UNIQUE COMMON FIXED POINT THEOREMS 
FOR FOUR MAPS IN DISLOCATED QUASI b-METRIC SPACES

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ABSTRACT

In this paper, we prove two common fixed point theorems for four mappings in dislocated quasi b-metric spaces.

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1. INTRODUCTION

Zeyada et.al [12] initiated the concept of dislocated quasi metric spaces and generalized the results of Hitzler and Seda [5] in dislocated quasi metric spaces. The notion of b-metric space was introduced by Czerwic [3] in connection with some problems concerning with the convergence of non measurable functions with respect to measure. Recently Klinkeam and Suanoom [7] introduced the concept of dislocated quasi b-metric spaces and which generalize b-metric spaces [3] and quasi b-metric spaces [10] and proved some fixed point theorems in it by using cyclic contractions. The authors [1, 4, 7, 8, 9, 11] etc. Obtained fixed, common fixed points theorems in dislocated quasi b-metric spaces using various contraction conditions for single and two maps. In this paper, we prove two common fixed point theorems for four maps in dislocated quasi b-metric spaces and we also give examples to support our theorems.

First we recall some known definitions and lemmas.

Definition 1.1: Let X be a non-empty set, s ≥ 1 (a fixed constant) and d: X×X→[0,∞) be a function. consider the following condition on d.

(1.1.1) d(x, x) = 0, ∀x ∈ X,
(1.1.2) d(x, y) = d(y, x) = 0 ⇒ x = y, ∀x, y ∈ X,
(1.1.3) d(x, y) = d(y, x), ∀x, y ∈ X,
(1.1.4) d(x, y) ≤ d(x, z) + d(z, y), ∀x, y, z ∈ X,
(1.1.5) d(x, y) ≤ s[d(x, z) + d(z, y)], ∀x, y, z ∈ X.

(i) If d satisfies (1.1.2), (1.1.3) and (1.1.4) then d is called a dislocated metric and (X, d) is called a dislocated metric space.
(ii) If d satisfies (1.1.1), (1.1.2) and (1.1.4) then d is called a quasi metric and (X, d) is called a quasi metric space.
(iii) If d satisfies (1.1.2) and (1.1.4) then d is called a dislocated quasi metric or dq-metric and (X, d) is called a dislocated quasi metric space.
(iv) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.4) then d is called a metric and (X, d) is called a metric space.
(v) If d satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.5) then d is called a b-metric and (X, d) is called a b-metric space.
(vi) If d satisfies (1.1.4) and (1.1.5) then d is called a dislocated quasi b-metric and (X, d) is called a dislocated quasi b-metric space or dq-metric space.

Definition 1.2: Let (X, d) be a dq b-metric space. A sequence {x_n} in (X, d) is said to be
(i) dq b convergent if there exists some point x ∈ X such that \( \lim_{n \to \infty} d(x, x_n) = 0 = \lim_{n \to \infty} d(x_n, x) \).
In this case x is called a dq b-limit of \{x_n\} and we write \( x_n \to x \) as \( n \to \infty \).
(ii) Cauchy sequence if \( \lim_{n,m \to \infty} d(x_n, x_m) = 0 = \lim_{m \to \infty} d(x_m, x_n) \).
The space \((X, d)\) is called complete if every Cauchy sequence in \(X\) is \(d\)-\(b\)-convergent.

One can prove easily the following

**Lemma 1.3:** Let \((X, d)\) be a \(d\)-\(b\)-metric space and \(\{x_n\}\) be \(d\)-\(b\)-convergent to \(x\) in \(X\) and \(y \in X\) be arbitrary. Then

\[
\frac{1}{s}d(x, y) \leq \liminf_{n \to \infty} d(x_n, y) \leq \limsup_{n \to \infty} d(x_n, y) \leq s(d(x, y) \quad \text{and} \quad \frac{1}{s}d(y, x) \leq \liminf_{n \to \infty} d(y, x_n) \leq \limsup_{n \to \infty} d(y, x_n) \leq s(d(x, y)).
\]

Note: \(\frac{1}{s}d(x, y) \leq \max\{d(x, z), d(z, y)\} \forall x, y, z \in X.\)

**Definition 1.4:** [6] Let \(X\) be a non-empty set and \(S, T: X \to X\) be given self maps on \(X\). The pair \((S, T)\) is said to be weakly compatible if \(STx = TSx\) whenever there exists \(x \in X\) such that \(Sx = Tx\).

