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A NEW CLASS OF PSEUDO CYCLIC ASSOCIATION SCHEME

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ABSTRACT

A class of pseudo cyclic association scheme is obtained on p^2 elements where p is a prime number, which is contained in a broad class of strongly balanced association schemes that seem to be related to H-matrices. Also from this class we have constructed an infinite series of 2- (p^2, $p^2(p+1)$, p^2-1, p-1, p-2) –design and a set of four Williamson type matrices of order p^2.

Keywords: *Association Schemes (AS), Pseudo Cyclic Association Scheme (PCAS), Block Circulant Matrix, Generalized Orthogonal Combinatorial Matrix(GOCM), Row and column regular GOCM.*

INTRODUCTION

Bose and Shimamoto [2] gave the first formal definition of AS and listed four types of ASs with two associate classes. Some series of ASs are those of Bose and Connor [3], Vartak [20], Ragavarao and Chandrasekhara rao [12]. Roy [13], Hinkelmann and Kempthorne [7], Kusomoto [8], Surendran [19] obtained series of higher class ASs which can be obtained from trivial schemes by iterated crossing and nesting. Yamamoto *et al.* [21] listed six infinite family of ASs the cyclic schemes, the factorial schemes, iterated nested schemes, hypercubic schemes, triangular type and square schemes**.** For a recent account of AS vide Baily [1]. Here we introduce a new class of pseudo cyclic association schemes (PCAS) which contains the family constructed here. In the construction of the family of PCAS the notion of GOCM [16] introduced by Singh *et al.* is used.

A series of non symmetric BIBDs are due to Bose [4], Srikhande [14], Sinha and Singh [15] Logan *et al.* [10], etc. Also some known infinite class consists of doubly Resolvable-BIBD (RBIBD) (p^n,p,1) for p a prime number and integer n>3 [5] and RBIBD (k(k+1), k, k-1) either k or k+1 is a prime power and k>3 [9]. It appears that a series of BIBDs from association schemes has not been constructed. In this note we construct a new series of PCAS's which yields a series of BIBDs.

Abbreviations: J_n is all ones matrix of order n, J_{mn} is m×n all ones matrix, I_n is the unit matrix of order n, $K_n = J_n - I_n$. Sometimes J_n, I_n, K_n will be abbreviated as J, I, K respectively.

First we define the following terms.

1.1Association Scheme (AS): A d-class association scheme with vertex set X of order v is a sequence of non-zero {0, 1}-matrices $A_0, A_1, A_2, \ldots, A_d$ with rows and column indexed by X, such that

(i) $A_0 = I$,
(ii) $A_i^T = A_i$ for all i ∈ {0, 1, 2,,………,d}
(iii) $A_0 + A_1 + A_2 + \ldots + A_d = J$ **,**
(iv) A_iA_j lies in the real span of $A_0, A_1, A_2, \ldots, A_d$: $A_iA_j = \sum_{k=0}^{d} P_{ij}^{\ k} A_k$.
(v) (vide Godsil and Song [6]).

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1.2. Pseudo Cyclic Association Scheme (PCAS): Let an AS be defined by (d+1) {0, 1} matrices $A_0, A_1, A_2, \ldots, A_d$. $A_0 = I$ having parameters P_{ij}^k (i, j, k = 0, 1, 2, . . .,d). The AS will be called pseudo cyclic association scheme (PCAS) if $\sum_{l=1}^{d} P_{ll}^{j}$ is same for j∈ {1,2,. . . .,d},

1.3 Block Circulant Matrix: Let $A_1, A_2, \ldots, A_m$ be square matrices each of order n. By a block circulant matrix M of type (m, n) is an mn× mn matrix of the form M = Circ ($A_1, A_2, \ldots, A_m$). A block circulant matrix is not necessarily a circulant matrix.

1.4 Williamson-type matrices: Four {1-1} matrices A, B, C, D of order v satisfying

(i) $MN^T = NM^T$, M,N∈{A,B,C,D} and
(ii) $AA^T + BB^T + CC^T + DD^T = 4vI_v$ are called Williamson-type matrices.

