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A NOTE ON PARALLEL BINOMIAL EXPANSION AND ITS MULTINOMIAL EXTENSION

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ABSTRACT

The note gives an application of Binomial theorem for multiplying any two digited numbers to itself any number of times which we call Parallel Binomial Expansion. The result has an easy extension to the multinomial case.

Keywords: *Binomial theorem, Parallel Binomial Expansion, place value, Parallel Multinomial Expansion.*

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1. INTRODUCTION

Binomial theorem states that $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_ny^n$ where n is a positive integer. The nomenclature "Binomial theorem" is credited to Abramowitz and Stegun (1972, p. 10). From Wolfram's Mathworld, we get some more interesting facts. The binomial theorem was known for the case n=2 by Euclid, around 300 BC, and stated in its modern form by Pascal in a posthumous pamphlet published in 1665. Pascal's pamphlet, together with his correspondence on the subject with Fermat beginning in 1654 (and published in 1679) is the basis for naming the arithmetical triangle in his honor.

Newton in 1676 showed the formula also holds for negative integers -n,

$$(x+a)^{-n} = \sum_{k=0}^{\infty} {}^{-n}C_k . x^k a^{-n-k}$$

This is the so-called negative binomial series that converges for $|x| < a$.

In fact, the generalization

$$(1+z)^a = \sum_{k=0}^{\infty} {}^aC_k . z^k$$

holds for all complex z with $|z| < 1$" (http://mathworld.wolfram.com/BinomialTheorem.html)

See also Coolidge (1949) and Courant and Robbins (1996).

The following discussion applies the Binomial theorem for multiplying any two digited number to itself any number of times. The result, which we call Parallel Binomial Expansion, has an immediate extension to the multinomial case.

PARALLEL BINOMIAL EXPANSION

Parallel binomial expansion states that binomial expansion can be applied in case of place value system also for multiplying any two digited number xy to itself n number of times.

place values→					hundreds	tens	units
$(xy)^n =$	${}^nC_0x^n$	${}^nC_1.x^{n-1}.y$	${}^nC_2.x^{n-2}.y^2$	…………………………….	${}^nC_{n-2}.x^2.y^{n-2}$	${}^nC_{n-1}.x.y^{n-1}$	${}^nC_n.y^n$

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PROCEDURE:

- Calculations must start from extreme right i.e. from UNIT'S PLACE.

- From the resultant of term at UNIT'S PLACE only last digit is to be written as the right most digit(at unit's place) of the final result and rest must be "carried over" to the TEN'S PLACE calculation where the carried over part is added to the resultant of the term at ten's place and from the result obtained from it the last digit is written at the ten's place of the final result. This method is continued in a similar manner until the left most term is calculated where the entire resultant number is written at the left most part of the final result.

We illustrate the procedure with some examples.

First Example:

place values→		THOUSANDS	HUNDREDS	TENS	UNITS
$(27)^4 =$	${}^4C_0.2^4$	${}^4C_1.2^3.7$	${}^4C_2.2^2.7^2$	${}^4C_3.2.7^3$	${}^4C_4.7^4$
$=$	1.2^4	$4.2^3.7$	$6.2^2.7^2$	$4.2.7^3$	7^4
$=$	16+37 = $\boxed{53}$	224+147 =37[1]	1176+298 =147[4]	2744+240 =298[4]	2401=240[1]
$=$	$\boxed{53}$ 1441				

Second Example:

place values→		TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(52)^5 =$	${}^5C_0.5^5$	${}^5C_1.5^4.2$	${}^5C_2.5^3.2^2$	${}^5C_3.5^2.2^3$	${}^5C_4.5.2^4$	${}^5C_5.2^5$
$=$	1.5^5	$5.5^4.2$	$10.5^3.2^2$	$10.5^2.2^3$	$5.5.2^4$	1.2^5
$=$	3125+677= $\boxed{3802}$	6250+520 =677 [0]	5000+204 =520[4]	2000+40 =204[0]	400+3 =40[3]	32=3[2]
$=$	$\boxed{3802}$ 04032					

2. HOW DOES PARALLEL BINOMIAL EXPANSION WORK?

We write a two digited number xy as xy= 10x + y

We have, $(xy)^n = (10x + y)^n$

Applying Binomial theorem, we get

$$(10x + y)^n = {}^nC_0.(10x)^n + {}^nC_1.(10x)^{n-1}.y + {}^nC_2.(10x)^{n-2}.y^2 + \ldots\ldots$$

$$\ldots\ldots + {}^nC_{n-1}.(10x).y^{n-1} + {}^nC_n.y^n$$

$$= 10^n.{}^nC_0.x^n + 10^{n-1}.{}^nC_1.x^{n-1}.y + 10^{n-2}.{}^nC_2.x^{n-2}.y^2 + \ldots\ldots$$

$$\ldots\ldots + 10.{}^nC_{n-1}.x.y^{n-1} + {}^nC_n.y^n$$

$$= 10^n.a_r + 10^{n-1}.a_{r-1} + 10^{n-2}.a_{r-2} + \ldots\ldots$$

$$\ldots\ldots + 10.a_1 + a_0$$

Where $a_r = {}^nC_{n-r}.x^r.y^{n-r}$ (r=power of 10 in the respective term)

