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UNIQUE COMMON FIXED POINT THEOREMS FOR FOUR MAPS IN DISLOCATED QUASI b-METRIC SPACES

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ABSTRACT

In this paper, we prove two common fixed point theorems for four mappings in dislocated quasi b-metric spaces.

***Subject Classification:** 54H25, 47H10.*

***Keywords:** Dislocated quasi b-metric, weakly compatible pair of maps, Cauchy sequence.*

1. INTRODUCTION

Zeyada *et.al* [12] initiated the concept of dislocated quasi metric spaces and generalized the results of Hitzler and Seda [5] in dislocated quasi metric spaces. The notion of b-metric space was introduced by Czerwic [3] in connection with some problems concerning with the convergence of non measurable functions with respect to measure. Recently Klin-eam and Suanoom [7] introduced the concept of dislocated quasi b-metric spaces and which generalize b-metric spaces [3] and quasi b-metric spaces [10] and proved some fixed point theorems in it by using cyclic contractions. The authors [1, 4, 7, 8, 9, 11] etc. Obtained fixed, common fixed points theorems in dislocated quasi b-metric spaces using various contraction conditions for single and two maps.

In this paper, we prove two common fixed point theorems for four maps in dislocated quasi b-metric spaces and we also give examples to support our theorems.

First we recall some known definitions and lemmas.

Definition 1.1: Let X be a non-empty set, $s\geq 1$(a fixed constant) and d: $X\times X\to[0,\infty)$ be a function. consider the following condition on d.

(1.1.1) $d(x,x) = 0, \forall x \in X$,
(1.1.2) $d(x,y) = d(y,x) = 0 \implies x = y, \forall x, y \in X$,
(1.1.3) $d(x,y) = d(y,x), \forall x, y \in X$,
(1.1.4) $d(x,y) \leq d(x,z) + d(z,y), \forall x, y, z \in X$,
(1.1.5) $d(x,y) \leq s[d(x,z) + d(z,y)], \forall x, y, z \in X$.

(i) If d satisfies (1.1.2),(1.1.3) and (1.1.4) then d is called a dislocated metric and (X, d) is called a dislocated metric space.
(ii) If d satisfies (1.1.1),(1.1.2) and (1.1.4) then d is called a quasi metric and (X, d) is called a quasi metric space.
(iii) If d satisfies (1.1.2) and (1.1.4) then d is called a dislocated quasi metric or dq-metric and (X, d) is called a dislocated quasi metric space.
(iv) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.4) then d is called a metric and (X, d) is called a metric space.
(v) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.5) then d is called a b-metric and (X, d) is called a b-metric space.
(vi) if d satisfies (1.1.2) and (1.1.5) then d is called a dislocated quasi b-metric and (X, d) is called a dislocated quasi b-metric space or dq-metric space.

Definition 1.2: Let (X, d) be a dq b- metric space. A sequence $\{x_n\}$ in (X, d) is said to be
(i) dq b − convergent if there exists some point $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$.
In this case x is called a dq b-limit of $\{x_n\}$ and we write $x_n\to x$ as $n\to\infty$.
(ii)Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = 0 = \lim_{m,n\to\infty} d(x_m, x_n)$.

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The space (X, d) is called complete if every Cauchy sequence in X is dq b-convergent.

One can prove easily the following

Lemma 1.3: Let (X, d) be a dq b-metric space and $\{x_n\}$ be dq b-convergent to x in X and $y \in X$ be arbitrary.Then
$$\frac{1}{s}d(x,y) \le \lim_{n\to\infty}\inf d(x_n,y) \le \lim_{n\to\infty}\sup d(x_n,y) \le sd(x,y) \quad \text{and}$$
$$\frac{1}{s}d(y,x) \le \lim_{n\to\infty}\inf d(y,x_n) \le \lim_{n\to\infty}\sup d(y,x_n) \le sd(y,x).$$

Note: $\frac{1}{2s}d(x,y) \le \max\{d(x,z),\ d(z,y)\}\ \forall x,y,z \in X.$

Definition 1.4: [6] Let X be a non-empty set and S, T: X→X be given self maps on X. The pair (S, T) is said to be weakly compatible if STx=TSx whenever there exists x∈X such that Sx=Tx.

Definition 1.5: [2] Let X be a non-empty set and f, g: X→X be mappings. If there exists x ∈ X such that fx = gx. Then x is called a Coincidence point of f and g and fx is called a point of Coincidence of f and g.