1.5 Row generalized orthogonal combinatorial matrix: (Row GOCM) [18].
Row generalized orthogonal combinatorial matrix (Row GOCM) over left modules on the ring of m x m matrices over ℤ.We consider left modules M_j of m x n_j (0,1) matrices on the ring of m x m matrices over ℤ, (j = 1,2,. . .,s) such modules are left m-modules. Let N = (N_{ij}) , i=1,2,3………,l and j=1,2,3,…….,s be a block matrix, where N_{ij} are m x n_j matrix from M_j, j=1, 2, ..,s where $\sum_{j=1}^{s} n_j$= n. Clearly blocks of j[th] column of N ∈ M_j. Let $R_i = (N_{i1}\ N_{i2} \ldots N_{is})$ be the i[th] block row of N. We define inner product of two block rows R_i and R_j as $R_i o R_j = R_i R_j^T = R_{ij} = \sum_{k=1}^{l} N_{ik} N_{jk}^T \in M_j$.N is called a row GOCM if there exists fixed integers r, $\lambda_1, \lambda_2, \lambda_3$ such that
$R_{ij} = \sum_{k=1}^{l} N_{ik} N_{jk}^T = rI_m + \lambda_1 K_m$, whenever i = j
and $=\lambda_2 I_m + \lambda_3 K_m$ whenever i ≠ j
l, s, m, $n_1, n_2, \ldots, n_s$, r, $\lambda_1, \lambda_2, \lambda_3$, will be called parameters of the row GOCM.

If $n_1 = n_2 = \ldots = n_s = n$, then l, s, m, n, r, $\lambda_1, \lambda_2, \lambda_3$, will be called parameters of the row GOCM.

When m = n, row GOCM will has square blocks with parameters l, s, n, r, $\lambda_1, \lambda_2, \lambda_3$.

Remark 1: N will be an orthogonal matrix if l = s, m = n= 1, r = 1,$\lambda_1 = \lambda_2 = \lambda_3 = 0$

1.6 Row and column Regular {0, 1} GOCM:
An lm×sm {0, 1} GOCM N will be called a row and column regular GOCM if sum of elements in any row is the same as well as sum of elements in any column is the same, or if $NJ_{sm} = rJ_{lm,sm}$ and $J_{lm}N = kJ_{lm,sm}$, where $J_{lm,sm}$ is lm×sm all ones matrix and r, k are fixed non negative integer.

Remark 2: A square matrix is called regular if its row sum is equal to its column sum.

2. ROW GOCM AND ITS REDUCTION TO AN INCIDENCE MATRIX OF A RECTANGULAR DESIGN

Theorem 2.1: A row GOCM N with constant column sum k is in general a rectangular design (RD).

Proof: We have $NN^T = \begin{pmatrix} rI_m + \lambda_1 K_m & \cdots & \lambda_2 I_m + \lambda_3 K_m \\ \vdots & \ddots & \vdots \\ \lambda_2 I_m + \lambda_3 K_m & \cdots & rI_m + \lambda_1 K_m \end{pmatrix}$

$$= r(I_m \times I_l) + \lambda_1(I_m \times K_l) + \lambda_2(K_m \times I_l) + \lambda_3(K_m \times K_l).$$

Let $B_0=(I_m\times I_l)$,$B_1= (I_m\times K_l)$,$B_2 = (K_m\times I_l)$,$B_3 = (K_m\times K_l)$.

These are the association matrices of at most three class association scheme. Using the properties of Kronecker product it is easy to verify the postulates of AS.

3. CONSTRUCTION THEOREMS

Theorem 3.1: A row and column regular {0, 1} GOCM N with parameters l, s, m, r, $\lambda_1, \lambda_2, \lambda_3$ is the incidence matrix of a (lm ,sm, r, lr/s, λ)-design if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$

Proof: If N is the incidence matrix of a (v, b, r, k, λ)-design, then
$NN^T=(r-\lambda) I_v + \lambda J_v$ and $J_vA = KJ_{v,b}$. More over λ (v-1) = r (k -1) and bk =vr. (vide R. Mathon & A.Rosa [11])
N is a row and column regular GOCM and $\lambda_1 = \lambda_2 = \lambda_3$ then $NN^T = rI_{lm} + \lambda K_{lm}$ hence by theorem 3.1, N is the incidence matrix of a (lm, sm, r, lr/s, λ)-design.