Just as 1000 x 5 + 100 x 6 + 10 x 7 + 8 = 5678,

Similarly in the PLACE VALUE SYSTEM,

$(10x + y)^n = 10^n.a_r + 10^{n-1}.a_{r-1} + 10^{n-2}.a_{r-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\ldots\ldots\ldots\ldots\ldots + 10.a_1 + a_0$

place value→					**THOUSANDS**	**HUNDREDS**	**TENS**	**UNITS**
$(10x + y)^n =$	a_r	a_{r-1}	a_{r-2}	…………….	a_3	a_2	a_1	a_0

place value→					**THOUSANDS**	**HUNDREDS**	**TENS**	**UNITS**
$(x\ y)^n =$	a_r	a_{r-1}	a_{r-2}	………………	a_3	a_2	a_1	a_0

place value→					**HUNDREDS**	**TENS**	**UNITS**
$(xy)^n =$	${}^nC_0x^n$	${}^nC_1x^{n-1}y$	${}^nC_2x^{n-2}y^2$	……………………	${}^nC_{n-2}x^2y^{n-2}$	${}^nC_{n-1}xy^{n-1}$	${}^nC_ny^n$

3. A FEW DIRECT FORMULAE:

- $(ab)^2 = a^2 \mid (2 \times a \times b) \mid b^2$

  Eg. $(98)^2 = 9^2 \mid (2 \times 9 \times 8) \mid 8^2$

  $= 81+15=96 \mid 144+6=15[0] \mid 6[4]$
  $=9604$

- $(ab)^3 = a^3 \mid 3 \times a^2 \times b \mid 3 \times a \times b^2 \mid b^3$

  Eg. $(98)^3 = 9^3 \mid 3 \times 9^2 \times 8 \mid 3 \times 9 \times 8^2 \mid 8^3$

  $= 729+212=941 \mid 1944+177=212[1] \mid 1728+51=177[9] \mid 51[2]$
  $= 941192$

PARALLEL MULTINOMIAL EXPANSION

Parallel Multinomial Expansion states that multinomial theorem can be applied in case of place value system also for multiplying any number to itself any number of times.

place value→					**HUNDREDS**	**TENS**	**UNITS**
$(abcde.....)^n =$	T_X	$T_{(X-1)}$	$T_{(X-2)}$	…………………	T_2	T_1	T_0

- Here value of X is the highest value on the power of 10 in all the terms in multinomial theorem expression.
- Here T_x , $T_{(X-1)}$ etc are the expressions of a, b, c, d,……….. and so on.
- The above expression can be understood by the examples given later.

4. MECHANISM OF PARALLEL MULTINOMIAL EXPANSION

- **For a three digited number abc**

$(abc)^n = (100a + 10b + c)^n$

$$= \Sigma \frac{n!}{p!.q!.r!} . 100^p .a^p . 10^q .b^q .c^r$$

$$= \Sigma \frac{n!}{p!.q!.r!} . a^p. b^q .c^r . 100^p. 10^q$$

$$= \Sigma \frac{n!}{p!.q!.r!} . a^p. b^q .c^r .10^{(2p+q)} \qquad \text{(where } p +q +r=n)$$

$= \Sigma\, T_{(2p+q)}.\, 10^{(2p+q)}$

- With the different values of(2p+q) say
  x, (x-1) ,(x-2),……..,3,2,1,0

  $(abc)^n = 10^X.T_x + 10^{(x-1)}.T_{(X-1)} + 10^{(X-2)}.T_{(X-2)} + \ldots\ldots\ldots + 10^2.T_2 + 10.T_1 + T_0$

And $T_{(2p+q)} = \dfrac{n!}{p!.q!.r!} .\ a^p.\, b^q.c^r$;where X=highest value of (2p+q)

since 1000 x 5 + 100 x 6 + 10 x 7 + 8 = 5678

Similarly in the **PLACE VALUE SYSTEM:**

place value→					HUNDREDS	TENS	UNITS
$(abc)^n =$	T_X	$T_{(X-1)}$	$T_{(X-2)}$	………	T_2	T_1	T_0

Table 1 gives the various combinations of p, q, r and the respective value of (2p+q) are given as (take n=3)

