International Journal of Mathematical Archive-2(9), 2011, Page: 1657-1661
Available online through www.ijma.info ISSN 2229 – 5046

Ovoids in the Hyperbolic Quadrics $Q^+ (7, q)$

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(Received on: 14-05-11; Accepted on: 18-09-11)

ABSTRACT

In this paper we present a solution of the problem related to the construction of ovoids in $Q^+ (7, q)$, see [1]. For this purpose we use the point-line geometry $D_{4,2}(q)$ as an isomorphic to the finite classical polar space $Q^+(7,q)$ $(\Omega^+(8,q))$. Further, an upper bounds for the size of ovoids in $Q^+ (7, q)$ is obtained.

Keywords: *Finite classical polar spaces- ovoids-point-line geometry*

2000 MSC: *05B25, 51E20.*

1. INTRODUCTION:

Many authors interested in the existence and non-existence of ovoids in finite classical polar spaces. In [2] Thas proved that $Q^-(2n+1,q)$, n>2, has no ovoid. In [4] the nonsingular hyperbolic quadric $Q^+(5, q)$ has an ovoid, and under a certain condition $Q^+(7, q)$ has an ovoid. In this paper we construct an ovoid for $Q^+(7, q)$ by using the correspondence between the point–line geometry $D_{4,2}(q)$ and the hyperbolic classical polar spaces $\Omega^+(8, q)$, see [5]. For the isomorphism between the non singular hyperbolic quadrics $Q^+ (7, q)$ and the classical polar space $\Omega^+ (8, q)$, see [3]. Further we present an upper pound for the ovoid.

Basic definitions.

Let P be a finite classical polar spaces of rank $r \geq 2$. **An ovoid** O of P is a pointset of P, which has exactly one point in common with every maximal totally isotropic subspace or maximal singular subspace of P, see [4].

The following definitions can be found in [6].

A given set I, **a geometry** Γ over I is an ordered triple $\Gamma= (X, *, D)$, where X is a set, D is a partition $\{X_i\}$ of X indexed by I, X_i are called components, $*$ is a symmetric and reflexive relation on X called incidence relation such that: A point-line geometry (P, L) is simply a geometry for which $|I| = 2$, one of the two types is called points, in this notation the points are the members of P and the other type is called lines. Lines are the members of L. If $p\in P$ and $l\in L$, then $p * l$ if and only if $p\in l$. In point-line geometry (P, L), it's said that two points of P are collinear if and only if they are incident with a common line.

A subspace of a point-line geometry $\Gamma = (P, L)$ is a subset $X\subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. **A hyperplane** of point line geometry is a proper subspace meets each line in at least one point. $\langle X\rangle$ means the intersection over all subspaces containing X, where $X\subseteq P$. Lines incident with more than two points are called thick lines, but those incident with exactly two points are called thin lines.

$x^\perp$ means a set of all points in P collinear with x, including x itself. A clique of P is a set of points in which every pair of points are collinear. A partial linear space is a point-line geometry (P, L), in which every pair of points are incident with at most one line and all lines have cardinality at least 2. A point line geometry $\Gamma = (P, L)$ is called singular or (linear) if every pair of points is incident with a unique line.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there is a chain of distinct subspaces $\varnothing \neq X_0 \subset X_1 \subset ... \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$, for example rank $(\varnothing) = -1$, rank$(\{p\}) = 0$ where p is a point and rank(L) = 1 where L a line.

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In a point-line geometry $\Gamma = (P, L)$, a path of length n is a sequence of n+1 $(x_0, x_1,.., x_n)$ where, (x_i,x_{i+1}) are collinear, x_0 is the initial point and x_n is the end point. A geodesic from a point x to a point y is a path of minimal possible length with initial point x and end point y. This length is denoted by d_Γ (x, y). Diameter of Γ is the maximal distance between the points of Γ, i.e, diameter (Γ) = maximum $\{d(x, y), x, y\in \Gamma\}$. A geometry Γ is called connected if and only if for any two of its points are connected by a bath. A subset X of P is said to be convex if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry Γ=(P, L) satisfying the Buekenhout-Shult axiom:

For each point-line pair (p, l) with p not incident with l; p is collinear with one or all points of l, that is $|p^\perp\cap l| = 1$ or else $p^\perp\supset l$. Clearly this axiom is equivalent to saying that $p^\perp$ is a geometric hyperplane of Γ for every point $p\in$ P.