Now we prove our main result.

2. MAIN RESULT

We need the following definition

Definition 2.1: For the fixed constant $s \ge 1$, let Φ_s denote the set of all functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the following

(ϕ_1) : ϕ is monotonically non-decreasing ,

(ϕ_2): $\sum_{n=1}^{\infty} s^n\phi^n(t) < \infty$ for all $t > 0$,

(ϕ_3) : $\phi(t) < t$ for $t > 0$.

Clearly (ϕ_1) and (ϕ_3) implies $\phi(0) = 0$.

Theorem 2.2: Let (X, d) be a complete dislocated quasi b-metric space with fixed constant s ≥ 1 and f, g, S, T: X→X be continuous mappings satisfying

(2.2.1) $d(fx,gy) \le \phi\left(\max\left\{d(Sx,Ty),\ \frac{1}{2s}d(Sx,fx),\ \frac{1}{2s}d(Ty,gy),\ \frac{1}{2s}d(Sx,gy),\ \frac{1}{2s}d(Ty,fx)\right\}\right)$
$\forall\, x,y \in X$, where $\phi \in \Phi_s$,

(2.2.2) $d(gx,fy) \le \phi\left(\max\left\{d(Tx,Sy),\ \frac{1}{2s}d(Tx,gx),\ \frac{1}{2s}d(Sy,fy),\ \frac{1}{2s}d(Sy,gx),\ \frac{1}{2s}d(Tx,fy)\right\}\right)$
$\forall\, x,y \in X$, where $\phi \in \Phi_s$,

(2.2.3) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$,

(2.2.4) $fS = Sf$ and $gT = Tg$.

Then f, g, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$.

Define $y_{2n} = fx_{2n} = Tx_{2n+1}, y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, n=0,1,2.......

Case-(i): Suppose $\max\{d(y_{n-1},y_n),d(y_n,y_{n-1})\} = 0$ for some n.
Without loss of generality assume that n=2m.

Then $y_{2m-1} = y_{2m}$.

Using (2.2.1), (2.2.2) and (ϕ_1), we get

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &= d(fx_{2m}, gx_{2m+1}) \\ &\le \phi\left(\max\left\{d(y_{2m-1}, y_{2m}),\ \tfrac{1}{2s}d(y_{2m-1}, y_{2m}),\ \tfrac{1}{2s}d(y_{2m}, y_{2m+1}), \tfrac{1}{2s}d(y_{2m-1}, y_{2m+1}), \tfrac{1}{2s}d(y_{2m}, y_{2m})\right\}\right) \\ &\le \phi\left(\max\left\{\begin{matrix} d(y_{2m-1}, y_{2m}),\ \ d(y_{2m-1}, y_{2m}),\ \ d(y_{2m}, y_{2m+1}), \\ \max\{d(y_{2m-1}, y_{2m}),\ d(y_{2m}, y_{2m+1})\},\ \ \max\{d(y_{2m}, y_{2m-1}),\ d(y_{2m-1}, y_{2m})\}\end{matrix}\right\}\right), \text{ from Note} \\ &= \phi(\max\{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m-1}), d(y_{2m}, y_{2m+1})\}) \end{aligned}$$

and

$$\begin{aligned}
d(y_{2m+1}, y_{2m}) &= d(gx_{2m+1}, fx_{2m})\\
&\le \phi(\max\{d(y_{2m}, y_{2m-1}), \tfrac{1}{2s}d(y_{2m}, y_{2m+1}), \tfrac{1}{2s}d(y_{2m-1}, y_{2m}), \tfrac{1}{2s}d(y_{2m-1}, y_{2m+1}), \tfrac{1}{2s}d(y_{2m}, y_{2m})\})\\
&\le \phi\left(\max\left\{\begin{matrix} d(y_{2m}, y_{2m-1}), & d(y_{2m}, y_{2m+1}), & d(y_{2m-1}, y_{2m}), \\ \max\{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\}, & & \max\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m})\}\end{matrix}\right\}\right)\\
&= \phi(\max\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\}).
\end{aligned}$$

Thus

$$\begin{aligned}
\max\{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\} &\le \phi\left(\max\left\{\begin{matrix} d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m-1}), \\ d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\end{matrix}\right\}\right) \\
&= \phi(\max\{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\})
\end{aligned} \tag{1}$$

From (ϕ_3) and (1.1.2), we have $y_{2m} = y_{2m+1}$. Thus $y_{2m-1} = y_{2m} = y_{2m+1}$.