**Definition 1.5:** [2] Let \(X\) be a non-empty set and \(f, g: X \to X\) be mappings. If there exists \(x \in X\) such that \(fx = gx\). Then \(x\) is called a Coincidence point of \(f\) and \(g\) and \(fx\) is called a point of Coincidence of \(f\) and \(g\).

Now we prove our main result.

### 2. MAIN RESULT

We need the following definition

**Definition 2.1:** For the fixed constant \(s \geq 1\), let \(\Phi_s\) denote the set of all functions \(\phi: [0, \infty) \to [0, \infty)\) satisfying the following

\((\phi_1)\) : \(\phi\) is monotonically non-decreasing ,

\((\phi_2)\): \(\sum_{n=1}^{\infty} s^n \phi^n(t) < \infty\) for all \(t > 0\),

\((\phi_3)\) : \(\phi(t) < t\) for \(t > 0\).

Clearly \((\phi_1)\) and \((\phi_3)\) implies \(\phi(0) = 0\).

**Theorem 2.2:** Let \((X, d)\) be a complete dislocated quasi \(b\)-metric space with fixed constant \(s \geq 1\) and \(f, g, S, T: X \to X\) be continuous mappings satisfying

\[
(2.2.1) \quad d(fx, gy) \leq \phi \left( \max \left\{ d(Sx, Ty), \frac{1}{2s}d(Sx, fx), \frac{1}{2s}d(Ty, gy), \frac{1}{2s}d(Sy, gy), \frac{1}{2s}d(Tx, fx) \right\} \right)
\]

\(\forall x, y \in X\), where \(\phi \in \Phi_s\),

\[
(2.2.2) \quad d(gx, fy) \leq \phi \left( \max \left\{ d(Tx, Sy), \frac{1}{2s}d(Tx, gx), \frac{1}{2s}d(Sy, fy), \frac{1}{2s}d(Sy, gx), \frac{1}{2s}d(Tx, fy) \right\} \right)
\]

\(\forall x, y \in X\), where \(\phi \in \Phi_s\),

\[
(2.2.3) \quad f(X) \subseteq T(X) \quad \text{and} \quad g(X) \subseteq S(X).
\]

\[
(2.2.4) \quad fS = SF \quad \text{and} \quad gT = TG.
\]

Then \(f, g, S\) and \(T\) have a unique common fixed point in \(X\).

**Proof:** Let \(x_0 \in X\).

Define \(y_{2n} = f x_{2n} = Tx_{2n+1}, y_{2n+1} = g x_{2n+1} = Sx_{2n+2}, \quad n=0,1,2,\ldots\)

**Case-(i):** Suppose \(\max \{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} = 0\) for some \(n\).

Without loss of generality assume that \(n=2m\).

Then \(y_{2m+1} = y_{2m}\).

Using \((2.2.1), (2.2.2)\) and \((\phi_1)\), we get

\[
d(y_{2m}, y_{2m+1}) = d(fx_{2m}, gx_{2m+1})
\]

\[
\leq \phi \left( \max \left\{ \max \{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\}, \frac{1}{2s}d(y_{2m-1}, y_{2m}), \frac{1}{2s}d(y_{2m}, y_{2m+1}), \frac{1}{2s}d(y_{2m}, y_{2m+1}) \right\} \right),
\]

from Note

\[
= \phi \left( \max \left\{ d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1}) \right\} \right).
\]

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and 
\[ d(y_{2m+1}, y_{2m}) = d(gx_{2m+1}, fx_{2m}) \]
\[ \leq \phi(\max\{d(y_{2m}, y_{2m-1}), y_{2m-1}, d(y_{2m}, y_{2m+1}), y_{2m+1}\}) \]
\[ \leq \phi(\max\{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1}), y_{2m-1}, y_{2m+1}\}) \]
\[ = \phi(\max\{d(y_{2m}, y_{2m-1}), d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\}) \]

Thus
\[ \max\{d(y_{2m}, y_{2m+1}), d(y_{2m+1}, y_{2m})\} \leq \phi(\max\{d(y_{2m-1}, y_{2m}), d(y_{2m}, y_{2m+1})\}) \]

From (1.1) and (1.1.2), we have \( y_{2m} = y_{2m+1} \). Thus \( y_{2m+1} = y_{2m+2} \).