Theorem 3.2: A series of pseudo cyclic association schemes on p^2elements, where p is a prime.
There exists a (p+1) class association scheme A(p+1) defined by the non-zero {0, 1}matrices of order p^2, A_0 =I, A_i (i = 1, 2, 3,…….,p+1) satisfying $A_i^2 = (p-1)I + (p-2)A_i$ (1)
And $A_iA_j= \sum_{k\in M-\{i,j\}} A_k$, M = {1, 2, 3,……….,p+1} (2)

Proof: For a given prime p we construct following {0, 1} regular matrices of order p^2 as
$A_0 = I = \mathrm{Circ}(I_p, O, \ldots, O)$,
$A_1 = \mathrm{Circ}(K_p, O, \ldots, O)$,
$A_2 = \mathrm{Circ}(O, I_p, \ldots, I_p)$
and $A_{i+2} = \mathrm{Circ}(O, (\alpha^i)^1, (\alpha^i)^2, \ldots, (\alpha^i)^{p-1})$, i = 1,2,3,. . . .,p-1 and α is the {0,1} circulantmatrix such that $\alpha^p = I_p$.

we introduce $B_1 = \mathrm{Circ}(J_p, O, \ldots, O)$ and $B_{r+1} = [\alpha^{(r-1)(j-i)})]$, i, j = 1, 2,. . . . .,p and r=1, 2,. . .,p

Now $B_1^2 = \mathrm{Circ}(J_p^2, O, \ldots, O)$
$= \mathrm{Circ}(pJ_p, O, \ldots, O)$
$= p\mathrm{Circ}(J_p, O, \ldots, O) = pB_1$

$B_2^2 = \mathrm{Circ}(I_p, I_p, \ldots, I_p)\,\mathrm{Circ}(I_p, I_p, \ldots, I_p)$
$= \mathrm{Circ}(pI_p, pI_p, \ldots, pI_p) = pB_2$

and $B_{r+1}^2 = [\alpha^{(r-1)(j-i)}]^2$
$= [\sum_{j=1}^{p} \alpha^{(r-1)(j-i)-(r-1)(j-k)}]$
$= [\sum_{j=1}^{p} \alpha^{(r-1)(k-i)}] = [p\,\alpha^{(r-1)(k-i)}] = p[\alpha^{(r-1)(k-i)}] = pB_{r+1}$

Putting $B_i = I + A_i$, it can be verified that A_i satisfies the condition
$A_i^2 = (p-1)I + (p-2)A_i$.

Next we show in two cases that $A_iA_j = \sum_{k\in M-\{i,j\}} A_k$, M = {1, 2, 3,……….,p+1

Case-(i): $B_1B_{r+1} = \mathrm{Circ}(J_p, O, \ldots, O)\,\mathrm{Circ}(I_p, \alpha^{r-1}, (\alpha^{r-1})^2, \ldots, (\alpha^{r-1})^{p-1})$
$= \mathrm{Circ}(J_p, J_p, \ldots, J_p)$

Also $B_{r+1}B_1 = \mathrm{Circ}(J_p, J_p, \ldots, J_p)$
$= I + A_1 + A_2 + \ldots + A_{p+1}$

Therefore, $(I + A_1)(I + A_{r+1}) = (I + A_{r+1})(I + A_1) = I + A_1 + A_2 + \ldots + A_{p+1}$

Or $A_{r+1}A_1 = A_1A_{r+1} = (A_1 + A_2 + \ldots + A_{p+1}) - A_1 - A_{r+1}$, r = 1, 2, . . .,p

Case-(ii): Let $1\le r < s\le p$
$B_{r+1}B_{s+1} = [\alpha^{(r-1)(j-i)}][\alpha^{(s-1)(j-k)}] = [\sum_{j=1}^{p} \alpha^{(r-1)(j-i)-(s-1)(j-k)}]$,
the superscript (r-1)(j-i)-(s-1)(j-k) = (r-s)j + k(s-1) –i(r-1) is a number mod p. Since r, s, i, k are fixed integer r<s and j varies, the superscript takes all values mod p from 0, 1,......(p-1),

Therefore $B_{r+1}B_{s+1} = J$
i.e.$(I + A_{r+1})(I + A_{s+1}) = I + A_1 + A_2 + \ldots + A_{r+1} + \ldots + A_{s+1} \ldots + A_{p+1}$

i.e.$A_{r+1}A_{s+1} = (A_1 + A_2 + \ldots + A_{p+1}) - A_{r+1} - A_{s+1}$

Therefore $A_iA_j = \sum_{k\in M-\{i,j\}} A_k$, M = {1, 2, 3,……….,p+1}

It is clear that the matrices constructed by the above method satisfy the condition of the theorem. Now we have
(i) $A_0 = I$
(ii) $A_i^T = A_i$
(iii) $A_0 + A_1 + \ldots + A_{p+1} = J$,
(iv) $A_iA_j = \sum_{k=0}^{p+1} P_{ij}^k A_k$,p_{ij}^k are integers