Continuing in this way we have $y_{2m-1} = y_{2m} = y_{2m+1} = \cdots$

Thus $y_{n-1} = y_n = y_{n+1} = \cdots$

Hence $\{y_n\}$ is a constant Cauchy sequence.

Case-(ii): suppose $\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_n)\} \neq 0$ for all n.
As in (1), we have

$$\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} \le \phi\left(\max\left\{\begin{matrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), \\ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\end{matrix}\right\}\right) \tag{2}$$

If $\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\} \le \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\}$,
then from (2), using (ϕ_3) , we get
$\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} = 0$, which is a contradiction to Case (ii).

Hence $\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\} > \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\}$.

Now from (2), $\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} \le \phi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1})\})$ (3)

This is true for n = 1, 2, 3 ....

$$\begin{aligned}
\text{Hence } \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_n)\} &\le \phi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n-1})\})\\
&\dots\dots\dots\dots\\
&\le \phi^n(\max\{d(y_0, y_1), d(y_1, y_0)\})
\end{aligned} \tag{4}$$

Now for all positive integers n and p, consider, using (4),

$$\begin{aligned}
d(y_n, y_{n+p}) &\le sd(y_n, y_{n+1}) + s^2d(y_{n+1}, y_{n+2}) + \dots + s^pd(y_{n+p-1}, y_{n+p})\\
&\le s\phi^n(t) + s^2\phi^{n+1}(t) + \dots + s^p\phi^{n+p-1}(t), \text{ where } t = \max\{d(y_0, y_1), d(y_1, y_0)\}\\
&\le s^n\phi^n(t) + s^{n+1}\phi^{n+1}(t) + \dots + s^{n+p-1}\phi^{n+p-1}(t), \text{ since } s \ge 1\\
&\le \textstyle\sum_{i=n}^{n+p-1} s^i\phi^i(t) \le \sum_{i=n}^{\infty} s^i\phi^i(t) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $\sum_{i=n}^{\infty} s^i\phi^i(t)$ converges for all $t > 0$.

Thus we have $\lim_{n\to\infty} d(y_n, y_{n+p}) = 0$.

Also using (4), we have

$$\begin{aligned}
d(y_{n+p}, y_n) &\le sd(y_{n+p}, y_{n+1}) + sd(y_{n+1}, y_n)\\
&\le s^2d(y_{n+p}, y_{n+2}) + s^2d(y_{n+2}, y_{n+1}) + sd(y_{n+1}, y_n)\\
&\le s^3d(y_{n+p}, y_{n+3}) + s^3d(y_{n+3}, y_{n+2}) + s^2d(y_{n+2}, y_{n+1}) + sd(y_{n+1}, y_n)\\
&\dots\dots\dots\dots\\
&\le s^{p-1}d(y_{n+p}, y_{n+p-1}) + s^{p-1}d(y_{n+p-1}, y_{n+p-2}) + \dots + s^2d(y_{n+2}, y_{n+1}) + sd(y_{n+1}, y_n)\\
&\le s^{p-1}\phi^{n+p-1}(t) + s^{p-1}\phi^{n+p-2}(t) + \dots + s^2\phi^{n+1}(t) + s\phi^n(t)\\
&\le s^{n+p-1}\phi^{n+p-1}(t) + s^{n+p-2}\phi^{n+p-2}(t) + \dots + s^{n+1}\phi^{n+1}(t) + s^n\phi^n(t) \quad \text{since } s \ge 1.\\
&= \textstyle\sum_{i=n}^{n+p-1} s^i\phi^i(t) \le \sum_{i=n}^{\infty} s^i\phi^i(t) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence we have $\lim_{n\to\infty} d(y_{n+p}, y_n) = 0$.

Thus $\{y_n\}$ is a Cauchy sequence in X.

Since X is a complete dislocated quasi b – metric space, there exists $z \in X$ such that $\{y_n\}$ converges to z.

Since S and f are continuous and Sf = fS, we have
$Sz = \lim_{n\to\infty} Sy_{2n} = \lim_{n\to\infty} Sfx_{2n} = \lim_{n\to\infty} fSx_{2n} = \lim_{n\to\infty} fy_{2n-1} = fz.$

Similarly, since T and g are continuous and Tg = gT, we have Tz = gz.