Continuing in this way we have
\[ y_{2m-1} = y_{2m} = y_{2m+1} = \cdots \]

Thus \( y_{n-1} = y_n = y_{n+1} = \cdots \)

Hence \( \{y_n\} \) is a constant Cauchy sequence.

**Case-(ii):** suppose \( \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_n)\} \neq 0 \) for all \( n \).

As in (1), we have
\[ \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} \leq \phi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}) \]

If \( \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \leq 0 \), then from (2), using (\( \phi_3 \)), we get
\[ \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} = 0, \text{ which is a contradiction to Case (ii).} \]

Hence \( \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} > \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} \).

Now from (2), \( \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n})\} \leq \phi(\max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\}) \)

This is true for \( n = 1, 2, 3, \ldots \)

Hence \( \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_n)\} \leq \phi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}) \)

\[ \cdots \cdots \cdots \]

\[ \leq \phi(\max\{d(y_0, y_1), d(y_1, y_0)\}) \]

Now for all positive integers \( n \) and \( p \), consider, using (4),
\[ d(y_{np}, y_{np+1}) \leq s^p d(y_{np}, y_{np+1}) + s^{p+1} d(y_{np+1}, y_{np+2}) + \ldots + s^{np+1} d(y_{np+p}, y_{np+p+1}) \]

since \( s^p \phi(t) \) converges for all \( t > 0 \).

Thus we have \( \lim_{n \to \infty} d(y_n, y_{n+p}) = 0 \).

Therefore, we have \( \lim_{n \to \infty} d(y_n, y_{n+p}) = 0 \).

Also using (4), we have
\[ d(y_{np+1}, y_{np+2}) \leq s^p d(y_{np+1}, y_{np+2}) + s^{p+1} d(y_{np+2}, y_{np+3}) + \ldots + s^{np+1} d(y_{np+p}, y_{np+p+1}) + s^{np+2} d(y_{np+p+1}, y_{np+p+2}) \]

since \( s^p \phi(t) \) converges for all \( t > 0 \).

Hence we have \( \lim_{n \to \infty} d(y_{np+1}, y_{np+2}) = 0 \).
Thus \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is a complete dislocated quasi b – metric space, there exists \( z \in X \) such that \( \{y_n\} \) converges to \( z \).

Since \( S \) and \( f \) are continuous and \( Sf = fS \), we have

\[
S_z = \lim_{n \to \infty} Sy_{2n} = \lim_{n \to \infty} Sfx_{2n} = \lim_{n \to \infty} fy_{2n-1} = fz.
\]

Similarly, since \( T \) and \( g \) are continuous and \( Tg = gT \), we have \( Tz = gz \).

Using (2.2.1), (2.2.2), \((\varphi_1)\) and Note, we get

\[
d(Sz, Tz) = d(fz, gz) \leq \varphi\left( \max\left\{d(Sz, Sz), \frac{1}{2s}d(Sz, Tz), \frac{1}{2s}d(Tz, Sz), \frac{1}{2s}d(Sz, Sz) \right\} \right)
\]

which in turn yields from \((\varphi_3)\) and (1.1.2) that \( Sz = Tz \).

Let \( \alpha = Sz = Tz \). Then \( S\alpha = S(Sz) = S(fz) = f(Sz) = f\alpha \) and \( T\alpha = T(Tz) = T(gz) = g(Tz) = g\alpha \).