Thus A_0 =I, A_i (i = 1,2,3,…….,p+1) form a (p+1)-class Pseudo Cyclic Association Scheme by 1.1

Remark 3: The parameters P_{ij}^{k} of the PCAS are given by $p_{ii}^{0} = n_i = (p-1)$, $p_{ii}^{k} = (p-2)\delta_{ik}$
And $p_{ij}^{k} = 1$, if $k \neq i,j$
0, if $k = i,j$ $(i, j, k = 1, 2, \ldots, p+1)$

3.4 Construction of 2- (p^2, $p^2(p+1)$, p^2-1, p-1, p-2)-design.
For a given prime $p>2$, we apply the above theorem to construct a GOCM, which will be an incidence matrix of a 2- (p^2, $p^2(p+1)$, p^2-1, p-1, p-2)-design

Theorem: If $A_1, A_2, \ldots, A_{p+1}$ be {0, 1} block matrices of order p^2 as defined in theorem 1,then $p^2 \times p^2(p+1)$ {0,1} matrix $N = [A_1\ A_2 \ldots A_{p+1}]$ is a GOCM which is the incidence matrix of a 2-(p^2, $p^2(p+1)$, p^2-1, p-1,p-2)-design.

Proof: We have $N = [A_1\ A_2 \ldots A_{p+1}]$ and

$$NN^T = \sum_{k=1}^{p+1} A_k A_k^T$$
$$= \sum_{k=1}^{p+1} A_k A_k \quad \text{(since } A_k \text{ is symmetric for each k)}$$
$$= \sum_{k=1}^{p+1} A_k^2$$
$$= \sum_{k=1}^{p+1} (p-1)I + (p-2)A_k \text{, by (theorem2)}$$
$$= (p+1)(p-1)I + (p-2)K$$
$$= (p^2-1)I + (p-2)K$$

Therefore N is GOCM with only one block row by (1.5). Here $r = p^2-1$, $\lambda 1 = \lambda 2 = \lambda = p-2$ and also by construction A_i are regular and so G is row and column regular {0, 1} GOCM. Hence by theorem 3.1, N is the incidence matrix of a 2-(p^2, $p^2(p+1)$, p^2-1, p-1, p-2) design.

Example: Let p=3, we construct the regular {0, 1} Matrices A_0, A_1, A_2, A_3, A_4 as described in theorem 1.
$A_0 = \text{Circ}(I_3, O, O)$, $A_1 = \text{Circ}(K_3, O, O)$
$A_2 = \text{Circ}(O, I_3, I_3)$, $A_3 = \text{Circ}(O, \alpha, \alpha^2)$
$A_4 = \text{Circ}(O, \alpha^2, \alpha)$,Where $\alpha = \text{Circ}(1, 0, 0)$

It can be easily verified that the matrices A_0, A_1, A_2, A_3, A_4 satisfy the condition of theorem 1 and hence defines a PCAS. Also since $A_i^2 = 2I + A_i$ (i = 1, 2, 3, 4) and
$A_iA_j = \sum_{k \in M-\{i,j\}} A_k$, M = {1, 2, 3, 4} for i, j, k = 1, 2, 3, 4 the parameters p_{ij}^{k} of the PCAS are $p_{ii}^{0} = 2$, $p_{ii}^{k} = \delta ik$ and $p_{ij}^{k} = 1$, if $k \neq i,j$ and 0 otherwise.

