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ON HOMOMORPHISMS IN CI-ALGEBRAS

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ABSTRACT

In this paper we discuss homomorphism of CI-algebras, its examples and investigate some new properties. We consider some particular type of mappings defined on Cartesian product of CI–algebras.

***Keywords**: CI-algebra, BE-algebra, subalgebra, ideal, Cartesian product.*

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1. INTRODUCTION

In 1966, Y. Imai and K. Iseki ([2, 3]) introduced the notion of BCK/BCI-algebras. There exist several generalizations of BCK/BCI-algebras, such as BCH-algebras ([1]), BH-algebras ([4]), d-algebras ([8]), etc. As a dualization of a generalization of BCK-algebra ([5]), H.S. Kim and Y. H. Kim introduced the notion of BE-algebra ([6]). In 2010, B. L. Meng ([7]) introduced the notion of CI-algebras as a generalization of BE-algebras. The concept of Homomorphisms in CI-algebras was introduced by P.M.Sithar Selvam, T.Priya and T.Ramchandran ([10]). In this paper we discuss some special type of homomorphisms on CI-algebras and investigate some of its properties in details.

2. PRELIMINARIES

Definition 2.1 ([6]): A system $(X; *, 1)$ of type $(2, 0)$ consisting of a non-empty set X, a binary operation $*$ and a fixed element 1 is called a BE–algebra if the following conditions are satisfied:

1. (BE 1) $x * x = 1$
2. (BE 2) $x * 1 = 1$
3. (BE 3) $1 * x = 1$
4. (BE 4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

Definition 2.2 ([7]): A system $(X; *, 1)$ consisting of a non–empty set X, a binary operation $*$ and a fixed element 1, is called a CI–algebra if the following conditions are satisfied:

1. (CI 1) $x * x = 1$
2. (CI 2) $1 * x = x$
3. (CI 3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$

In X, we can define a binary relation $\leq$ by $x \leq y$ iff $x * y = 1$.

Example 2.3: Let $X = R^+ = \{x \in R: x > o\}$
For $x, y \in X$, we define
$x * y = y \,.\, \frac{1}{x}$
Then $(X; *, 1)$ is a CI–algebra

Example 2.4: The simplest example of a BE–algebra and a CI –algebra are the following.

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Let X = {0, 1}. We consider binary operations $*$ and o given by the Cayley tables

$*$	0	1
0	1	1
1	0	1

Table-(2.4(a))

o	0	1
0	1	0
1	0	1

Table-(2.4(b))

Then (i) $(X; *, 1)$ is a BE–algebra,
(ii) $(X; o, 1)$ is a CI–algebra but not a BE–algebra.

In X, we can define a binary relation $\leq$ by $x \leq y$ iff $x * y = 1$.

Lemma 2.5 ([7]): In a CI–algebra $(X; *, 1)$ following results are true:
(1) $x * ((x * y) * y) = 1$
(2) $(x * y) * 1 = (x * 1) * (y * 1)$ for all $x, y \in X$

Definition 2.6 ([7]): Let $(X; *, 1)$ be a CI–algebra.
(a) A non-empty subset I of X is said to be an ideal of X if it satisfies the following conditions:
(i) $x \in X$ and $a \in I$ imply $x * a \in I$, i.e., $X * I \subseteq I$
(ii) $x \in X$ and $a \in I$, $b \in I$ imply $(a * (b * x)) * x \in I$
(b) A non-empty subset A of X is called a sub–algebra of X if $x \in A$ and $y \in A$ imply $x * y \in A$.
It is easy to see that X is a trivial ideal (resp. sub–algebra) of X.

Note 2.7: Taking $x = a$ in (i) we see that if I is an ideal in X then $1 \in I$.

Theorem 2.8 ([9]): Let $(X; *, 1)$ be a system consisting of a non-empty set X, a binary operation $*$ and a fixed element Let $Y = X \times X$. For $u = (x_1, x_2)$, $v = (y_1, y_2)$ a binary operation $\otimes$ is defined in Y as

$$u \otimes v = (x_1 * y_1, x_2 * y_2)$$

Then $(Y; \otimes, (1, 1))$ is a CI- algebra iff $(X; *, 1)$ is a CI-algebra .

Corollary 2.9 ([9]): If $(X; *, 1)$ and $(Y; o, e)$ are two CI–algebras, then $Z = X \times Y$ is also a CI–algebra under the binary operation defined as follows:

For $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in Z,

$$u \otimes v = (x_1 * x_2, y_1 \text{ o } y_2)$$

Here the distinct element of Z is (1, e).