Using (2.2.1), (2.2.2), (ϕ_1) and Note, we get
$d(Sz , Tz) = d(fz , gz)$
$\leq \phi\left(\max\left\{d(Sz,Tz),\ \frac{1}{2s}d(Sz,Sz),\ \frac{1}{2s}d(Tz,Tz),\ \frac{1}{2s}d(Sz,Tz),\ \frac{1}{2s}d(Tz,Sz)\right\}\right)$
$\leq \phi(\max\{d(Sz, Tz), d(Tz, Sz)\})$
and
$d(Tz, Sz) \leq \phi(\max\{d(Sz, Tz), d(Tz, Sz)\}).$

Thus $\max\{d(Sz,Tz), d(Tz, Sz)\} \leq \phi(\max\{d(Sz, Tz), d(Tz, Sz)\})$

which in turn yields from (ϕ_3) and (1.1.2) that Sz = Tz.

Let $\alpha = Sz = Tz$. Then $S\alpha = S(Sz) = S(fz) = f(Sz) = f\alpha$ and $T\alpha = T(Tz) = T(gz) = g(Tz) = g\alpha$.

Now using (2.2.1), (2.2.2), (ϕ_1) and from Note, we have
$d(S\alpha , \alpha) = d(f\alpha , gz)$
$\leq \phi\left(\max\left\{d(S\alpha,\alpha),\ \frac{1}{2s}d(S\alpha,S\alpha),\ \frac{1}{2s}d(\alpha,\alpha),\ \frac{1}{2s}d(S\alpha,\alpha),\ \frac{1}{2s}d(\alpha,S\alpha)\right\}\right)$
$\leq \phi(\max\{d(S\alpha, \alpha), d(\alpha, S\alpha)\})$
and
$d(\alpha, S\alpha) \leq \phi(\max\{d(S\alpha, \alpha), d(\alpha, S\alpha)\}).$

Thus we have $\max\{d(S\alpha, \alpha),d(\alpha, S\alpha)\} \leq \phi(\max\{d(S\alpha, \alpha), d(\alpha, S\alpha)\})$

which in turn yields from (ϕ_3) and (1.1.2) that $S\alpha = \alpha$.

Similarly we can show that $T\alpha = \alpha$.

Thus $f\alpha = S\alpha = \alpha = T\alpha = g\alpha$.

Hence α is a common fixed point of f, g, S and T.

One can prove the uniqueness of common fixed point of f, g, S and T using (2.2.1) and (2.2.2).

Now we give an example to illustrate the Theorem 2.2.

Example 2.3: Let **X= [**0, 1**]** and $d(x, y) = (x+2y)^2$.

Let f, g, S, T: X→X be defined by $fx = \frac{x}{8}$, $gx = \frac{x}{12}$, $Sx = \frac{x}{2}$ and $Tx = \frac{x}{3}$.

Let $\phi:[0,\infty)\to[0,\infty)$ be defined by $\phi(t) = \frac{t}{4}$, for $t \in [0,\infty)$.

Then it is clear that $d(x, y) = d(y, x) = 0 \Rightarrow x = y$

Also $d(x, y) = (x+2y)^2 \leq [(x + 2z) + (z + 2y)]^2 \leq 2[(x + 2z)^2 + (z + 2y)^2] = s[d(x,z) + d(z,y)]$, where s = 2

Thus d is a dislocated quasi b – metric with s = 2.

Consider $d(fx, gy) = \left(\frac{x}{8} + \frac{2y}{12}\right)^2$
$$= \left(\frac{3x + 4y}{24}\right)^2$$
$$= \left(\frac{\frac{x}{2}+\frac{2y}{3}}{4}\right)^2$$

$$= \frac{d(Sx,Ty)}{16}$$
$$\leq \frac{1}{4} d(Sx,Ty)$$
$$\leq \frac{1}{4} \max\left\{d(Sx,Ty),\ \frac{1}{2s}d(Sx,fx),\ \frac{1}{2s}d(Ty,gy),\ \frac{1}{2s}d(Sx,gy),\ \frac{1}{2s}d(Ty,fx)\right\}$$
$$= \phi\left(\max\left\{d(Sx,Ty),\ \frac{1}{2s}d(Sx,fx),\ \frac{1}{2s}d(Ty,gy),\ \frac{1}{2s}d(Sx,gy),\ \frac{1}{2s}d(Ty,fx)\right\}\right).$$

