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A Study of *n*-dimensional Minkowski Space with *t-topology*

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ABSTRACT

The topological properties, namely Hausdorffness, regularity, normality, metrizability, compactness, local compactness, paracompactness, first countability, second countability and separability of the n-dimensional Minkowski space with t-topology are studied and proved [the results are stated in "t-topology on the n-dimensional Minkowski space", Journal of Mathematical Physics, Volume 50, (2009) without proofs]. Further, the results obtained and the techniques used in proving these are compared with the corresponding known results and techniques for the n-dimensional Euclidean space.

Keywords— *t-topology, Minkowski space, Euclidean space, Topological properties, n-dimensional space.*

I. INTRODUCTION

In Mathematics, Minkowski space or Minkowski spacetime is the setting in which Einstein theory of special relativity is most conveniently formulated. In this setting the *three* ordinary dimensions of space are combined with the *single* dimension of time to form a *four*-dimensional manifold for representing a spacetime. Minkowski space is often contrasted with Euclidean space. While the Euclidean space has only spacelike dimensions, a Minkowski space also has one timelike dimension. Therefore, the symmetry group of a Euclidean space is the Euclidean group and for a Minkowski space, it is the Poincare group. Lorentz invariant quantization of string theory can be developed using *four*-dimensional Minkowski space.

In Special relativity (in Minkowski spacetime), the metric signature (+, -, -, -) or (-, +, +, +) signifies that there are three types of vectors, timelike, spacelike or null (sometimes called as lightlike). Any Lorentz transformation preserves the type of the vector. That is, a spacelike vector cannot be transformed into a null or timelike vector and likewise for the other two. A Euclidean metric does not have this structure. Any Lorentz transformation preserves the type of a null (lightlike) vector encodes the statement that "the speed of light is the same for all observers".

The most natural topology on R^n is the Euclidean topology. However it does not take into account one of the important features of the Minkowski space, that is, its causal structure of spacetime and the homeomorphism group of the Minkowski space with Euclidean topology is too large to be of any physical significance. So the non-Euclidean topologies arising out of the quadratic form as Minkowski space has become the focus of the present day research. Fine topology [10], proposed by Zeeman, was the first non-Euclidean topology. This topology incorporates the causal structure of Minkowski spacetime, as well as, has its homeomorphic group isomorphic to the group generated by the Lorentz group, translations and dilations. The other interesting non-Euclidean topologies on Minkowski space are *t*-topology, *s*-topology, space topology, time topology, order topology, etc [9, 10].

The present paper is focused on the study and proofs of topological properties of the *n*-dimensional Minkowski space with the non-Euclidean *t*-topology.

II. NOTATION AND PRELIMINARIES

Let R and N denote the set of real numbers and natural numbers, respectively. Let R^n be the n-dimensional real coordinate space.

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A. Topological Space

A topological space is a set X together with T, a collection of subsets of X, satisfying the following axioms:

- The empty set and X are in T.
- The union of any collection of sets in T is also in T.
- The intersection of any finite collection of sets in T is also in T.

The collection T is called a topology on X. The elements of X are usually called points, though they can be any mathematical objects. The sets in T are the open sets, and their complements in X are called closed sets. A set may be neither closed nor open, either closed or open, or both. A set that is both closed and open is called a clopen set.

Suppose that T and T' are two topologies on a given set X. If T is a subset of T', we say that T' is finer than T; if T' properly contains T, we say that T' is strictly finer than T. We also say that T is coarser than T' or strictly coarser, in these two respective situations. We say that T is comparable with T' if either T is a subset of T' or T' is a subset of T.

Topological spaces can be broadly classified, up to homeomorphism, by their topological properties. A topological property is a property of spaces that is invariant under homeomorphisms. Some common topological properties are separation axioms, countability conditions, connectedness, compactness, metrizability, etc [4].