Now using (2.2.1), (2.2.2), \((\varphi_1)\) and from Note, we have

\[
d(S\alpha, \alpha) = d(f\alpha, gz) \leq \varphi\left( \max\left\{d(S\alpha, \alpha), \frac{1}{2s}d(S\alpha, S\alpha), \frac{1}{2s}d(\alpha, \alpha), \frac{1}{2s}d(S\alpha, \alpha), \frac{1}{2s}d(\alpha, S\alpha) \right\} \right)
\]

which in turn yields from \((\varphi_3)\) and (1.1.2) that \( S\alpha = \alpha \).

Similarly we can show that \( T\alpha = \alpha \).

Hence \( \alpha \) is a common fixed point of \( f, g, S \) and \( T \).

One can prove the uniqueness of common fixed point of \( f, g, S \) and \( T \) using (2.2.1) and (2.2.2).

Now we give an example to illustrate the Theorem 2.2.

**Example 2.3:** Let \( X = [0, 1] \) and \( d(x, y) = (x+2y)^2 \).

Let \( f, g, S, T : X \to X \) be defined by \( fx = \frac{x}{8} \), \( gx = \frac{x}{12} \), \( Sx = \frac{x}{2} \) and \( Tx = \frac{x}{3} \).

Let \( \phi : [0, \infty) \to [0, \infty) \) be defined by \( \phi(t) = \frac{t}{4} \), for \( t \in [0, \infty) \).

Then it is clear that \( d(x, y) = d(y, x) = 0 \Rightarrow x = y \)

Also \( d(x, y) = (x+2y)^2 \leq [(x + 2z) + (z + 2y)]^2 \leq 2[(x + 2z)^2 + (z + 2y)^2] = s[d(x, z) + d(z, y)] \), where \( s = 2 \)

Thus \( d \) is a dislocated quasi b – metric with \( s = 2 \).

Consider \( d(fx, gy) = \left(\frac{x}{8} + \frac{2y}{12}\right)^2 = \left(\frac{3x + 4y}{24}\right)^2 = \left(\frac{x + 2y}{4}\right)^2 \).
\[
\begin{align*}
&= \frac{d(Sx, Ty)}{16} \\
&\leq \frac{1}{4} d(Sx, Ty) \\
&\leq \frac{1}{4} \max \left\{ d(Sx, Ty), \frac{1}{2s} d(Sx, fx), \frac{1}{2s} d(Ty, gy), \frac{1}{2s} d(Sx, gy), \frac{1}{2s} d(Ty, fx) \right\} \\
&= \phi \left( \max \left\{ d(Sx, Ty), \frac{1}{2s} d(Sx, fx), \frac{1}{2s} d(Ty, gy), \frac{1}{2s} d(Sx, gy), \frac{1}{2s} d(Ty, fx) \right\} \right).
\end{align*}
\]

Similarly we can show that
\[
d(gx, fy) \leq \phi \left( \max \left\{ d(Tx, Sy), \frac{1}{2s} d(Tx, gx), \frac{1}{2s} d(Sy, fy), \frac{1}{2s} d(Sy, gx), \frac{1}{2s} d(Tx, fy) \right\} \right).
\]

Clearly \( f(X) = [0, \frac{1}{8}] \subseteq [0, \frac{1}{3}] = T(X) \) and \( g(X) = [0, \frac{1}{12}] \subseteq [0, \frac{1}{2}] = S(X) \).

It is also clear that \( Sf = fS \) and \( Tg = gT \).

For \( t > 0 \),
\[
\text{Consider } \sum_{n=1}^{\infty} s^n \phi^n(t) = \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{n=1}^{\infty} \frac{1}{4^n} = t \left( \frac{\frac{1}{2}}{1 - \frac{1}{4}} \right) = t < \infty.
\]

Thus all conditions of Theorem 2.2 are satisfied. Clearly 0 is the unique common fixed point of \( f \), \( g \), \( S \) and \( T \).

In the similar lines of proof of Theorem 2.2, we prove the following.

