ON HOMOMORPHISMS IN CI-ALGEBRAS

PULAK SABHAPANDIT*
Department of Mathematics,
Biswanath College, Biswanath Charial, Assam, India.

KULAJIT PATHAK
Department of Mathematics, B. H. College, Howly, Assam, India.

(Received On: 20-01-18; Revised & Accepted On: 15-02-18)

ABSTRACT

In this paper we discuss homomorphism of CI-algebras, its examples and investigate some new properties. We consider some particular type of mappings defined on Cartesian product of CI–algebras.

Keywords: CI-algebra, BE-algebra, subalgebra, ideal, Cartesian product.

Mathematics Subject Classification: 06F35, 03G25, 08A30.

1. INTRODUCTION

In 1966, Y. Imai and K. Iseki ([2, 3]) introduced the notion of BCK/BCI-algebras. There exist several generalizations of BCK/BCI-algebras, such as BCH-algebras ([1]), BH-algebras ([4]), d-algebras ([8]), etc. As a dualization of a generalization of BCK-algebra ([5]), H.S. Kim and Y. H. Kim introduced the notion of BE-algebra ([6]). In 2010, B. L. Meng ([7]) introduced the notion of CI-algebras as a generalization of BE-algebras. The concept of Homomorphisms in CI-algebras was introduced by P.M.Sithar Selvam, T.Priya and T.Ramchandran ([10]). In this paper we discuss some special type of homomorphisms on CI-algebras and investigate some of its properties in details.

2. PRELIMINARIES

Definition 2.1 ([6]): A system $(X; \ast, 1)$ of type $(2, 0)$ consisting of a non-empty set $X$, a binary operation $\ast$ and a fixed element $1$ is called a BE–algebra if the following conditions are satisfied:

1. $(BE\ 1)\ x \ast x = 1$
2. $(BE\ 2)\ x \ast 1 = 1$
3. $(BE\ 3)\ 1 \ast x = 1$
4. $(BE\ 4)\ x \ast (y \ast z) = y \ast (x \ast z)$ for all $x, y, z \in X$.

Definition 2.2 ([7]): A system $(X; \ast, 1)$ consisting of a non–empty set $X$, a binary operation $\ast$ and a fixed element $1$, is called a CI–algebra if the following conditions are satisfied:

1. $(CI\ 1)\ x \ast x = 1$
2. $(CI\ 2)\ 1 \ast x = x$
3. $(CI\ 3)\ x \ast (y \ast z) = y \ast (x \ast z)$ for all $x, y, z \in X$

In $X$, we can define a binary relation $\leq$ by $x \leq y$ iff $x \ast y = 1$.

Example 2.3: Let $X = R^+ = \{x \in R: x > 0\}$

For $x, y \in X$, we define $x \ast y = y \cdot \frac{1}{x}$

Then $(X; \ast, 1)$ is a CI–algebra.

Example 2.4: The simplest example of a BE–algebra and a CI –algebra are the following.
Let $X = \{0, 1\}$. We consider binary operations $*$ and $\circ$ given by the Cayley tables

\[
\begin{array}{c|cc}
 & 0 & 1 \\
\hline
0 & 1 & 1 \\
1 & 0 & 1
\end{array}
\quad \begin{array}{c|cc}
 & 0 & 1 \\
\hline
0 & 1 & 0 \\
1 & 0 & 1
\end{array}
\]

Then (i) $(X; *, 1)$ is a BE–algebra,

(ii) $(X; \circ, 1)$ is a CI–algebra but not a BE–algebra.

In $X$, we can define a binary relation $\leq$ by $x \leq y$ iff $x * y = 1$.

Lemma 2.5 ([7]): In a CI–algebra $(X; *, 1)$ following results are true:

1. $x * ((x * y) * y) = 1$
2. $(x * y) * 1 = (x * 1) * (y * 1)$ for all $x, y \in X$

Definition 2.6 ([7]): Let $(X; *, 1)$ be a CI–algebra.

(a) A non-empty subset $I$ of $X$ is said to be an ideal of $X$ if it satisfies the following conditions:
   (i) $x \in X$ and $a \in I$ imply $x * a \in I$, i.e., $X * I \subseteq I$
   (ii) $x \in X$ and $a \in I, b \in I$ imply $(a * (b * x)) * x \in I$

(b) A non-empty subset $A$ of $X$ is called a sub–algebra of $X$ if $x \in A$ and $y \in A$ imply $x * y \in A$.

It is easy to see that $X$ is a trivial ideal (resp. sub–algebra) of $X$.

Note 2.7: Taking $x = a$ in (i) we see that if $I$ is an ideal in $X$ then $1 \in I$.