Similarly we can show that
$$d(gx,fy) \leq \phi\left(\max\left\{d(Tx,Sy),\ \frac{1}{2s}d(Tx,gx),\ \frac{1}{2s}d(Sy,fy),\ \frac{1}{2s}d(Sy,gx),\ \frac{1}{2s}d(Tx,fy)\right\}\right).$$

Clearly $f(X) = [0, \frac{1}{8}] \subseteq [0, \frac{1}{3}] = T(X)$ and $g(X) = [0, \frac{1}{12}] \subseteq [0, \frac{1}{2}] = S(X)$.

It is also clear that Sf = fS and Tg = gT.

For t > 0,

Consider $\sum_{n=1}^{\infty} s^n \phi^n(t) = \sum_{n=1}^{\infty} 2^n \frac{t}{4^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} t = t\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = t < \infty.$

Thus all conditions of Theorem 2.2 are satisfied. Clearly 0 is the unique common fixed point of f, g, S and T.

In the similar lines of proof of Theorem 2.2, we prove the following.

Theorem 2.4: Let (X, d) be a complete dislocated quasi b-metric space with fixed constant $s \geq 1$and f, g: X⟶X be continuous mappings satisfying
(2.4.1) $d(fx,gy) \leq \phi\left(\max\left\{d(x,y),\ d(x,fx),\ d(y,gy),\ \frac{1}{2s}d(x,gy),\ \frac{1}{2s}d(y,fx)\right\}\right)$ $\forall x,y \in X$, where $\phi \in \Phi_s$,
(2.4.2) $d(gx,fy) \leq \phi\left(\max\left\{d(x,y),\ d(x,gx),\ d(y,fy),\ \frac{1}{2s}d(y,gx),\ \frac{1}{2s}d(x,fy)\right\}\right)$ $\forall x,y \in X$, where $\phi \in \Phi_s$.
Then f and g have a unique common fixed point .

Proof: As in Theorem 2.2, we can show that $\{x_n\}$ is convergent to $z \in X$, where $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, n = 0,1,2.....and $x_0 \in X$ is arbitrary.

Since f is continuous and $x_n \to z$, we have
$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = f\left(\lim_{n\to\infty} x_n\right) = fz.$$

Similarly, since g is continuous we have z = gz.

Thus z is a common fixed point of f and g.

Consider $d(z,z) = d(fz,gz) \leq \phi\left(\max\left\{d(z,z),\ d(z,z),\ d(z,z),\ \frac{1}{2s}d(z,z),\ \frac{1}{2s}d(z,z)\right\}\right) = \phi(d(z,z))$

From (ϕ_3) follows that d(z, z) = 0

Thus d(z, z) = 0 whenever z is a common fixed point of f and g.

Now suppose that w is another common fixed point of f and g.

Then d(w, w) = 0.

Now consider d(z, w) = d(fz, gw)
$$\leq \phi\left(\max\left\{d(z,w),\ d(z,z),\ d(w,w),\ \frac{1}{2s}d(z,w),\ \frac{1}{2s}d(w,z)\right\}\right)$$
$$\leq \phi(\max\{d(z,w), d(w,z)\})$$
and
d(w, z) = d(gw, fz)
$$\leq \phi\left(\max\left\{d(w,z),\ d(w,w),\ d(z,z),\ \frac{1}{2s}d(z,w),\ \frac{1}{2s}d(w,z)\right\}\right)$$
$$\leq \phi(\max\{d(z,w), d(w,z)\}).$$

Hence $\max\{d(z, w), d(w, z)\} \leq \phi(\max\{d(z, w), d(w, z)\})$

which in turn yields from (ϕ_3) and (1.1.2) that $w = z$.

Hence z is the unique common fixed point of f and g.

Theorem 2.5: Let (X, d) be a complete dislocated quasi b-metric space with fixed constant $s \geq 1$and f, g : X→X be continuous mappings satisfying
(2.5.1) $d(fx, fy) \leq \phi\left(\max\left\{d(gx, gy),\ d(gx, fx),\ d(gy, fy),\ \frac{1}{2s}d(gx, fy),\ \frac{1}{2s}d(gy, fx)\right\}\right)$ $\forall\, x, y \in X$, where $\phi \in \Phi_s$,
(2.5.2) $f(X) \subseteq g(X)$ and $fg = gf$.
Then f and g have a unique common fixed point.