B. Topological Properties

- Hausdorff Space: A topological space X is called a Hausdorff space if for each pair x_1, x_2 of each distinct points of X, there exists neighbourhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.
- Regular Space: Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B [4].
- Normal Space: The space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exists disjoint open sets containing A and B respectively [4].
- Compact Space: A space X is said to be compact if every open covering A of X contains a finite sub collection that also covers X [4].
- Paracompact Space: A space X is said to be paracompact if every open covering A of X has a locally finite open refinement B that covers X [4].
- Local Compact Space: A space X is said to be locally compact at x if there is some compact subspace C of X that contains a neighbourhood of x. If X is locally compact at each of its points, X is said to be simply locally compact [4].
- Connected Space: The space X is said to be connected if there does not exist a separation of X. A separation of X is a pair U, V of disjoint non empty open subsets of X whose union is X [4].
- Path Connected Space: A space X is said to be path connected if every pair of points of X can be joined by a path in X. Given points x and y of the space X, a path in X from x to y is a continuous map $f:[a,b]\rightarrow X$ of some closed interval in the real line into X such that $f(a) = x$ and $f(b) = y$ [4].
- Local Connected Space: A space X is said to be locally connected at x if for every neighbourhood U of x, there is a connected neighbourhood V of x contained in U. If X is locally connected at each of its points it is said to be locally connected [4].
- Separable Space: A space having a countable dense subset is often said to be separable [4].
- First Countable Space: A space X that has a countable basis at each of its points is said to satisfy the first countability axiom. A space X is said to have a countable basis at the point x if there is a countable collection $\{U_n\}_{n\in Z^+}$ of neighbourhood U of x contains at least one of these sets U_n [4].
- Second Countable Space: If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be second countable [4].
- Metrizable Space: If X is a topological space, X is said to be metrizable if there exists a metric d on the set X that induces the topology of X. A metric space is a metrizable space X together with a specific metric d that gives the topology of X [4].

C. Euclidean Space

The n-dimensional Euclidean space, is the set R^n together with the distance function, which is obtained by defining the distance between two points $(x_1, x_2, x_{3,\dots,} x_n)$ and $(y_1, y_2, y_{3,\dots,} y_n)$ to the square root of (sigma)$(x_i - y_i)^2$, where the sum is over $i = 1, 2, 3, \dots, n$. This distance function is called the Euclidean metric. That is,

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

Euclidean Space is a topological space with the natural topology induced by the metric. The metric topology on R^n is called the Euclidean topology [4].

D. Minkowski Space

For $n \in N$ and $n > 1$, the n-dimensional real vector space R^n with the bilinear form $g:R^n \text{x} R^n \rightarrow R$, satisfying the following properties: (1) for all $x, y \in R^n$, $g(x, y) = g(y, x)$, i.e., the bilinear form is symmetric, (2) if for all $y \in R^n$, $g(x, y) = 0$, then $x = 0$, i.e., the bilinear form is non-degenerate, and (3) there exists a basis $\{e_0, e_1, e_2, \dots, e_{n-1}\}$ for R^n with

$$g(e_i, e_j) \equiv \eta_{ij} = \begin{cases} 1, i = j = 0 \\ -1, i = j = 1, ..., n-1 \\ 0, i \neq j \end{cases}$$

is called the n-dimensional Minkowski space, denoted by the symbol M. The bilinear form g is called the Lorentz inner product and the matrix $(\eta_{ij})_{nxn}$ is known as the Minkowski metric. Thus the standard Lorentzian or Minkowski inner product is an indefinite inner product with one positive or timelike direction and many negative or spacelike directions. If x and y are two vectors such that $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, then the inner product can be defined as

$$<x, y> = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$$

This is not an inner product in usual sense, since it is not positive definite [1].

E. Lines and Cones

A straight line is called a timelike straight line or light ray or spacelike straight line, accordingly as it is parallel to a timelike or lightlike or spacelike vector [1].

An event $x \in M$ is called the timelike, lightlike (also called null), or spacelike, accordingly as $Q(x)$ is positive, zero or negative. The sets $C^T(x) = \{y\in M: y = x \text{ or } Q(y- x) > 0\}$, $C^L(x) = \{y\in M: Q(x, y) = 0\}$, $C^S(x) = \{y\in M: y = x \text{ or } Q(y- x) < 0\}$ are likewise respectively, called the time cone, light cone (or null cone) and space cone at x [1].

