

## AN ANALYTICAL TECHNIQUE FOR FINDING EXACT SOLUTION OF TWO-DIMENSIONAL DIFFUSION EQUATIONS

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### ABSTRACT

The variational iteration method (VIM) is a powerful tool for solving large amount of problems. In this article, the variational iteration method has been used to obtain exact solutions of the two-dimensional diffusion equation. The idea of variational iteration method was first introduced by He in 1997 [1]. The variational iteration method, a correction functional is constructed by a general Lagrange multiplier which can be identified via a variational theory. The variational iteration method has successfully been applied to many situations [2-5]. Two illustrative examples are given to demonstrate the effectiveness of the present method.

**Keywords and Phrases:** Homotopy perturbation method; Two-dimensional diffusion equation

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### 1. INTRODUCTION

Consider the following two-dimensional diffusion equation with the following initial condition

$$\frac{\partial u}{\partial t} = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (1.1)$$

With initial condition

$$u(x, y, 0) = f(x, y), \quad 0 \leq x, y \leq 1. \quad (1.2)$$

In this article, we use the variational iteration method to solve this kind of equations. To illustrate its basic idea of the method, we consider the following general nonlinear equation

$$Lu(t) + Nu(t) = g(t). \quad (1.3)$$

Where,  $L$  is a linear operator,  $N$  is a nonlinear operator and  $g(t)$  an inhomogeneous term.

According to the variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi) (Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)) d\xi, \quad (1.4)$$

Where  $\lambda$  is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and  $\tilde{u}_n$  is a restricted variation which means  $\delta \tilde{u}_n = 0$  [6].

It is obvious now that the main steps of He's variational iteration method require first the determination of the Lagrangian multiplier  $\lambda$  that will be identified optimally. Having determined the Lagrange multiplier, the successive approximations  $u_{n+1}, n \geq 0$ . Of the solution  $u$  will be readily obtained upon using any selective

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function  $u_0$ . Consequently, the solution

$$u(x) = \lim_{n \rightarrow \infty} u_n(x). \quad (1.5)$$

In other words, the correction functional (1.4) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

## 2. ANALYSIS OF TWO-DIMENSIONAL DIFFUSION EQUATION:

Now for solving Eq. (1.1), by VIM, we construct a correction functional in the following form

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda(\xi) \left\{ \frac{\partial u_n}{\partial \xi}(x, y, \xi) - \alpha \left( \frac{\partial^2 \tilde{u}_n}{\partial x^2}(x, y, \xi) + \frac{\partial^2 \tilde{u}_n}{\partial y^2}(x, y, \xi) \right) \right\} d\xi, \quad (2.1)$$

After some calculations, we obtain the following stationary conditions:

$$\begin{aligned} \lambda'(\xi) &= 0, \\ 1 + \lambda(\xi) \Big|_{\xi=x} &= 0, \end{aligned} \quad (2.2)$$

The Lagrange multiplier, can be easily identified as  $\lambda = -1$ .

Substituting the identified multiplier in to equation (2.1), we would have the following iteration formula

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \left\{ \frac{\partial u_n}{\partial \xi}(x, y, \xi) - \alpha \left( \frac{\partial^2 u_n}{\partial x^2}(x, y, \xi) + \frac{\partial^2 u_n}{\partial y^2}(x, y, \xi) \right) \right\} d\xi, \quad (2.3)$$

We start with the initial approximation of  $u(x, y, 0)$  given by Eq. (1.2). Using the above iteration formula, we can obtain the other components as follows:

$$\begin{aligned} u_0(x, y, t) &= u(x, y, 0) = f(x, y), \\ u_1(x, y, t) &= u_0(x, y, t) - \int_0^t \left\{ \frac{\partial u_0}{\partial \xi}(x, y, \xi) - \alpha \left( \frac{\partial^2 u_0}{\partial x^2}(x, y, \xi) + \frac{\partial^2 u_0}{\partial y^2}(x, y, \xi) \right) \right\} d\xi, \\ u_2(x, y, t) &= u_1(x, y, t) - \int_0^t \left\{ \frac{\partial u_1}{\partial \xi}(x, y, \xi) - \alpha \left( \frac{\partial^2 u_1}{\partial x^2}(x, y, \xi) + \frac{\partial^2 u_1}{\partial y^2}(x, y, \xi) \right) \right\} d\xi, \\ &\vdots \end{aligned} \quad (2.4)$$

## 3. ILLUSTRATIVE EXAMPLES:

**Example: 1** For the first test problem consider (1.1) with  $\alpha=1$  and  $f(x, y) = \exp(x+y)$ , the exact solution is given with

$$u(x, y, t) = \exp(x+y+2t). \quad (3.1)$$

For solving by the variational iteration method we obtain the recurrence relation

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t \left\{ \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) \right\} d\xi, \quad (3.2)$$

Let us start with an initial approximation  $u_0(x, y, t) = f(x, y) = \exp(x + y)$ . applying the iteration formula (2.4). We can obtain directly the other components as

$$\begin{aligned} u_1(x, y, t) &= \exp(x + y) + 2t \exp(x + y), \\ u_2(x, y, t) &= \exp(x + y) + 2t \exp(x + y) + 2t^2 \exp(x + y), \\ u_3(x, y, t) &= \exp(x + y) + 2t \exp(x + y) + 2t^2 \exp(x + y) + \frac{4}{3} t^3 \exp(x + y), \\ &\vdots \\ u_n(x, y, t) &= \sum_{k=0}^n \frac{(2t)^k}{k!} \exp(x + y), \\ &\vdots \end{aligned} \quad (3.3)$$

And exact solution will be as

$$u(x, y, t) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{(2t)^k}{k!} \exp(x + y) = \exp(x + y + 2t). \quad (3.4)$$

**Example: 2** For the first test problem consider (1.1) with  $\alpha = 1$  and  $f(x, y) = (1 + y) \exp(x)$ , the exact solution is given with

$$u(x, y, t) = (1 + y) \exp(x + t). \quad (3.5)$$

For solving by the variational iteration method we obtain the recurrence relation

$$\begin{aligned} u_{n+1}(x, y, t) &= u_n(x, y, t) \\ &- \int_0^t \left\{ \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) \right\} d\xi, \end{aligned} \quad (3.6)$$

Let us start with an initial approximation  $u_0(x, y, t) = f(x, y) = (1 + y) \exp(x)$ . applying the iteration formula (2.4). We can obtain directly the other components as

$$\begin{aligned} u_1(x, y, t) &= (1 + y) \exp(x) + (1 + y) \exp(x) t, \\ u_2(x, y, t) &= (1 + y) \exp(x) + (1 + y) \exp(x) t + \frac{1}{2} (1 + y) \exp(x) t^2, \\ u_3(x, y, t) &= (1 + y) \exp(x) + (1 + y) \exp(x) t + \frac{1}{2} (1 + y) \exp(x) t^2 + \frac{1}{6} (1 + y) \exp(x) t^3, \\ &\vdots \\ u_n(x, y, t) &= u_n(x, y, t) = \sum_{k=0}^n \frac{t^k}{k!} (1 + y) \exp(x), \\ &\vdots \end{aligned} \quad (3.7)$$

And exact solution will be as

$$u(x, y, t) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{t^k}{k!} (1 + y) \exp(x) = (1 + y) \exp(x + t). \quad (3.8)$$

#### 4. CONCLUSION:

In this paper, variation iteration method (VIM) has been successfully applied to find the exact solution of two-dimensional diffusion equation. The obtained solution shows that the method is very convenient and effective to solve wide classes of problems. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Also VIM provides more realistic series solutions that converge very rapidly in real physical problems.

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