**Theorem 2.4:** Let \((X, d)\) be a complete dislocated quasi b-metric space with fixed constant \( s \geq 1 \) and \( f, g : X \to X \) be continuous mappings satisfying
\[
(2.4.1) \quad d(fx, gy) \leq \phi \left( \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{1}{2s} d(x, gy), \frac{1}{2s} d(y, fx) \right\} \right) \quad \forall x, y \in X, \text{ where } \phi \in \Phi_s,
\]
\[
(2.4.2) \quad d(gx, fy) \leq \phi \left( \max \left\{ d(x, y), d(x, gx), d(y, fy), \frac{1}{2s} d(y, gx), \frac{1}{2s} d(x, fy) \right\} \right) \quad \forall x, y \in X, \text{ where } \phi \in \Phi_s.
\]

Then \( f \) and \( g \) have a unique common fixed point.

**Proof:** As in Theorem 2.2, we can show that \( \{x_n\} \) is convergent to \( z \in X \), where \( x_{2n+1} = fx_{2n} \), \( x_{2n+2} = gx_{2n+1} \), \( n = 0, 1, 2, .... \) and \( x_0 \in X \) is arbitrary.

Since \( f \) is continuous and \( x_n \to z \), we have
\[
z = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} fx_{2n} = f \left( \lim_{n \to \infty} x_n \right) = fz.
\]

Similarly, since \( g \) is continuous we have \( z = gz \).

Thus \( z \) is a common fixed point of \( f \) and \( g \).

Consider \( d(z, z) = d(fz, gz) \leq \phi \left( \max \left\{ d(z, z), d(z, z), d(z, z), \frac{1}{2s} d(z, z), \frac{1}{2s} d(z, z) \right\} \right) = \phi(d(z, z)) \)

From (\( \phi \)) follows that \( d(z, z) = 0 \)

Thus \( d(z, z) = 0 \) whenever \( z \) is a common fixed point of \( f \) and \( g \).

Now suppose that \( w \) is another common fixed point of \( f \) and \( g \).

Then \( d(w, w) = 0 \).

Now consider \( d(z, w) = d(fz, gw) \)
\[
\leq \phi \left( \max \left\{ d(z, w), d(z, z), d(w, w), \frac{1}{2s} d(z, w), \frac{1}{2s} d(w, z) \right\} \right)
\]
and
\[
d(w, z) = d(gw, fz)
\leq \phi \left( \max \left\{ d(w, z), d(w, w), d(z, z), \frac{1}{2s} d(w, w), \frac{1}{2s} d(w, z) \right\} \right)
\leq \phi \left( \max \left\{ d(z, w), d(w, z) \right\} \right).
\]
Hence \(\max\{d(z, w), d(w, z)\} \leq \phi(\max\{d(z, w), d(w, z)\})\)

which in turn yields from \((\phi_1)\) and \((1.1.2)\) that \(w = z\).

Hence \(z\) is the unique common fixed point of \(f\) and \(g\).

**Theorem 2.5:** Let \((X, d)\) be a complete dislocated quasi b-metric space with fixed constant \(s \geq 1\) and \(f, g : X \to X\) be continuous mappings satisfying

\[
(2.5.1) \quad d(fx, fy) \leq \phi\left(\max\left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2s}d(gx, fy), \frac{1}{2s}d(gy, fx)\right\}\right) \quad \forall \ x, y \in X, \text{ where } \phi \in \Phi_s,
\]

\[(2.5.2) \quad f(X) \subseteq g(X) \text{ and } fg = gf.\]

Then \(f\) and \(g\) have a unique common fixed point.

**Proof:** As in Theorem 2.2, we can show that \(\{gx_n\}\) is convergent to \(z \in X\), where \(fx_n = gx_{n+1}, n = 0, 1, 2, \ldots\) and \(x_0 \in X\) is arbitrary.

Since \(f\) and \(g\) are continuous and \(fg = gf\), we have \(fz = \lim_{n \to \infty} fgx_n = \lim_{n \to \infty} gfx_n = gz\).

Thus \(fz\) is a point of coincidence of \(f\) and \(g\).

Consider \(d(fz, fz) \leq \phi\left(\max\left\{d(fz, fz), d(fz, fz), d(fz, fz), \frac{1}{2s}d(fz, fz), \frac{1}{2s}d(fz, fz)\right\}\right) = \phi(d(fz, fz))\)

which in turn yields from \((\phi_3)\) that \(d(fz, fz) = 0\).