F. Topologies on M

- Euclidean Topology on M: For $x \in R^n$ and $\varepsilon > 0$, $N_\varepsilon^E(x)$ denotes the Euclidean open ball about x of radius ε given by the set $\{y\in R: d_E(x, y) < \varepsilon\}$, where $d_E(x, y)$ is the Euclidean distance between x and y. The topology generated by the basis $B = \{ N_\varepsilon^E(x): \varepsilon > 0, x \in M\}$ is called the Euclidean topology on M [1].
- t-topology on M: The collection $N(x) = \{ N_\varepsilon^t(x): \varepsilon > 0\}$, where $N_\varepsilon^t(x) = N_\varepsilon^E(x) \cap C^T(x)$, forms a local base for the family of neighbourhoods of $x \in M$. The topology generated by these neighbourhood systems is called the t-topology: $N_\varepsilon^t(x)$ will be called the t-neighbourhood of radius ε. Hence U, a subset of M is open with respect to t-topology if and only if for each $x\in U$, there exists some $N_\varepsilon^t(x)$ such that $N_\varepsilon^t(x)$ is a subset of U. It thus follows that $N_\varepsilon^E(x)$ and $N_\varepsilon^t(x)$ are open in M with t-topology, $N_\varepsilon^t(x)$-$\{x\}$ is open in M with Euclidean topology, while $N_\varepsilon^t(x)$ is not open in M with Euclidean topology. Hence $\{ N_\varepsilon^t(x): \varepsilon > 0, x \in M\}$ is a basis for t-topology and the t-topology is strictly finer than the Euclidean topology on M [1].

III. TOPOLOGICAL PROPERTIES OF M^t

In this section, topological properties namely Hausdorffness, regular, normal, metrizable, compact, locally compact, paracompact, first countable, second countable and separable of the n-dimensional Minkowski space with t-topology are discussed. Also alternate proofs are obtained for Hausdorffness and compactness.

A. Topological Properties

It is well known that R^n with Euclidean topology is Hausdorff, regular, normal, metrizable, not compact, locally compact, paracompact, first countable, second countable and separable. The proofs for these properties for n-dimensional Minkowski space with t-topology are as follows:

***1) Proposition*:** Let M be the n-dimensional Minkowski space. Then M^t is Hausdorff.

***Proof*:** Let $x, y \in M$ such that $x \neq y$. We obtain two non empty, disjoint open sets in M containing x and y respectively. Choose $U = N_\varepsilon^t(x)$ and $V = N_\varepsilon^t(y)$ where $\varepsilon = \frac{1}{2}d(x,y)$, for all $x, y \in M$. Since $N_\varepsilon^t(x)$ and $N_\varepsilon^t(y)$ are the t-neighbourhoods of x and y with radius ε respectively, hence $x \in U$ and $y \in V$ respectively. By the definition U and V are open in M^t. We assert that they are disjoint. For if $z \in U\cap V$, then $z \in N_\varepsilon^t(x) \cap N_\varepsilon^t(y)$, for some $x \in U$, $y \in V$. That is, $z \in \{N_\varepsilon^E(x) \cap C^T(x)\} \cap \{ N_\varepsilon^E(y) \cap C^T(y)\}$, for some $x \in U$, $y \in V$. That implies $z \in N_\varepsilon^E(x)$ and $z \in N_\varepsilon^E(y)$, that implies $N_\varepsilon^E(x)=\{z: d(x, z) < \varepsilon\}$ and $N_\varepsilon^E(y)=\{z: d(y, z) < \varepsilon\}$. But since $\varepsilon = \frac{1}{2}d(x,y)$ and $x \neq y$, the above statement is a contradiction, hence $U \cap V = \emptyset$. This proves that, n-dimensional Minkowski space with t-topology i.e., M^t is Hausdorff.

***Alternate Proof*:** We know that, R^n is Hausdorff and Hausdorffness is preserved under finer topology. Since t-topology is finer than Euclidean topology, hence M^t is also Hausdorff.

2) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not regular.*

Proof: Assume that M^t is regular. Let $x\in M$ and λ be a light ray passing through x. Let $F\equiv\lambda$-$\{x\}$, then F is a closed set in M^t such that $x \notin F$. There do not exist two disjoint open sets G_x and G_F such that $x\in G_x$ and $F\subseteq G_F$. Hence M^t is not regular.

3) Lemma: *The subspace topology induced on a light ray induced from M^t is discrete.*
Proof: Let λ be a light ray and $x \in \lambda$. Then a neighbourhood of x in λ is given by $N_\varepsilon^t(x) \cap \lambda = \{x\}$. Hence x is open in λ. Therefore λ is discrete.

Remark: The above lemma intuitively means that the track of a photon is not continuous.

4) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not normal.*
Proof: Let M^t be a normal space. We know that a topological space which satisfies the conditions of T_4 is known as a normal space and that which satisfies the conditions of T_3 is known as regular space. Also $T_4 \Rightarrow T_3$[4]. Thus M^t is regular, a contradiction. Hence M^t is not normal.

5) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not metrizable.*
Proof: Let M^t be metrizable. Then "Every metrizable space is normal" [4]. This implies that M^t is normal, a contradiction. This proves that M^t is not metrizable.

6) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not compact.*
Proof: Let M^t be compact. Then since "Every compact Hausdorff space is normal" [4] and M^t is Hausdorff, hence M^t is normal, a contradiction. This proves that M^t is not compact.

Alternate Proof: Let M^t be compact. Since compactness is preserved under coarser topology and Euclidean topology on M is coarser than t-topology on M, hence R^n is compact, a contradiction. This proves that M^t is not compact.

7) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not locally compact.*
Proof: Let M^t be locally compact. Then since "Every locally compact Hausdorff space is regular" [4] and M^t is Hausdorff, hence M^t is regular, a contradiction. This proves that M^t is not locally compact.

8) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not paracompact.*
Proof: Let M^t be paracompact. Then since "Every paracompact Hausdorff space is normal" [4] and M^t is Hausdorff, hence M^t is normal, a contradiction. This proves that M^t is not paracompact.

9) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is first countable.*
Proof: Let $x \in M$. Then the collection of open sets $N(x) = \{N_\varepsilon^t(x)$: $\varepsilon > 0$, ε being any rational$\}$, where $N_\varepsilon^E(x) \cap C^T(x)$, containing a point x is called a base at x since for each open set $N_\varepsilon^t(x)$ containing x there is a $N_\varepsilon^E(x) \in N(x)$ such that $x \in N_\varepsilon^E(x)$ is a subset of $N_\varepsilon^t(x)$. Since set of rational numbers is countable hence $N(x)$ is countable base at x. This proves that M^t is first countable.

10) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is not second countable.*
Proof: Let M^t be second countable. Then light ray λ is also second countable, since subspace of a second countable space is second countable. By Lemma 3, λ is discrete. Since λ is uncountable, λ cannot be second countable, a contradiction. Hence M^t is not second countable.

11) Proposition: *Let M be the n-dimensional Minkowski space. Then M^t is separable.*
Proof: We know that K^n is countable in M, since finite product of countable sets is countable. Now for proving that K^n is dense in M^t we consider two cases: Case (1) $x \in K^n$: then clearly $K^n \cap N_\varepsilon^t(x) \neq \varnothing$, Case (2) $x \notin K^n$: assume that $K^n \cap N_\varepsilon^t(x) = \varnothing$. This implies that K^n is not dense in R^n, is a contradiction. Hence M^t is separable.

IV. CONCLUSIONS

It is natural to expect that the topological properties which hold good for R^n may not hold for M^t. It has been observed that the topological properties of the n-dimensional Minkowski space with t-topology are not same as that of the n-dimensional Euclidean space. M^t and R^n are both Hausdorff, not compact, first countable and separable. While R^n is regular, normal, locally compact, paracompact, metrizable and second countable, M^t is not. The techniques and methods used for the two spaces are different.

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