Proof: As in Theorem 2.2, we can show that $\{gx_n\}$ is convergent to $z \in X$, where $fx_n = gx_{n+1}$, n = 0,1,2.....and $x_0 \in X$ is arbitrary.

Since f and g are continuous and $fg = gf$, we have $fz = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gz$.
Thus fz is a point of coincidence of f and g.

Consider $d(fz, fz) \leq \phi\left(\max\left\{d(fz, fz),\ d(fz, fz),\ d(fz, fz),\ \frac{1}{2s}d(fz, fz),\ \frac{1}{2s}d(fz, fz)\right\}\right) = \phi(d(fz, fz))$

which in turn yields from (ϕ_3) that $d(fz, fz) = 0$.

Thus if fz is a point of coincidence of f and g then $d(fz, fz) = 0$.

Suppose fw is another point of coincidence of f and g. Then $d(fw, fw) = 0$.

From (2.5.1) and (ϕ_1), we have
$$d(fz, fw) \leq \phi\left(\max\left\{d(fz, fw),\ d(fz, fz),\ d(fw, fw),\ \frac{1}{2s}d(fz, fw),\ \frac{1}{2s}d(fw, fz)\right\}\right)$$
$$\leq \phi(\max\{d(fz, fw), d(fw, fz)\})$$
and
$$d(fw, fz) \leq \phi\left(\max\left\{d(fw, fz),\ d(fw, fw),\ d(fz, fz),\ \frac{1}{2s}d(fw, fz),\ \frac{1}{2s}d(fz, fw)\right\}\right)$$
$$\leq \phi(\max\{d(fz, fw), d(fw, fz)\}).$$
Thus we obtain
$\max\{d(fz, fw), d(fw, fz)\} \leq \phi(\max\{d(fz, fw), d(fw, fz)\})$

which in turn yields from (ϕ_3) and (1.1.2) that $fz = fw$.

Thus fz is the unique point of coincidence of f and g.

Let $\alpha = fz = gz$.

Since $fg = gf$ we have $f\alpha = fgz = gfz = g\alpha$.

Hence $f\alpha$ is a point of coincidence of f and g.

Thus $fz = f\alpha$ which implies that $\alpha = f\alpha = g\alpha$.

Hence α is a common fixed point of f and g.

Suppose β is another common fixed point of f and g.

That is $\beta = f\beta = g\beta$.

Hence $f\beta$ is a point of coincidence of f and g.

But fz is the unique point of coincidence of f and g.

Hence $f\beta = fz$ which implies that $\beta = \alpha$.

Thus α is the unique common fixed point of f and g.

Corollary 2.6: Let (X, d) be a complete dislocated quasi b-metric space with fixed constant $s \geq 1$and f: X→X be continuous mapping satisfying
(2.6.1) $d(fx, fy) \leq \phi\left(\max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2s}d(x, fy), \frac{1}{2s}d(y, fx)\right\}\right)$ $\forall\, x, y \in X$, where $\phi \in \Phi_s$.
Then f have a unique common fixed point in X .

Proof: It follows from Theorem 2.5.

Now by replacing the continuities of all mappings and completeness of space X by weakly compatibility pairs of mappings and completeness of one of subspace and using some other contractive conditions, we prove a common fixed point theorem for four maps in dislocated quasi b-metirc spaces. Actually we prove the following Theorem.

Theorem 2.7: Let (X, d) be a dislocated quasi b-metric space with fixed constant $s \geq 1$and f, g, S, T: X→X be mappings satisfying
(2.7.1) $d(fx, gy) \leq \phi\left(\frac{1}{2s^2}\max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\}\right)$
$\forall\, x, y \in X$, where $\phi \in \Phi_s$ and ϕ is continuous,
(2.7.2) $d(gx, fy) \leq \phi\left(\frac{1}{2s^2}\max\{d(Tx, Sy), d(Tx, gx), d(Sy, fy), d(Sy, gx), d(Tx, fy)\}\right)$
$\forall\, x, y \in X$, where $\phi \in \Phi_s$ and ϕ is continuous,
(2.7.3) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$,
(2.7.4) One of S(X) and T(X) is a complete subspace of X and
(2.7.5) The pairs (f, S) and (g, T) are weakly compatible.
Then f, g, S and T have a unique common fixed point in X.