Note 2.10: The above result can be extended for finite numbers of CI-algebras.

Theorem 2.11 ([7]): Let $(X; *, 1)$ be a BE-algebra and let $a \notin X$. A binary operation o is defined on $X \cup \{a\}$ as follows:
For any $x, y \in X \cup \{a\}$,

$$x \text{ o } y = \begin{cases} x * y & \text{if } x, y \in X \\ a & \text{if } x = a, y \neq a \\ a & \text{if } x \neq a, y = a \\ 1 & \text{if } x = y = a \end{cases}$$

Then $(X \cup \{a\}; o, 1)$ is a CI–algebra

3.1 HOMOMORPHISMS IN CI-ALGEBRAS

Definition 3.1 ([37]): Let $(X; *, 1)$ and $(Y; o, e)$ be CI-algebras and let $f: X \to Y$ be a mapping. Then f is said to be a homomorphism if

$$f(x * y) = f(x) \text{ o } f(y) \text{ for all } x, y \in X.$$

Proposition 3.2: Let $f: (X; *, 1) \to (Y; o, e)$ be a homomorphism. Then
(a) $f(1) = e$, and (b) $x \leq y \Rightarrow f(x) \leq f(y)$.

Proof: (a) Since $1 * 1 = 1$, we see that

$$f(1 * 1) = f(1) \Rightarrow f(1) \text{ o } f(1) = f(1)$$
$$\Rightarrow e = f(1).$$

(b) let $x \leq y$. Then $x * y = 1$. So

$$f(x * y) = f(1) = e$$
$$\Rightarrow f(x) \text{ o } f(y) = e \Rightarrow f(x) \leq f(y).$$

Example 3.3: Let $(X; *, 1)$ be a BE-algebra and let $(Y; o, e)$ be CI-algebra defined in theorem (2.11) where $Y=X\cup\{t\}$, $t \notin X$. Let f be a homomorphism defined on X. Let $f^1: Y \to Y$ be defined as

$$f^1(x) = f(x) \text{ if } x \in X \text{ and } f^1(t) = t.$$

Now for $x, y \in X$, $f^1(x \text{ o } y) = f(x * y) = f(x) * f(y) = f^1(x) * f^1(y)$
$= f^1(x) \text{ o } f^1(y)$.

For $x \in X$, we have $f^1(x \text{ o } t) = f^1(t) = t$,
and $f^1(x) \text{ o } f^1(t) = f^1(x) \text{ o } t = t$.

Also $f^1(t \text{ o } x) = f^1(t) = t$,
and $f^1(t) \text{ o } f^1(x) = t \text{ o } f^1(x) = t$.

So f^1 is a homomorphism.

Definition 3.4 ([37]): Let $f: (X; *, 1) \to (Y; o, e)$ be a homomorphism. Then the kernel of f, denoted as ker f, is defined as Ker $f = \{x \in X: f(x) = e\}$.

Proposition 3.5: Let $f: (X; *, 1) \to (Y; o, e)$ be a homomorphism. If $f(X) \subseteq B(Y)$ then ker f is an ideal of X.

Proof: Let $x \in X$ and $a \in \ker f$. Then

$$f(x * a) = f(x) \text{ o } f(a) = f(x) \text{ o } e = e,$$

Since $f(x) \in B(Y)$. So $x * a \in \ker f$.

Again let $a, b \in \ker f$ and $x \in X$.

$$\begin{aligned} \text{Then}\quad f((a * (b * x)) * x) &= (f(a) \text{ o } (f(b) \text{ o } f(x))) \text{ o } f(x) \\ &= (e \text{ o } (e \text{ o } f(x))) \text{ o } f(x) \\ &= (e \text{ o } f(x)) \text{ o } f(x) \\ &= f(x) \text{ o } f(x) = e. \end{aligned}$$

This implies that $(a * (b * x)) * x \in \ker f$.

Hence ker f is an ideal.

Definition 3.6: Let $f, g \in F(X)$. Then composite of f and g, denoted as $f \bullet g$, is defined as

$$(f \bullet g)(x) = f(g(x))$$

Proposition 3.7: Composition of two homomorphisms is a homomorphism.

Proof: Let f and g be homomorphisms in F(X). Then we have

$$\begin{aligned} (f \bullet g)(x * y) &= f(g(x * y)) \\ &= f(g(x) * g(y)) \\ &= f(g(x)) * f(g(y)) \\ &= (f \bullet g)(x) * (f \bullet g)(y). \text{ for all } x, y \in X. \end{aligned}$$

Hence $f \bullet g$ is a homomorphism.