A point-line geometry Γ = (P, L) is called a projective plane only in case if satisfies the following conditions:

(i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line
(ii) Every two lines intersect in one point
(iii) There are four points no three of them are on a line

A point-line geometry Γ = (P, L) is called a projective space if the following conditions are satisfied:

(i) Every two points lie exactly on one line
(ii) If l_1, l_2 are two lines $l_1\cap l_2 \neq \varnothing$, then $\langle l_1, l_2\rangle$ is a projective plane. ($\langle l_1,l_2\rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

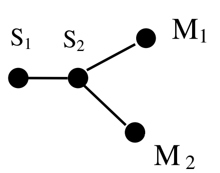
A point-line geometry Γ = (P, L) is called **a parapolar space** only in case it satisfies the following properties:

(i) Γ is a connected gamma space
(ii) for every line l, $l^\perp$ s not a singular subspace
(iii) for every pair of non-collinear points x, y; $x^\perp \cap y^\perp$ is either empty, a single point, or a non-degenerate polar space of rank at least 2

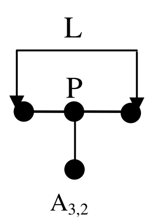
If x, y are distinct points in P and if $|x^\perp \cap y^\perp| = 1$, then (x, y) is called a special pair and if $x^\perp \cap y^\perp$ is a polar space, hence (x, y) is called a polar pair (or a symplectic pair). A parapolar space is called **a strong** parapolar space if it has no special pairs.

Now we present a definition and the construction of the point-line geometry $D_{4,2}$(q) to be isomorphic to the hyperbolic classical polar space $\Omega^+(8, q)$.

Construction of $D_{4,2}$(q)



Polar space Δ



$D_{4,2}$(q)

The geometry $D_{4,2}$ was defined as an isomorphic to the classical polar space $\Delta=\Omega^+(8, F)$ that comes from a vector space of dimension 8 over a finite field GF(q) with a symmetric bilinear form. The set S_1 consists of all totally isotropic 1-dimentional subspaces of the vector space V and S_2 consists of all totally 2-dimensional subspaces of V. The two classes M_1, M_2 consist of maximal totally isotropic 4-dimensional subspaces. Two 4-subspaces fall in the same class if their intersection is of even dimension. Then the geometry $D_{4,2}$ (F) is a point-line geometry (P, L), whose set of points P is corresponding to the class S_2, and whose each line is corresponding to the totally isotropic (1, 4)-dimensional subspaces (A, B) and $A\subseteq B$. A point C is incident with a line (A, B) if and only if $A\subset C\subset B$ as a subspaces of V.

To define the co linearity, let C_1 and C_2 be two point (the points are the T.I 2-spaces), then C_1 is collinear to C_2 if and only if the intersection of C_1 and C_2 is a T.I 1-dimensional space, $C_1\cap C_2$ in addition to the complement of C_1 and C_2 must form a T.I 3-dimensional space and then contained in a T.I 4-space. The elements of the class M_2 are corresponding to the class of geometries of type $A_{3,2}$ that are convex polar spaces of rank 2 and then they represent symplecta in the geometry $D_{4,2}$. Then the symplecta of $D_{4,2}$ (F) are the Grassmannians of type $A_{3,2}$(F) that are corresponding to the collection of TI 4-dimensional spaces.

Notation: Let the map Ψ: P→ V defined above, i.e., Ψ (p) is the T.I. 2-dimensional subspace corresponding to the point p. We will use Ψ for the rest of the geometry; for example Ψ ($A_{3,2}$) is the T.I. 4-dimensional subspace corresponding to a geometry of type $A_{3,2}$. The inverse map Ψ^{-1} will be used for the inverse; for example Ψ^{-1}(C) is the point corresponding to the T.I. 2-dimensional subspace C.

2. OLD RESULTS:

Let V be a vector space over a finite field F=GF(q), q is a prime power. The following are finite classical polar spaces:

1- Symplectic Geometry $W_n(q)$ is the point-line geometry (P, L), where P is the set of all 1-dimensional subspaces $\langle x\rangle$ of V for which B(x, x)=0, and L is the set of all 2-dimensional subspaces $\langle x, y\rangle$ for which B(x, y)=0, for a symplectic bilinear form B. In this case n is even, the polar space is of rank n/2.

2- Hypebolic Geometry Ω^+(n, q) is the point-line geometry (P, L), where P is the set of all 1-dimensional subspaces $\langle x\rangle$ of V for which B(x, x)=0, and L is the set of all 2-dimensional $\langle x, y\rangle$ for which B(x, y)=0, for a hyperbolic bilinear form B. In this case n is even, the polar space is of rank n/2.

3- Elliptic Geometry Ω^-(n, q) is the point-line geometry (P, L), where P is the set of all 1-dimensional subspaces $\langle x\rangle$ of V for which B(x, x)=0, and L is the set of all 2-dimensional $\langle x, y\rangle$ for which B(x, y)=0, for elliptic bilinear form B. In this case n is even, the polar space is of rank (n/2)-1.