Theorem 2.8 ([9]): Let $(X; *, 1)$ be a system consisting of a non-empty set $X$, a binary operation $*$ and a fixed element $1$. Let $Y = X \times X$. For $u = (x_1, x_2), v = (y_1, y_2)$ a binary operation $\otimes$ is defined in $Y$ as

$u \otimes v = (x_1 * y_1, x_2 * y_2)$

Then $(Y; \otimes, (1, 1))$ is a CI–algebra iff $(X; *, 1)$ is a CI-algebra.

Corollary 2.9 ([9]): If $(X; *, 1)$ and $(Y; \circ, e)$ are two CI–algebras, then $Z = X \times Y$ is also a CI–algebra under the binary operation defined as follows:

For $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $Z$,

$u \circ v = (x_1 * x_2, y_1 \circ y_2)$

Here the distinct element of $Z$ is $(1, e)$.

Note 2.10: The above result can be extended for finite numbers of CI-algebras.

Theorem 2.11 ([7]): Let $(X; *, 1)$ be a BE-algebra and let $a \notin X$. A binary operation $\circ$ is defined on $X \cup \{a\}$ as follows:

For any $x, y \in X \cup \{a\}$,

$x \circ y = \begin{cases}x * y & \text{if } x, y \in X \\a & \text{if } x = a, y \neq a \\a & \text{if } x \neq a, y = a \\
1 & \text{if } x = y = a\end{cases}$

Then $(X \cup \{a\}; \circ, 1)$ is a CI–algebra.

3.1 HOMOMORPHISMS IN CI-ALGEBRAS

Definition 3.1 ([37]): Let $(X; *, 1)$ and $(Y; \circ, e)$ be CI-algebras and let $f : X \rightarrow Y$ be a mapping. Then $f$ is said to be a homomorphism if

$f(x * y) = f(x) \circ f(y)$ for all $x, y \in X$.

Proposition 3.2: Let $f : (X; *, 1) \rightarrow (Y; \circ, e)$ be a homomorphism. Then

(a) $f(1) = e$, and (b) $x \leq y \Rightarrow f(x) \leq f(y)$.

Proof: (a) Since $1 * 1 = 1$, we see that

$f(1 * 1) = f(1) \Rightarrow f(1) \circ f(1) = f(1)$

$\Rightarrow e = f(1)$.

(b) Let $x \leq y$. Then $x * y = 1$. So

$f(x * y) = f(1) = e$

$\Rightarrow f(x) \circ f(y) = e \Rightarrow f(x) \leq f(y)$. 

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Example 3.3: Let \((X; \ast, 1)\) be a BE-algebra and let \((Y; o, e)\) be CI-algebra defined in theorem (2.11) where \(Y=XU\{t\}\), \(t \notin X\). Let \(f\) be a homomorphism defined on \(X\). Let \(\tilde{f}^1: Y \to Y\) be defined as
\[
\tilde{f}^1(x) = f(x) \text{ if } x \in X \text{ and } \tilde{f}^1(t) = t.
\]
Now for \(x, y \in X\),
\[
\tilde{f}^1(x \circ y) = f(x \ast y) = f(x) \ast f(y) = \tilde{f}^1(x) \ast \tilde{f}^1(y).
\]
For \(x \in X\), we have \(\tilde{f}^1(x \circ t) = \tilde{f}^1(t) = t\),
and \(\tilde{f}^1(x) \circ \tilde{f}^1(t) = \tilde{f}^1(x) \circ t = t\).
Also \(\tilde{f}^1(t \circ x) = \tilde{f}^1(t) = t\),
and \(\tilde{f}^1(t) \circ \tilde{f}^1(x) = t \circ \tilde{f}^1(x) = t\).
So \(\tilde{f}^1\) is a homomorphism.

Definition 3.4 ([37]): Let \(f: (X; \ast, 1) \to (Y; o, e)\) be a homomorphism. Then the kernel of \(f\), denoted as \(\ker f\), is defined as \(\ker f = \{x \in X: f(x) = e\}\).

Proposition 3.5: Let \(f: (X; \ast, 1) \to (Y; o, e)\) be a homomorphism. If \(f(X) \subseteq B(Y)\) then \(\ker f\) is an ideal of \(X\).

Proof: Let \(x \in X\) and \(a \in \ker f\). Then
\[
f(x \ast a) = f(x) \circ f(a) = f(x) \circ e = e,
\]
Since \(f(x) \in B(Y)\). So \(x \ast a \in \ker f\).

Again let \(a, b \in \ker f\) and \(x \in X\).
Then
\[
(f((a \ast (b \ast x)) \ast x)) = (f((a) \circ ((b) \circ (f(x)))) \circ f(x))
\]
\[
= (e \circ (e \circ f(x))) \circ f(x)
\]
\[
= (e \circ f(x)) \circ f(x)
\]
\[
= f(x) \circ f(x) = e.
\]
This implies that \((a \ast (b \ast x)) \ast x \in \ker f\).
Hence \(\ker f\) is an ideal.