Thus if \(fz\) is a point of coincidence of \(f\) and \(g\) then \(d(fz, fz) = 0\).

Suppose \(fw\) is another point of coincidence of \(f\) and \(g\) then \(d(fw, fw) = 0\).

From \((2.5.1)\) and \((\phi_1)\), we have

\[
d(fz, fz) \leq \phi\left(\max\left\{d(fz, fz), d(fz, fz), d(fw, fw), \frac{1}{2s}d(fz, fz), \frac{1}{2s}d(fw, fz)\right\}\right)
\]

and

\[
d(fw, fz) \leq \phi\left(\max\left\{d(fw, fz), d(fw, fw), d(fz, fz), \frac{1}{2s}d(fw, fz), \frac{1}{2s}d(fz, fw)\right\}\right)
\]

Thus we obtain

\[
\max\{d(fz, fz), d(fw, fz)\} \leq \phi(\max\{d(fz, fz), d(fw, fz)\})
\]

which in turn yields from \((\phi_1)\) and \((1.1.2)\) that \(fz = fw\).

Thus \(fz\) is the unique point of coincidence of \(f\) and \(g\).

Let \(\alpha = fz = gz\).

Since \(fg = gf\) we have \(f\alpha = fgz = gfz = g\alpha\).

Hence \(f\alpha\) is a point of coincidence of \(f\) and \(g\).

Thus \(fz = f\alpha\) which implies that \(\alpha = f\alpha = g\alpha\).

Hence \(\alpha\) is a common fixed point of \(f\) and \(g\).

Suppose \(\beta\) is another common fixed point of \(f\) and \(g\).

That is \(\beta = f\beta = g\beta\).

Hence \(f\beta\) is a point of coincidence of \(f\) and \(g\).

But \(fz\) is the unique point of coincidence of \(f\) and \(g\).

Hence \(f\beta = fz\) which implies that \(\beta = \alpha\).

Thus \(\alpha\) is the unique common fixed point of \(f\) and \(g\).
Corollary 2.6: Let $(X, d)$ be a complete dislocated quasi b-metric space with fixed constant $s \geq 1$ and $f : X \to X$ be continuous mapping satisfying

$$(2.6.1) \quad d(fx, fy) \leq \phi \left( \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{2s}d(x, fy), \frac{1}{2s}d(y, fx) \right\} \right) \quad \forall x, y \in X,$$

where $\phi \in \Phi_s$.

Then $f$ have a unique common fixed point in $X$.

Proof: It follows from Theorem 2.5.

Now by replacing the continuities of all mappings and completeness of space $X$ by weakly compatibility pairs of mappings and completeness of one of subspace and using some other contractive conditions, we prove a common fixed point theorem for four maps in dislocated quasi b-metric spaces. Actually we prove the following Theorem.

Theorem 2.7: Let $(X, d)$ be a dislocated quasi b-metric space with fixed constant $s \geq 1$ and $f, g, S, T : X \to X$ be mappings satisfying

$$(2.7.1) \quad d(fx, gy) \leq \phi \left( \frac{1}{2s} \max \left\{ d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx) \right\} \right) \quad \forall x, y \in X,$$

$$(2.7.2) \quad d(gx, fy) \leq \phi \left( \frac{1}{2s} \max \left\{ d(Tx, Sy), d(Tx, gx), d(Sy, fy), d(Sy, gx), d(Tx, fy) \right\} \right) \quad \forall x, y \in X,$$

$$(2.7.3) \quad f(X) \subseteq T(X) \quad \text{and} \quad g(X) \subseteq S(X),$$

$$(2.7.4) \quad \text{One of } S(X) \text{ and } T(X) \text{ is a complete subspace of } X \quad \text{and}$$

$$(2.7.5) \quad \text{The pairs } (f, S) \text{ and } (g, T) \text{ are weakly compatible.}$$

Then $f, g, S$ and $T$ have a unique common fixed point in $X$.

Proof: As in proof of Theorem 2.2 the sequence $\{y_n\}$ is Cauchy in $X$, where $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2,...$.

Suppose $S(X)$ is complete subspace of $X$.