We have $N = [A_1\ A_2\ A_3\ A_4]_{9x36}$

$$= \begin{matrix} K_3 & 0 & 0 & 0 & I_3 & I_3 & 0 & \propto & \propto^2 & 0 & \propto^2 & \propto \\ 0 & K_3 & 0 & I_3 & 0 & I_3 & \propto^2 & 0 & \propto & \propto & 0 & \propto^2 \\ 0 & 0 & K_3 & I_3 & I_3 & 0 & \propto & \propto^2 & 0 & \propto^2 & \propto & 0 \end{matrix}$$

Also $NN^T = 8I_9 + K_9$

Thus N is the incidence matrix of a BIBD with parameters
$v = p^2 = 9$, $b = p^2(p+1) = 36$, $r = p^2-1 = 8$, $k = p-1 = 2$, $\lambda = p-2 = 1$ [vide theorem 3.1]

3.5 Construction of H-Matrix

Theorem: The existence of a PCAS with parameters v, P_{ij}^{k}(i, j, k = 1, 2, 3, ...,m) satisfying m Diophantine equations
$2\sum_{i=1}^{m} P_{ii}^{k} + (a_0a_k + b_0b_k + c_0c_k + d_0d_k) + \sum_{1 \le i,j \le m}(\sum_{a,b,c,d} a_ia_j)P_{ij}^{k} = 0$,
Where a_i, b_i, c_i, d_i =1 or -1, $(a_0 = b_0 = c_0 = d_0 = 1)$ implies the existence of Williamson type matrices
$A = \sum_0^m a_iA_i$, $B = \sum_0^m b_iA_i$, $C = \sum_0^m c_iA_i$, $D = \sum_0^m d_iA_i$.

Proof: Here we have,

$$A^2 = a_o^2 I_v + \sum_{i=1}^{m} a_i^2 A_i^2 + 2a_o \sum_{k=1}^{m} a_k A_k + \sum_{i,j=1}^{m} a_i a_j A_i A_j$$

Simplifying in view of theorem(1) and (2) we get,

$$A^2 = I_v + \sum_{i=1}^{m} n_i I_v + \sum_{i=1}^{m}\sum_{k=1}^{m} P_{ii}^{k} A_k + 2\sum_{k=1}^{m} a_o a_k A_k + 2\sum_{k=1}^{m}\sum_{i,j=1}^{m} a_i a_j P_{ij}^{k} A_k$$

Writing similar expressions for B^2, C^2, D^2 and adding all the four we get
$A^2 + B^2 + C^2 + D^2 = 4vI_v$. In view of 1.4 A, B, C, D are Williamson type matrices.

Example: We construct an H-Matrix of order $4x3^2$

Let p = 3, We construct the 4 class PCAS A_0, A_1, A_2, A_3, A_4 as in above example. Here $v = p^2 = 9$, $p_{ii}^0 = 2$, $p_{ii}^k = \delta ik$ and $p_{ij}^k = 1$, if $k \neq i,j$ and 0otherwise.

The 4 Diophantine equations

$$2\sum_{i=1}^{4} P_{ii}^k + (a_0a_k + b_0b_k + c_0c_k + d_0d_k) + \sum_{1\le i,j\le 4}(\sum_{a,b,c,d} a_ia_j)P_{ij}^k = 0,$$

is satisfied if we take

$[a_1\ a_2\ a_3\ a_4] = [-1\ -1\ -1\ -1]$,

$[b_1\ b_2\ b_3\ b_4] = [-1\ -1\ 1\ -1]$,$[c_1\ c_2\ c_3\ c_4] = [-1\ 1\ -1\ 1]$

and $[d_1\ d_2\ d_3\ d_4] = [1\ -1\ -1\ 1]$

Then the Williamson-type matrices are

$A = A_0 - A_1 - A_2 - A_3 - A_4$, $B = A_0 - A_1 - A_2 + A_3 - A_4$, $C = A_0 - A_1 + A_2 - A_3 + A_4$,

$D = A_0 + A_1 - A_2 - A_3 + A_4$

And

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}$$

is the Hadamard matrix.

Note: It appears that H-Matrix of order $4x5^2$ and $4x7^2$ is not possible by the above theorem, however some H-matrix from 2-associate PCAS has been constructed in the paper of Singh and Sinha, vide [17]

Remark: Almost all the symmetric AS which were used to construct H-Matrices are pseudo cyclic AS's or weakly pseudo cyclic AS's. We hope that in future more H-Matrices will be constructed from such association schemes. Some examples

1. Williamson's circulant AS used to construct H-Matrices are pseudocycle.
2. Singh (et al.) used weakly pseudo cyclic AS.
3. Singh and Prasad's family of pseudo cyclic AS's contain one AS which can be used to construct an H-Matrix of order 36.
4. Classen used non symmetric AS to construct an H-Matrix of Bush-type.

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