Table 1: Table showing combinations of p,q,r and the respective value of (2p+q) for n=3

All possibilities of p, q, r where p+q+r=3	Values of 2p+q	Expression of $T_{(2p+q)}$
p=0 , q=0 , r=3;	0	c^3
p=0 , q=1 , r=2;	1	$3bc^2$
p=0 , q=2 , r=1;	2	$3b^2c$
p=0 , q=3 , r=0;	3	b^3
p=1, q=0 , r=2;	2	$3ac^2$
p=1 , q=1 , r=1;	3	$6abc$
p=1 , q=2 , r=0;	4	$3ab^2$
p=2 , q=0 , r=1;	4	$3a^2c$
p=2 , q=1 , r=0;	5	$3a^2b$
p=3 , q=0 , r=0;	6	a^3

Here X=6. Therefore, the arrangement will be as

$(abc)^3 = 10^6.T_6 + 10^5.T_5 + 10^4.T_4 + 10^3.T_3 + 10^2.T_2 + 10.T_1 + T_0$

Since,

$T_{(2p+q)} = \dfrac{n!}{p!.q!.r!} .\ a^p.\, b^q.c^r$

Putting the value of n=3 and various values of p, q, r we get $T_{(2p+q)}$. Since there are two terms containing (2p+q) = 3 so these two terms will be added as (b^3 + 6abc). Similar process will be done for (2p+q) =4.

General formula for cube of a three digited number

place value→	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(abc)^3 =$	a^3	$3a^2b$	$3a^2c + 3ab^2$	$6abc + b^3$	$3ac^2 + 3b^2c$	$3bc^2$	c^3

Example: Take a=1, b=2, c=3. Writing the value of $T_{(2p+q)}$ in their respective place values for an illustration:

place values→	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(123)^3 =$	1	6	9+12	36+8	27+36	54	27

- Now we give the desired expansion with proper carry over method **:-**

place values→	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(123)^3$ =	1	6+2 =8	21+5 =2[6]	44+6 =5[0]	63+5 =6[8]	54+2 =5[6]	2[7]

Hence

$(123)^3$ = 1860867

- **For a four digited number abcd**

$(abcd)^n = (1000a + 100b + 10c + d)^n$

$$= \Sigma \frac{n!}{p!.q!.r!.s!} \cdot 1000^p .a^p.\ 100^q .b^q.\ 10^r c^r .d^s$$

$$= \Sigma \frac{n!}{p!.q!.r!.s!} \cdot a^p.\ b^q .c^r .d^s.\ 1000^p.\ 100^q .10^r$$

$$= \Sigma \frac{n!}{p!.q!.r!.s!} \cdot a^p.\ b^q .c^r.d^s .10^{(3p+2q+r)}$$

(where $p + q + r + s = n$)

$= \Sigma\ T_{(3p+2q+r)}.\ 10^{(3p+2q+r)}$

With the different values of (3p+2q+r) say

x, (x-1) ,(x-2),……..,3,2,1,0

$(abcd)^n = 10^X.T_x + 10^{(x-1)} .T_{(X-1)} + 10^{(X-2)} .T_{(X-2)} + \ldots\ldots + 10^2.T_2 + 10 .T_1 + T_0$

And $T_{(3p+2q+r)} = \frac{n!}{p!.q!.r!,s!} \cdot a^p.\ b^q .c^r,\ d^s$;

Where X=highest value of (3p+2q+r);

since 1000 x 5 + 100 x 6 + 10 x 7 + 8 = 5678

Similarly in PLACE **VALUE SYSTEM:**

place values→					HUNDREDS	TENS	UNITS
$(abcd)^n$ =	T_X	$T_{(X-1)}$	$T_{(X-2)}$	……	T_2	T_1	T_0

Similarly, for a five digited number abcde,

$(abcde)^n = (10000a + 1000b + 100c + 10d + e)^n$

$$= \Sigma \frac{n!}{p!.q!.r!.s!.t!} \cdot 10000^p .a^p.\ 1000^q .b^q.\ 100^r c^r .10^s d^s .e^t$$

$$= \Sigma \frac{n!}{p!.q!.r!.s!.t!} \cdot a^p.\ b^q .c^r .d^s .e^t.\ 10000^p.\ 1000^q .100^r.10^s$$

$$= \Sigma \frac{n!}{p!.q!.r!.s!.t!} \cdot a^p.\ b^q .c^r .d^s .e^t.10^{(4p+3q+2r+s)}$$