Proof: As in proof of Theorem 2.2 the sequence $\{y_n\}$ is Cauchy in X, where $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, n = 0, 1, 2.....

Suppose S(X) is complete subspace of X.

Since $y_{2n+1} = Sx_{2n+2} \subseteq SX$, there exist $z, u \in X$ such that $y_{2n+1} \to z = Su$.

By Lemma 1.3, (2.7.1), (ϕ_1) and continuity of ϕ, we get

$$\begin{aligned}
\frac{1}{s}d(fu, z) &\leq \lim_{n\to\infty}\inf d(fu, gx_{2n+1})\\
&\leq \lim_{n\to\infty}\inf \phi\left(\frac{1}{2s^2}\max\{d(z, y_{2n}), d(z, fu), d(y_{2n}, y_{2n+1}), d(z, y_{2n+1}), d(y_{2n}, fu)\}\right)\\
&\leq \lim_{n\to\infty}\inf \phi\left(\frac{1}{2s^2}\max\{d(z, y_{2n}), d(z, fu), 2s\max\{d(y_{2n}, z), d(z, y_{2n+1})\}, d(z, y_{2n+1}), d(y_{2n}, fu)\}\right)\\
&\leq \phi\left(\frac{1}{2s^2}\max\{0, d(z, fu), 0, 0, d(z, fu)\}\right)\\
&\leq \phi\left(\frac{1}{s}d(z, fu)\right)\\
&\leq \phi\left(\frac{1}{s}\max\{d(z, fu), d(fu, z)\}\right) \qquad (1)
\end{aligned}$$

Also we can show that $\frac{1}{s}d(z, fu) \leq \phi\left(\frac{1}{s}\max\{d(z, fu), d(fu, z)\}\right)$ (2)

From (1) and (2) $\frac{1}{s}\max\{d(fu, z), d(z, fu)\} \leq \phi\left(\frac{1}{s}\max\{d(z, fu), d(fu, z)\}\right)$

which in turn yields from (ϕ_3) and (1.1.2) that fu = z. Thus Su = z = fu.

Since (f, S) is weakly compatible, we have Sz = S(Su) = S(fu) = f(Su) = fz.

By Lemma 1.3, (2.7.1), (ϕ_1) and continuity of ϕ, we obtain

$$\begin{aligned}
\frac{1}{s}d(Sz, z) = \frac{1}{s}d(fz, z) &\leq \lim_{n\to\infty}\inf d(fz, gx_{2n+1})\\
&\leq \lim_{n\to\infty}\inf \phi\left(\frac{1}{2s^2}\max\{d(Sz, y_{2n}), d(Sz, Sz), d(y_{2n}, y_{2n+1}), d(Sz, y_{2n+1}), d(y_{2n}, Sz)\}\right)\\
&\leq \lim_{n\to\infty}\inf \phi\left(\frac{1}{2s^2}\max\left\{\begin{matrix}d(Sz, y_{2n}), 2s\max\{d(Sz, z), d(z, Sz)\}, 2s\max\{d(y_{2n}, z), d(z, y_{2n+1})\},\\ d(Sz, y_{2n+1}), d(y_{2n}, Sz)\end{matrix}\right\}\right)\\
&\leq \phi\left(\frac{1}{2s^2}\max\{sd(Sz, z), 2s\max\{d(Sz, z), d(z, Sz)\}, 0, 0, sd(Sz, z), sd(z, Sz)\}\right)\\
&\leq \phi\left(\frac{1}{s}\max\{d(Sz, z), d(z, Sz)\}\right) \qquad (3)
\end{aligned}$$

Also we can show that $\frac{1}{s}\,d(z,Sz) \le \phi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz)\}\right)$ (4)

From (3) and (4), $\frac{1}{s}\max\{d(Sz,z), d(z,Sz)\} \le \phi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz)\}\right)$

which in turn yields from (ϕ_3) and (1.1.2) that $Sz = z$.

Thus $Sz = z = fz$. (5)

Since $f(X) \subseteq T(X)$, there exists $\alpha \in X$ such that $T\alpha = fz$.