Notation 3.8: Let $f: X \to X$ be a homomorphism and let

$$B_f = \{x \in X: f(x) = x\}.$$

Proposition 3.9: B_f is a subalgebra of X.

Proof: Since $f(1) = 1$, $1 \in B_f$ and B_f is non-empty. Let $a, b \in B_f$.
Then $f(a) = a$ and $f(b) = b$.

So $f(a * b) = f(a) * f(b) = a * b$
$\Rightarrow a * b \in B_f$.
Hence the result.

Now we discuss some special type of homomorphisms on CI– algebras.

Let $(X; *, 1)$ be a CI–algebra and let $Y = X^n$ be the Cartesian product of X with itself n times. Then theorem (2.8) implies that Y is a CI-algebra under the binary operation $\otimes$ and fixed element $1^n = (1, 1, \ldots, 1)$.

Definition 3.10: The mappings P_k and P_{ij} defined on X^n into itself as

$$P_k(x_{1,\ldots\ldots}, x_k, \ldots\ldots\ldots, x_n) = (1, 1, \ldots\ldots, x_k, \ldots\ldots, 1)$$
$$P_{ij}(x_{1,\ldots\ldots}, x_i, \ldots\ldots, x_j, \ldots\ldots, x_n) = (1, 1, \ldots\ldots, x_i, 1, \ldots\ldots, x_j, \ldots\ldots, 1)$$ are called dual projection maps.

Theorem 3.11: P_k and P_{ij} are homomorphism on X^n.

Proof: Let $x = (x_{1,\ldots\ldots\ldots}, x_n)$ and $y = (y_{1,\ldots\ldots\ldots}, y_n)$ be elements of X^n. Then

$$\begin{aligned} P_k(x \otimes y) &= P_k(x_1 * y_1, \ldots\ldots, x_k * y_k, \ldots\ldots\ldots, x_n * y_n) \\ &= (1, \ldots\ldots, x_k * y_k, \ldots\ldots, 1) \\ &= (1, \ldots\ldots, x_k, \ldots\ldots, 1) \otimes (1, \ldots\ldots, y_k, \ldots\ldots, 1) \\ &= P_k(x) \otimes P_k(y). \end{aligned}$$

This implies that P_k is a homomorphism.

Similarly it can be proved that P_{ij} is a homomorphism.

Definition 3.12: Let $(X; *, 1)$ be a CI–algebra and let $Y = X^n$. Then forward shift with replacement 1 and backward shift with replacement 1, denoted as (F S 1) and (B S 1) respectively, are defined as

$$(F\,S\,1)(x) = (1, x_1, x_2, \ldots\ldots, x_{n-1})$$
$$(B\,S\,1)(x) = (x_2, x_3, \ldots\ldots\ldots, x_n, 1) \text{ for all } x = (x_1, x_2, \ldots\ldots\ldots, x_n) \in Y.$$

Theorem 3.13: (F S 1) and (B S 1) are homomorphisms on Y.

Proof: Let $\mathcal{u}, \mathcal{v} \in Y$. Then $\mathcal{u} = (x_1, \ldots\ldots\ldots\ldots, x_n)$ and $\mathcal{v} = (y_1, \ldots\ldots\ldots\ldots, y_n)$.
We have

$$\begin{aligned} (F\,S\,1)(\mathcal{u} \otimes \mathcal{v}) &= (1, x_1 * y_1, \ldots\ldots\ldots, x_{n-1} * y_{n-1}) \\ &= (1, x_1, \ldots\ldots\ldots\ldots, x_{n-1}) \otimes (1, y_1, \ldots\ldots\ldots\ldots, y_{n-1}) \\ &= ((F\,S\,1)(\mathcal{u})) \otimes ((F\,S\,1)(\mathcal{v})). \end{aligned}$$

Also

$$\begin{aligned} (B\,S\,1)(\mathcal{u} \otimes \mathcal{v}) &= (x_2 * y_2, \ldots\ldots, x_n * y_n, 1) \\ &= (x_2, \ldots\ldots\ldots\ldots, x_n, 1) \otimes (y_2, \ldots\ldots\ldots\ldots, y_n, 1) \\ &= ((B\,S\,1)(\mathcal{u})) \otimes ((B\,S\,1)(\mathcal{v})). \end{aligned}$$

Hence (F S 1) and (B S 1) are homomorphisms.

Note 3.14: If X contains 0 then (F S 0) and (B S 0) on Y are not homomorphisms on Y, since $0 * 0 = 1 \neq 0$.

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