Since $y_{2n+1} = Sx_{2n+2} \subseteq SX$, there exist $z, u \in X$ such that $y_{2n+1} \rightharpoonup z = Su$.

By Lemma 1.3, (2.7.1), $(\phi_1)$ and continuity of $\phi$, we get

$$\frac{1}{s}d(fu, z) \leq \liminf_{n \to \infty} d(fu, gx_{2n+1}) \leq \liminf_{n \to \infty} \phi \left( \frac{1}{2s} \max \left\{ d(z, y_{2n}), d(z, fu), d(y_{2n}, y_{2n+1}), d(z, y_{2n+1}), d(y_{2n}, fu) \right\} \right)$$

$$\leq \liminf_{n \to \infty} \phi \left( \frac{1}{2s} \max \left\{ 0, d(z, fu), 0, 0, d(z, fu) \right\} \right) \leq \phi \left( \frac{1}{s}d(z, fu) \right) \leq \phi \left( \frac{1}{s} \max \left\{ d(z, fu), d(fu, z) \right\} \right) \quad (1)$$

Also we can show that $\frac{1}{s}d(z, fu) \leq \phi \left( \frac{1}{s} \max \left\{ d(z, fu), d(fu, z) \right\} \right) \quad (2)$

From (1) and (2) $\frac{1}{s} \max \left\{ d(fu, z), d(z, fu) \right\} \leq \phi \left( \frac{1}{s} \max \left\{ d(z, fu), d(fu, z) \right\} \right)$

which in turn yields from $(\phi_1)$ and (1.1.2) that $fu = z$. Thus $Su = z = fu$.

Since $(f, S)$ is weakly compatible, we have $Sz = S(Su) = S(fu) = f(Su) = fz$.

By Lemma 1.3, (2.7.1), $(\phi_1)$ and continuity of $\phi$, we obtain

$$\frac{1}{s}d(Sz, z) = \frac{1}{s}d(fz, z) \leq \liminf_{n \to \infty} d(fz, gx_{2n+1}) \leq \liminf_{n \to \infty} \phi \left( \frac{1}{2s} \max \left\{ d(Sz, y_{2n}), d(Sz, Sz), d(y_{2n}, y_{2n+1}), d(Sz, y_{2n+1}), d(y_{2n}, Sz) \right\} \right)$$

$$\leq \liminf_{n \to \infty} \phi \left( \frac{1}{2s} \max \left\{ d(Sz, y_{2n}), 2s \max \left\{ d(Sz, z), d(z, Sz) \right\}, 2s \max \left\{ d(y_{2n}, z), d(z, y_{2n+1}) \right\} \right\} \right) \right) \leq \phi \left( \frac{1}{s} \max \left\{ d(Sz, z), 2s \max \left\{ d(Sz, z), d(z, Sz) \right\}, 0, 0, d(Sz, z), d(z, Sz) \right\} \right)$$

$$\leq \phi \left( \frac{1}{s} \max \left\{ d(Sz, z), d(z, Sz) \right\} \right) \quad (3)$$
Also we can show that \( \frac{1}{s} d(z, Sz) \leq \phi \left( \frac{1}{s} \max \{d(Sz, z), d(z, Sz)\} \right) \) \hspace{1cm} (4)

From (3) and (4), \( \frac{1}{s} \max \{d(Sz, z), d(z, Sz)\} \leq \phi \left( \frac{1}{s} \max \{d(Sz, z), d(z, Sz)\} \right) \)

which in turn yields from (\( \phi \)) and (1.1.2) that \( Sz = z \).

Thus \( Sz = z = fz \). \hspace{1cm} (5)

Since \( f(X) \subseteq T(X) \), there exists \( \alpha \in X \) such that \( T\alpha = fz \).