4- Orthogonal Geometry Ω(n, q) is the point-line geometry (P, L), where P is the set of all 1-dimensional subspaces $\langle x\rangle$ of V for which B(x, x)=0, and L is the set of all 2-dimensional $\langle x, y\rangle$ for which B(x, y)=0, for orthogonal bilinear form B. In this case n is odd, the polar space is of rank n/2.

5- Hermitian Geometry $H^+_n(q^2)$ is the point-line geometry (P, L), where P is the set of all 1-dimensional subspaces $\langle x\rangle$ of V for which B(x, x)=0, and L is the set of all 2-dimensional $\langle x, y\rangle$ for which B(x, y)=0, for a Hermitian bilinear form B. In this case n is odd, the polar space is of rank (n-1)/2.

For the result to come it is useful to present the following theorems that determine the numbers of points and the maximal totally isotropic spaces. For the proofs see [2].

2.1 Theorem: The numbers of points of the finite classical polar spaces are given by the following formulae:

$$\begin{gathered}
|W_{2n}(q)| = (q^{2n}-1)/(q-1),\\
|\Omega(2n+1)| = (q^{2n}-1)/(q-1),\\
|\Omega^+(2n,q)| = (q^{n-1}+1)(q^n-1)/(q-1),\\
|\Omega^-(2n,q)| = (q^{n-1}-1)(q^n+1)/(q-1),\\
|H^+(2n,q)| = (q^{2n}-1)(q^{2n}+1)/(q^2-1).
\end{gathered}$$

2.2 Theorem: The numbers of maximal totally singular subspaces of the finite classical polar spaces are given by the following formulae:

$$\begin{gathered}
\left|\sum(W_{2n}(q))\right| = (q+1)(q^2+1)\ \dots\ (q^{2n}+1),\\
\left|\sum(\Omega(2n+1,q))\right| = (q+1)(q^2+1)\ \dots\ (q^{n+1}+1),\\
\left|\sum(\Omega^+(2n,q))\right| = 2(q+1)(q^2+1)\ \dots\ (q^n+1),\\
\left|\sum(\Omega^-(2n,q))\right| = (q^2+1)(q^3+1)\ \dots\ (q^n+1),\\
\left|\sum(H^+(2n,q^2))\right| = (q+1)(q^3+1)\ \dots\ (q^n+1).
\end{gathered}$$

2.3 Proposition: [7]. The number of subspaces of dimension k in a vector space of dimension n over GF(q) is given by:

$$\frac{(q^n-1)(q^n-q)\ \dots\ (q^n-q^{k-1})}{(q^k-1)(q^k-q)\ \dots\ (q^k-q^{k-1})}$$

Remark: This number is called a *Gaussian coefficient*, and is denoted by: $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

2.4. Theorem: [7] Let V be equipped with a bilinear form then the number of Totally isotropic *k*-subspaces is the following:

$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^{n-i}+1)$ in the symplectic case W(2*n*,*q*).

$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^{n-i}+1)$ in the orthogonal case Ω(2*n*+1,*q*).

$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^{n-i-1}+1)$ in the hyperbolic case $\Omega^+(2n,q)$.

$\begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} (q^{n-i+1}+1)$ in the elliptic case $\Omega^-(2n+2,q)$.

The following theorem identifies the number of ovoids and spreads in the finite classical polar spaces. For the proof of that theorem see [2].

2.5 Theorem: [2] Let *O* be an ovoid and *S* be a spread of the finite classical polar space P. Then

$$
\begin{aligned}
&For\ P = W_n(q), && |O| = |S| = q^{(n+1)/2}+1,\\
&For\ P = \Omega(2n+1,q), && |O| = |S| = q^n+1,\\
&For\ P = \Omega^+(2n+2,q), && |O| = |S| = q^n+1,\\
&For\ P = \Omega^-(2n+2,q), && |O| = |S| = q^{n+1}+1,\\
&For\ P = H(2n,q), && |O| = |S| = q^{2n+1}+1,\\
&For\ P = H(2n+1,q^2), && |O| = |S| = q^{2n+1}+1,
\end{aligned}
$$

3. THE MAIN RESULT:

First we construct an ovoid in the classical polar space $\Omega^+(8, q)$ by constructing an ovoid in the point-line geometry $D_{4,2}(q)$ (which is isomorphic to $\Omega^+(8, q)$) and then it is considered an ovoid to $Q^+(7, q)$ which is isomorphic to $\Omega^+(8, q)$.

3.1 Theorem: Let P be the set of points of the point-line geometry $D_{4,2}(q)$. Then the set $\Delta^{\square}_2(p)$ forms an ovoid of the geometry $D_{4,2}$. Where $\Delta^{\square}_2(p)$ is the set of all points that are of distance at most 2 from the fixed point p, i.,e., $\Delta^{\square}_2(p) = \{x \in P: d(x, p) \leq 2\}$.

