $\text{clos}.B(x_0, r)^{d'}$ into X, satisfying the following conditions;

$$d(Tx, Ty) \le \alpha. d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{2.1}$$

Where non negative α, β, γ , such that, $0 \le \alpha + \beta + \gamma < 1$

In addition assume the following three properties hold:

$$d(x_0, Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \tag{2.2}$$

If $d \ngeq d'$

then T is uniformaly continuous from $(B(x_0, r), d)$ into (X, d') (2.3)

if $d \neq d'$ then T is continuous from $\left(\text{clos}.B(x_0, r)^{d'}, d'\right)$ into (X, d') (2.4)
then T has fixed point, that is there exists $x \in \text{clos}.B(x_0, r)^{d'}$ with $Tx = x$.

Proof: Let $x_1 = Tx_0$ then from (2.2), we have

$$d(x_0, x_1) = d(x_0, Tx_0) < \left(1 - \frac{\alpha+\beta}{1-\beta-\gamma}\right) r \le r$$

So that, $x_1 \in B(x_0, r)$

Next let $x_2 = Tx_1$ then we note that,

$$d(x_1, x_2) = d(Tx_0, Tx_1)$$

From (2.1)

$$d(Tx_0, Tx_1) \le \alpha\, d(x_0, x_1) + \beta[d(x_0, x_1) + d(x_1, x_2)] + \gamma\, d(x_0, x_2)$$

$$d(Tx_0, Tx_1) \le \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1-\frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$

Now

$$d(x_0, x_2) \le d(x_0, x_1) + d(x_1, x_2)$$
$$d(x_0, x_2) \le \left(1-\frac{\alpha+\beta}{1-\beta-\gamma}\right) r + \left(\frac{\alpha+\beta}{1-\beta-\gamma}\right)\left(1-\frac{\alpha+\beta}{1-\beta-\gamma}\right) r$$
$$d(x_0, x_2) \le \left(1-\frac{\alpha+\beta}{1-\beta-\gamma}\right) r \ \left(1+\frac{\alpha+\beta}{1-\beta-\gamma}\right)$$
$$d(x_0, x_2) < \left(1-\frac{\alpha+\beta}{1-\beta-\gamma}\right) r \ \left(1+\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]+\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^2+\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^3+\cdots\dots\dots\right)$$
$$d(x_0, x_2) < \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right) r \ \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$
$$d(x_0, x_2) < r$$

So that, $x_2 \in B(x_0, r)$

Proceeding inductively we obtain

$$d(x_{n+1}, x_n) \le \left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]^n d(x_0, x_1)$$
$$d(x_0, x_{n+1}) < \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^n r \ \left(1-\left[\frac{\alpha+\beta}{1-\beta-\gamma}\right]\right)^{-1}$$

It follows $d(x_0, x_{n+1}) < r$ and $x_{n+1} \in B(x_0, r)$

In this way we construct a sequence $\{x_n\}$ of elements of X, such that $\{x_n\}$ is a Cauchy sequence with respect to, d, which converges to x.

We claim that $\{x_n\}$ is a Cauchy sequence with respect to d'.

If $d \ge d'$ then this is trivial.

Next we suppose that, $d >\ne d'$

Let $\varepsilon > 0$ be given Now from (1.3) that there exists $\delta > 0$ such that,

$d'(Tx, Ty) < \varepsilon$ whenever $x, y \in B(x_0, r)$ and $d(x, y) < \delta$ (2.5)

From the above the sequence $\{x_n\}$ is a Cauchy sequence with respect to d, so we know that there exists N with

$$d(x_n, x_m) < \delta \text{ for all } n, m \ge N \tag{2.6}$$

Now from (2.5) and (2.6) implies

$d'(x_{n+1}, x_{m+1}) = d'(Tx_n, Tx_m) < \varepsilon$ whenever $n, m \ge N$

Which proves that $\{x_n\}$ is a Cauchy sequence with respect to d'.

Now since (X, d') is complete there exists $x \in \text{clos}.B(x_0, r)^{d'}$ with

$d'(x_n, x) \to 0$ and $n \to \infty$.

We claim that, $x = Tx$ (2.7)

First consider the case, when $d \ne d'$

$d'(x, Tx) \le d(x, x_n) + d(x_n, Tx) = d(x, x_n) + d(Tx_{n-1}, Tx)$

Let $n \to \infty$ and using (2.4), we obtain

$d'(x, Tx) \le d(x, x) + d(Tx, Tx)$

$d^{'}(x, Tx) = 0$

And thus (2.7) is true,

Next we suppose that $d = d'$ then

$d^{'}(x, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$

From (2.1),

$d^{'}(x, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, Tx) + \beta[d(x_{n-1}, Tx_{n-1}) + d(x, Tx)] + \gamma[d(x_{n-1}, Tx) + d(x, Tx_{n-1})]$

As $\to \infty$, $Tx_n = x = Tx$ and above inequality can be written as,

$$\left(1 - \left[\frac{\alpha + \beta}{1-\beta-\gamma}\right]\right) d(x, Tx) \leq 0$$

So that, $d(x, Tx) = 0$ and (2.7) holds.

This complete proof of the theorem

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