From (2.7.1) and (\( \phi \)) we have

\[
\begin{align*}
d(T\alpha, g\alpha) &= d(fz, g\alpha) \\
&\leq \phi \left( \frac{1}{2s} \max \{d(T\alpha, T\alpha), d(T\alpha, g\alpha), d(T\alpha, T\alpha)\} \right) \\
&\leq \phi \left( \frac{1}{2s} \max \{d(T\alpha, g\alpha)\} \right) \\
&\leq \phi \left( \frac{1}{2s} \max \{2s \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\}\} \right) \\
&\leq \phi \left( \frac{1}{s} \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \right) \\
&\leq \phi \left( \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \right) \\
&\leq \phi \left( \phi \left( \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \right) \right) \hspace{1cm} (6)
\end{align*}
\]

Similarly we have \( d(g\alpha, T\alpha) \leq \phi \left( \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \right) \) \hspace{1cm} (7)

From (6) and (7), \( \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \leq \phi \left( \max \{d(T\alpha, g\alpha), d(g\alpha, T\alpha)\} \right) \)

which in turn yields from (\( \phi \)) and (1.1.2) that \( T\alpha = g\alpha \).

Thus \( g\alpha = z = T\alpha \).

Since \( (g, T) \) is a weakly compatible pair, we have \( gz = Tz \).

From (2.7.1) and (\( \phi \)) we have

\[
\begin{align*}
d(z, gz) &= d(fz, gz) \\
&\leq \phi \left( \frac{1}{2s} \max \{d(z, gz), d(z, z), d(gz, gz), d(z, gz), d(gz, z)\} \right) \\
&\leq \phi \left( \frac{1}{2s} \max \{d(z, gz), 2s \max \{d(z, gz), d(gz, z)\}, 2s \max \{d(gz, z), d(z, gz)\}\} \right) \\
&\leq \phi \left( \frac{1}{s} \max \{d(z, gz), d(gz, z)\} \right) \\
&\leq \phi \left( \max \{d(z, gz), d(gz, z)\} \right) \hspace{1cm} (8)
\end{align*}
\]

Similarly we have \( d(gz, z) \leq \phi \left( \max \{d(gz, z), d(z, gz)\} \right) \) \hspace{1cm} (9)

From (8) and (9), \( \max \{d(z, gz), d(gz, z)\} \leq \phi \left( \max \{d(gz, z), d(z, gz)\} \right) \)

which in turn yields from (\( \phi \)) and (1.1.2) that \( gz = z \).

Hence \( Tz = gz = z \) \hspace{1cm} (10)

From (5) and (10) we have \( fz = Sz = z = Tz = gz \).

Thus \( z \) is a common fixed point of \( f, g, S \) and \( T \).

The uniqueness of common fixed point follows easily from (2.7.1) and (2.7.2).

Now we provide the following example to support our Theorem 2.7

**Example 2.8:** Let \( X = [0, 1] \) and \( d(x, y) = (x + 2y)^2 \).

Let \( f, g, S, T : X \to X \) be defined by \( fx = \frac{x^2}{16} \), \( gx = \frac{x^2}{24} \), \( Sx = \frac{x^2}{2} \) and \( Tx = \frac{x^2}{3} \).

Let \( \phi : [0, \infty) \to [0, \infty] \) be defined by \( \phi(t) = \frac{t}{\theta} \), for \( t \in [0, \infty) \).
As in Example 2.3, \( d \) is a dislocated quasi \( b \)– metric with \( s = 2 \).

Consider 
\[
\begin{align*}
d(fx, gy) &= \left( \frac{x^2}{16} + \frac{2y^2}{24} \right)^{\frac{2}{2}} \\
&= \left( 3x^2 + 4y^2 \right)^{\frac{2}{6 \times 8}} \\
&= \left( \frac{x^2 + 2y^2}{8} \right)^{\frac{2}{8}} \\
&= d(Sx, Ty)^{\frac{1}{2}} \\
&\leq \frac{1}{2s^2} \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\} \\
&= \phi \left( \frac{1}{2s^2} \max\{d(Sx, Ty), d(Sx, fx), d(Ty, gy), d(Sx, gy), d(Ty, fx)\} \right).
\end{align*}
\]

Thus (2.7.1) is satisfied.

Clearly one can verify the remaining conditions (2.7.2), (2.7.3), (2.7.4) and (2.7.5).

Clearly 0 is the unique common fixed point of \( f, g, S \) and \( T \).

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