GROUP ACTION ON FUZZY LEFT N-SUBGROUP OF A NEAR RING

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ABSTRACT


In this paper, the notion of Q-fuzzification of left N-subgroups is introduced in a near ring and investigated some related properties. Characterization of Q-fuzzy left N subgroup with respect to a triangular norm is given.

Keywords: Set action on a fuzzy set; group action on fuzzy left N-subgroup in a near-ring.

SECTION-1: INTRODUCTION


SECTION-2: BASIC DEFINITIONS AND PRELIMINARIES

Definition 2.1: A near ring is a non – empty set R with two binary operations + and . satisfying the following axioms: (1). (R, +) is a group; (2). (R, .) is a semigroup; (3). x. (y + z) = x. y +  x. z for all x, y, z, \ in R. Then it is called a left near – ring by (3). In this paper, it will use the word near- ring. Here xy denotes x.y; (2). x.0 = 0, and x(-y) = -(xy) for x, y in R.

Definition 2.2: A two sided R – subgroup of a near – ring R is a subset H of R such that (1). (H, +) is a subgroup of (R, +); (2). RH ⊆ H; (3). HR ⊆ H. If H satisfies (1) and (2), then it is a left R-subgroup of R, while if H satisfies (1) and (3), then it is a right R-subgroup of R.

Definition 2.3: A fuzzy set μ in a set R is a function μ: R → [0, 1].

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Definition 2.4: Let $G$ be any group. A mapping $\mu : G \to [0, 1]$ is a fuzzy group if (FG1). $\mu(xy) \geq \min \{\mu(x), \mu(y)\}$ and (FG2). $\mu(x^{-1}) = \mu(x)$, for all $x, y \in G$.

Definition 2.5: Let $(S, *)$ be a group, and $G$ be a non-empty set. Then $G$ acts on $S$ if there exists a function $*: G \times S \to S$ (denoted $*(g, s) = gs$ for all $g \in G$, and $s \in S$) such that $es = s$ and $(g + h) * s = g * (h * s)$ for all $s \in S$, and for all $g, h$ in $G$.

Definition 2.6: A map $f : R \to S$ is called homomorphism if $f(x + y) = f(x) + f(y)$ for all $x, y$ in $S$.

Definition 2.7: (T-norm) A triangular norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions for all $x, y, z$ in $[0, 1]$.

$(T1)$: $T(x, 1) = x$;

$(T2)$: $T(x, y) = T(y, x)$;

$(T3)$: $T(x, T(y, z)) = T(T(x, y), z)$;

$(T4)$: $T(x, y) \leq T(x, z)$ if $y \leq z$.

Definition 2.8: Let $(G, \Delta)$ a group acting on $R^\prime$-fuzzy group $B$ under a near-ring $(R^\prime, +, \cdot, \cdot)$, then the inverse image of $B$ under $\theta$ denoted by $\theta^{-1}(B)$ is fuzzy set in $(G, \Delta)$ defined by $\mu_{\theta^{-1}}(B) = \mu_{\theta}(\Delta(x))$; (ii) $(G, \Delta)$ be a group acting on $R$-fuzzy group $A$ under a near-ring $(R, +, \cdot)$.

Definition 2.9: Then the image of $A$ under $\theta$ denoted by $\theta(A)$, where $\mu_{\theta(A)}(y) = \{\sup \mu_{A(s)} : x \in \theta^{-1}(y) \neq \emptyset, 0\}$, otherwise. Then $\mu_{\theta(A)}(x * s) = \mu_{\theta}(x)(s)$ for all $s \in S$. Also $\mu_{\theta(A)}(y * s) = \{\sup \mu_{s}(x * s) : s \in \theta^{-1}(y) \neq \emptyset, 0\}$.

SECTION 3: GROUP ACTION ON FUZZY $N$-GROUP OF A NEAR-RING

Theorem 3.1: Let $T$ be a $t$-norm. Then every imaginable fuzzy left $N$-subgroup $\mu$ of a near ring $R$ acted by a group $(G, \Delta)$ is a fuzzy left $N$-subgroup of $R$ acted by $G$.

Proof: Assume $\mu$ is an imaginable fuzzy left $N$-subgroup of $R$. Then it gets that

$\mu(x * s) \leq T[\mu(x) * s, \mu(x * t)]$ and $\mu(x * s) \geq \mu(x * s)$ for all $s, t$ in $R$ and $x$ in $G$.

Since $\mu$ is imaginable, it follows that

$\min \{\mu(x * s), \mu(x * t)\} = T[\mu(x) * s, \mu(x * t)]$,

and so

$T[\min \{\mu(x * s), \mu(x * t)\}] \leq \min \{\mu(x * s), \mu(x * t)\}$.