(where $p + q + r + s + t = n$)
$= \Sigma\ T_{(4p+3q+2r+s)}.\ 10^{(4p+3q+2r+s)}$

With the different values of (4p+3q+2r+s) say

x , (x-1), (x-2),……..,3,2,1,0

$(abcde)^n = 10^X.T_x + 10^{(x-1)}.T_{(X-1)} + 10^{(X-2)}.T_{(X-2)} + ….. +10^2.T_2 + 10.T_1 + T_0$

And $T_{(4p+3q+2r+s)} = \frac{n!}{p!.q!.r!.s!.t!} . a^p. b^q.c^r.d^s.e^t$;

where X=highest value of (4p+3q+2r+s)

since 1000 x 5 + 100 x 6 + 10 x 7 + 8 = 5678

In **PLACE VALUE SYSTEM**, we have**:**

place values→					HUNDREDS	TENS	UNITS
$(abcde)^n =$	T_X	$T_{(X-1)}$	$T_{(X-2)}$	………	T_2	T_1	T_0

Table 2 gives the various combinations of p, q, r, s, t and the respective value of (4p+3q+2r+s) is given as (take n=2)

Table 2: Table showing combinations of p, q, r, s, t and the respective values of (4p+3q+2r+s) for n=2

All possibilities of p, q, r, s, t/p+ q+r+s+t=2	Value of (4p+3q+2r+s)	Expression of $T_{(4p+3q+2r+s)}$

p=0 , q=0 , r=0 , s=0 , t=2;
p=0 , q=0 , r=0, s=2 , t=0;
p=0 , q=0 , r=2, s=0 , t=0;
p=0 , q=2 , r=0, s=0 , t=0;
p=2, q=0 , r=0, s=0 , t=0;
	0
	2
	4
	6
	8
		e^2
		d^2
		c^2
		b^2
		a^2

p=1 , q=1 , r=0, s=0 , t=0;
p=1 , q=0 , r=1, s=0 , t=0;
p=1 , q=0 , r=0, s=1 , t=0;
p=1 , q=0 , r=0, s=0 , t=1;
	7
	6
	5
	4
		2ab
		2ac
		2ad
		2ae

p=0 , q=1 , r=1, s=0 , t=0;
p=0 , q=1 , r=0, s=1 , t=0;
p=0 , q=1 , r=0, s=0 , t=1;
	5
	4
	3
		2bc
		2bd
		2be

p=0 , q=0 , r=1, s=1 , t=0; p=0 , q=0 , r=1, s=0 , t=1;	3 2	2cd 2ce
p=0 , q=0 , r=0, s=1 , t=1;	1	2de

Here X=8. Therefore, the arrangement will be as
$(abcde)^2 = 10^8.T^8 + 10^7.T^7 + 10^6.T_6 + 10^5.T_5 + 10^4.T_4 + 10^3.T_3 + 10^2.T_2 + 10.T_1 + T_0$
Since,

$$T_{(4p+3q+2r+s)} = \frac{n!}{p!.q!.r!.s!.t!} . a^p. b^q.c^r.d^s.e^t$$

Put value of n=2 and various values of p, q, r, s, t to get $T_{(4p+3q+2r+s)}$;

- Terms with same value of (4p+3q+2r+s) will be added as shown in previous examples.

General formula for square of a five digited number abcde:

place values→	TEN CRORE	CRORE	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(abcde)^2=$	a^2	2ab	$b^2+ 2ac$	2ad+2bc	$c^2 + 2ae + 2bd$	2be + 2cd	$2ce + d^2$	2de	e^2

Example: (take a=1, b=2, c=3, d=4, e=5). Writing the value of $T_{(4p+3q+2r+s)}$ in their respective place values for illustration-

place values→	TEN CRORE	CRORE	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(12345)^2$=	1	4	4+6	8+12	9+10+16	20+24	30+16	40	25

- Now we give the desired expansion with proper carry over method-

place values→	TEN CRORE	CRORE	TEN LACS	LACS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
$(12345)^2$=	1	4+1 =5	10+2 =1[2]	20+3 =2[3]	35+4 =3[9]	44+5 =4[9]	46+4 =5[0]	40+2=4[2]	2[5]

Hence

$(12345)^3$ = 152399025

5. CONCLUDING REMARKS

From the previous examples, we observe that

- For a three digited number, the value of X depends on the values of (2p+q)
  For a four digited number, the value of X depends on the values of (3p+2q+r)
  For a five digited number, the value of X depends on the values of (4p+3q+2r+s) and so on.

- Calculating all the possibilities is not difficult because it follows a certain pattern. In the given two examples that pattern can be understood and by practice, finding all possibilities of p, q, r, s……. and $T_{(subscript)}$ become quite easier.

As future work, it may be rewarding to do a comparative study on computational complexity of an algorithm that achieves these expansions with the same for an algorithm that performs the products in question without using these expansions.

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