From (2.7.1) and (ϕ_1) we have

$$\begin{aligned}
d(T\alpha, g\alpha) &= d(fz, g\alpha)\\
&\le \phi\left(\frac{1}{2s^2}\max\{d(T\alpha,T\alpha),\ d(T\alpha,T\alpha),\ d(T\alpha,g\alpha),\ d(T\alpha,g\alpha),\ d(T\alpha,T\alpha)\}\right)\\
&\le \phi\left(\frac{1}{2s^2}\max\{d(T\alpha,T\alpha),\ d(T\alpha,g\alpha)\}\right)\\
&\le \phi\left(\frac{1}{2s^2}\max\{2s\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\},\ d(T\alpha,g\alpha)\}\right)\\
&\le \phi\left(\frac{1}{s}\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\}\right)\\
&\le \phi(\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\}) \qquad (6)
\end{aligned}$$

Similarly we have $d(g\alpha,T\alpha) \le \phi(\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\})$ (7)

From (6) and (7), $\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\} \le \phi(\max\{d(T\alpha,g\alpha),\ d(g\alpha,T\alpha)\})$

which in turn yields from (ϕ_3) and (1.1.2) that $T\alpha = g\alpha$.

Thus $g\alpha = z = T\alpha$.

Since (g, T) is a weakly compatible pair, we have $gz = Tz$.

From (2.7.1) and (ϕ_1) we have

$$\begin{aligned}
d(z,gz) &= d(fz, gz)\\
&\le \phi\left(\frac{1}{2s^2}\max\{d(z,gz),\ d(z,z),\ d(gz,gz),\ d(z,gz),\ d(gz,z)\}\right)\\
&\le \phi\left(\frac{1}{2s^2}\max\{d(z,gz),\ 2s\max\{d(z,gz),\ d(gz,z)\},2s\max\{d(gz,z),\ d(z,gz)\},\ \ d(z,gz),\ d(gz,z)\}\right)\\
&\le \phi\left(\frac{1}{s}\max\{d(z,gz),\ d(gz,z)\}\right)\\
&\le \phi(\max\{d(z,gz),\ d(gz,z)\}) \qquad (8)
\end{aligned}$$

Similarly we have $d(gz,z) \le \phi(\max\{d(gz,z),\ d(z,gz)\})$ (9)

From (8) and (9), $\max\{d(z,gz),\ d(gz,z)\} \le \phi(\max\{d(gz,z),\ d(z,gz)\})$

which in turn yields from (ϕ_3) and (1.1.2) that $gz = z$.

Hence $Tz = gz = z$ (10)

From (5) and (10) we have $fz = Sz = z = Tz = gz$.

Thus z is a common fixed point of f , g , S and T.

The uniqueness of common fixed point follows easily from (2.7.1) and (2.7.2).

Now we provide the following example to support our Theorem 2.7

Example 2.8: Let **X=[**0,1**]** and $d(x,y)=(x+2y)^2$.

Let f, g, S,T:X→X be defined by $fx = \frac{x^2}{16}$, $gx = \frac{x^2}{24}$, $Sx = \frac{x^2}{2}$ and $Tx = \frac{x^2}{3}$.

Let $\phi:[0,\infty)\to[0,\infty)$ be defined by $\phi(t) = \frac{t}{8}$, for $t \in [0,\infty)$.

As in Example 2.3, d is a dislocated quasi b – metric with s = 2.

Consider $d(fx, gy) = \left(\frac{x^2}{16} + \frac{2y^2}{24}\right)^2$

$$= \left(\frac{3x^2 + 4y^2}{6 \times 8}\right)^2$$
$$= \left(\frac{\frac{x^2}{2}+\frac{2y^2}{3}}{8}\right)^2$$
$$= \frac{d(Sx,Ty)}{64}$$
$$= \frac{1}{8}\frac{1}{2s^2}\, d(Sx, Ty)$$
$$\leq \frac{1}{8}\frac{1}{2s^2}\max\{d(Sx, Ty),\ d(Sx, fx),\ d(Ty, gy),\ d(Sx, gy),\ d(Ty, fx)\}$$
$$= \phi\left(\frac{1}{2s^2}\max\{d(Sx, Ty),\ d(Sx, fx),\ d(Ty, gy),\ d(Sx, gy),\ d(Ty, fx)\}\right).$$

Thus (2.7.1) is satisfied.

Clearly one can verify the remaining conditions (2.7.2), (2.7.3), (2.7.4) and (2.7.5).

Clearly 0 is the unique common fixed point of f, g, S and T.

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