Proof: Every maximal totally isotropic in $\Omega^+(8, q)$ corresponds to a line l in $D_{4,2}$, so to prove that $\Delta^{\square}_2(p)$ represents ovoid all what to do is to show that every maximal totally isotropic 4-space has 2-space (corresponds to a point r in the line l) such that $r \in \Delta^{\square}_2(p)$. Now the line l that is corresponding to the maximal totally isotropic 4-space is identified by the two points r and s such that $\Psi(r) = \langle x_1, x_3\rangle$ and $\Psi(s) = \langle x_2, x_3\rangle$ and let $\Psi(p) = \langle y_1, y_2\rangle$. Now if $\Psi(p) \subset \Psi(l)$, then $p \in l$ and $p \in \Delta^{\square}_2(p)$ (because d(p, p) = 0), so $l \cap \Delta^{\square}_2(p) \neq \varphi$. If Ψ (p) is not contained in $\Psi(l)$, then there are two cases:

1. $\Psi(l) \cap \Psi(p)$ = 1-sapace = $\langle x\rangle$, $x = x_3 = y_2$. If $y_1^{\perp} \cap \Psi(l) = \langle x, x_1, x_2\rangle$, then $\langle y_1, x, x_2\rangle$ forms a TI 3-space and p is collinear to s. This means that $s \in \Delta^{*}_2(p)$, then $l \cap \Delta^{\square}_2(p) \neq \varphi$.

2. $\Psi(l) \cap \Psi(p)$ = 0-sapace. If $y_1^{\perp} \cap \Psi(l) = \langle x_3, x_1, x_2\rangle$ and $y_2^{\perp} \cap \Psi(l) = \langle x_3, x_1, u\rangle$, then there is a point q such that $\Psi(q) = \langle x_3, y_2\rangle$. Now since $\langle y_1, y_2, x_3\rangle$ is a TI 3-space, the point q is collinear to the point sand since $\langle y_2, x_3, x_2\rangle$ form TI

3-spaces, the point q is collinear to the point p which means d (p, s) = 2. Then $s \in \Delta^{\square}_2(p)$, so $l \cap \Delta^{\square}_2(p) \neq \varphi$. Then every maximal totally isotropic in $\Omega^+(8, q)$ intersect the corresponding of $\Delta^{\square}_2(p)$ in exactly one 2-space. Which means that $\Delta^{\square}_2(p)$ corresponds to a void in $\Omega^+(8, q)$ and then a void to $Q^+(7, q)$.

Through the following theorem we present an upper pound for the ovoid that is corresponding to $\Delta^{\square}_2(p)$ in $Q^+(7, q)$.

3.2 Theorem. For any ovoid of $Q^+(7, q)$, q is finite, we have;

$$|O| \le (q^{n-1}+1)^2 (q+1)^2 + 1.$$

Proof: Let p a point such that the corresponding totally isotropic 2-space in $\Omega^+(8, q)$ is $\Psi(p) = <x_1, x_2>$. We give an upper pound for the number of point of the geometry that are at a distance at most 2. Let q be a point in the geometry such that $\Psi(q) = <y_1, y_2>$, then we have two cases:

1- d(p, q)=1, then $\Psi(p) \cap \Psi(q)$ = 1-space. Then the number of 2-space that intersect $\Psi(p)$ in 1-space is equal the number of 1-space in the 2-space $\Psi(p)$ and by Theorem 2.4 this number is given by the formula:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \prod_{i=0}^{0} (q^{n-i-1}+1) = (q+1)(q^{n-1}+1),$$

i.,e.,

$$|O| \le (q^{n-1}+1)\ (q+1)$$

2- d(p, q)=2, then there is a point r such that $\Psi(r) \cap \Psi(p)$=1-space and $\Psi(r) \cap \Psi(q)$=1-space. Then every 1-space in $\Psi(r) \cap \Psi(p)$ have $(q+1)\ (q^{n-1}+1)$ 1-spaces in $\Psi(r) \cap \Psi(q)$. then the total number of ways such that $\Psi(r) \cap \Psi(p)$=1-space and $\Psi(r) \cap \Psi(q)$=1-space is $(q+1)^2(q^{n-1}+1)^2$. Then the number of points q such that d(p, q)=2 is $(q+1)^2\ (q^{n-1}+1)^2$.

i.,e.,

$$|O| \le (q^{n-1}+1)^2\ (q+1)^2.$$

Now since $p \in \Delta_2(p)$ (d(p, p)=0), then

$$|O| \le (q^{n-1}+1)^2\ (q+1)^2 + 1.$$

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