Definition 3.6: Let \(f, g \in F(X)\). Then composite of \(f\) and \(g\), denoted as \(f \circ g\), is defined as
\[
(f \circ g)(x) = f(g(x))
\]

Proposition 3.7: Composition of two homomorphisms is a homomorphism.

Proof: Let \(f\) and \(g\) be homomorphisms in \(F(X)\). Then we have
\[
(f \circ g)(x \ast y) = f(g(x \ast y)) = f(g(x) \ast g(y)) = f(g(x)) \ast f(g(y)) = (f \circ g)(x) \ast (f \circ g)(y).
\]
for all \(x, y \in X\).
Hence \(f \circ g\) is a homomorphism.

Notation 3.8: Let \(f: X \to X\) be a homomorphism and let
\[
B_f = \{x \in X: f(x) = x\}.
\]

Proposition 3.9: \(B_f\) is a subalgebra of \(X\).

Proof: Since \(f(1) = 1\), \(1 \in B_f\) and \(B_f\) is non-empty. Let \(a, b \in B_f\).
Then \(f(a) = a\) and \(f(b) = b\).
So \(f(a \ast b) = f(a) \ast f(b) = a \ast b\)
\[\Rightarrow a \ast b \in B_f.
\]
Hence the result.
Now we discuss some special type of homomorphisms on CI- algebras. Let \((X; \ast, 1)\) be a CI–algebra and let \(Y = X^n\) be the Cartesian product of \(X\) with itself \(n\) times. Then theorem (2.8) implies that \(Y\) is a CI-algebra under the binary operation \(\otimes\) and fixed element \(1^n = (1, 1, \ldots, 1)\).
Definition 3.10: The mappings $P_k$ and $P_{ij}$ defined on $X^n$ into itself as

\[
P_k(x_1, \ldots, x_k, \ldots, x_n) = (1, 1, \ldots, x_k, \ldots, 1)
\]

\[
P_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = (1, 1, \ldots, x_i, 1, \ldots, x_j, \ldots, 1)
\]

are called dual projection maps.

Theorem 3.11: $P_k$ and $P_{ij}$ are homomorphism on $X^n$.

Proof: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be elements of $X^n$. Then

\[
P_k(x \otimes y) = P_k(x_1 \ast y_1, \ldots, x_k \ast y_k, \ldots, x_n \ast y_n)
\]

\[
= (1, \ldots, x_k \ast y_k, \ldots, 1)
\]

\[
= (1, \ldots, x_k, \ldots, 1) \otimes (1, \ldots, y_k, \ldots, 1)
\]

\[
= P_k(x) \otimes P_k(y).
\]

This implies that $P_k$ is a homomorphism.

Similarly it can be proved that $P_{ij}$ is a homomorphism.

Definition 3.12: Let $(X; \ast, 1)$ be a CI–algebra and let $Y = X^n$. Then forward shift with replacement 1 and backward shift with replacement 1, denoted as $(F S 1)$ and $(B S 1)$ respectively, are defined as

$(F S 1)(x) = (1, x_1, x_2, \ldots, x_{n-1})$

$(B S 1)(x) = (x_2, x_3, \ldots, x_n, 1)$ for all $x = (x_1, x_2, \ldots, x_n) \in Y$.

Theorem 3.13: $(F S 1)$ and $(B S 1)$ are homomorphisms on $Y$.

Proof: Let $\mathbf{u}, \mathbf{v} \in Y$. Then $\mathbf{u} = (x_1, \ldots, x_n)$ and $\mathbf{v} = (y_1, \ldots, y_n)$. We have

\[
(F S 1)(\mathbf{u} \otimes \mathbf{v}) = (1, x_1 \ast y_1, \ldots, x_{n-1} \ast y_{n-1})
\]

\[
= (1, x_1, \ldots, x_{n-1}) \otimes (1, y_1, \ldots, y_{n-1})
\]

\[
= ((F S 1)(\mathbf{u}) \otimes ((F S 1)(\mathbf{v})).
\]

Also

\[
(B S 1)(\mathbf{u} \otimes \mathbf{v}) = (x_2 \ast y_2, \ldots, x_n \ast y_n, 1)
\]

\[
= (x_2, \ldots, x_n, 1) \otimes (y_2, \ldots, y_n, 1)
\]

\[
= ((B S 1)(\mathbf{u}) \otimes ((B S 1)(\mathbf{v})).
\]

Hence $(F S 1)$ and $(B S 1)$ are homomorphisms.

Note 3.14: If $X$ contains 0 then $(F S 0)$ and $(B S 0)$ on $Y$ are not homomorphisms on $Y$, since $0 \ast 0 = 1 \neq 0$.

REFERENCES


Source of support: Nil, Conflict of interest: None Declared.

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