It also finds that $\mu(x * s) \geq T[\min \{\mu(x) * s, \mu(x * t)\}] = \min \{\min \{\mu(x) * s, \mu(x * t)\}\}$ for all $s, t$ in $R$, and $x$ in $G$.

Hence $G$ acts on $\mu$ ‘$\mu$’ is a fuzzy left $N$-subgroup of $R$ acted by $G$.

Theorem 3.2: If $\mu$ is fuzzy left $N$-subgroup of a near ring $(R, +, \cdot)$ acted by a group $(G, \Delta)$, and $Q$ is an endomorphism of $R$, then $\mu_{Q(\cdot)}$ is a fuzzy left $N$-subgroup of $R$ acted by $G$.

Proof: $\mu$ is fuzzy left $N$-subgroup of a near ring $(R, +, \cdot)$ acted by a group $(G, \Delta)$.

Define $\mu_{Q(\cdot)} : R \to R$ by $\mu_{Q(\cdot)}(x * s) = \mu(Q(x * s))$ for all $x$ in $G$, and $s$ in $R$. Clearly $G$ acts also on fuzzy subgroup $\mu_{Q(\cdot)}$ of $R$.

For any $s, t$ in $R$, and $x$ in $G$, it gets that

(i) $\mu_{Q(\cdot)}(x * n(s - t)) = \mu(Q(x * ((n(s - t)))))$

$= \mu(Q(x * s), Q(x * t))$

$\geq T[\mu(Q(x * s), Q(x * t))]$

$= T[\mu_{Q(\cdot)}(x * s), \mu_{Q(\cdot)}(x * t)]$. 

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Proof: Let \( f: R \rightarrow R' \) be an onto homomorphism of near rings and \( A \) be a fuzzy left \( N \)-subgroup of \( R' \) acted by \( G \). Let \( B \) be the pre-image of \( A \) under \( f \).

Then (i). \( B((x_a) = f^{-1}(x') \cap (x_a) \), \( f(x' - y') = A12 \).

(ii). \( B((x_a) = f^{-1}(x') \cap (x_a) \), \( f(x' - y') = A12 \).

Thus \( B \) is a fuzzy left \( N \)-subgroup of \( R \) acted by \( G \).

Theorem 3.3: Onto homomorphism pre-image of a fuzzy left \( N \)-subgroup of near ring \( R' \) acted by a group \((G, \Delta)\), is fuzzy left \( N \)-subgroup of a near-ring \( R \) acted by \( G \).

Proof: Let \( f: R \rightarrow R' \) be an onto homomorphism of near rings and \( A \) be a fuzzy left \( N \)-subgroup of \( R' \) acted by \( G \). Let \( B \) be the pre-image of \( A \) under \( f \).

Then (i). \( B((x_a) = f^{-1}(x') \cap (x_a) \), \( f(x' - y') = A12 \).

(ii). \( B((x_a) = f^{-1}(x') \cap (x_a) \), \( f(x' - y') = A12 \).

Thus \( B \) is a fuzzy left \( N \)-subgroup of \( R \) acted by \( G \).

Theorem 3.3: Onto homomorphism pre-image under supreme property of a fuzzy left \( N \)-subgroup of near ring \( R' \) acted by a group \((G, \Delta)\), is fuzzy left \( N \)-subgroup of a near-ring \( R' \) acted by \( G \).

Proof: Let \( f: R \rightarrow R' \) be an onto homomorphism on near-rings. Let \( \mu \) be fuzzy \( N \)-subgroup of \( R' \).

By supreme property, let \( x, y \in R' \) and \( x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y') \), be such that

\[
\mu(x * x_0) = \sup_{\mu(x*h) \in f^{-1}(x')} \mu(x*h) \geq \mu(x * y_0) = \sup_{\mu(x* h) \in f^{-1}(y')} \mu(x*h) \text{ respectively.}
\]

Then we can deduce that

(1). \( f^{\mu}(x * (n(x' - y'))) = \sup_{\mu(x*z) \in f^{-1}(x * (n(x' - y')))} \mu(x*z) \geq T\{ \sup_{\mu(x* x_0)} \sup_{\mu(x* y_0)} \}

(2). \( f^{\mu}(x * nx') = \sup_{\mu(x* x_0)} \sup_{\mu(x* x_0)} \mu(x* x_0) \geq \mu(x * y') = \mu(x* x_0)

Hence \( f^{\mu} \) is a fuzzy left \( N \)-subset of \( R \) acted by \( G \).

Theorem 3.5: Let \( T \) be a continuous \( t \)-norm and \( f \) be a homomorphism on a near ring \( R \). If \( \mu \) is fuzzy left \( N \)-subgroup of \( S \) acted by a group \((G, \Delta)\), then \( f^{\mu} \) is a fuzzy left \( N \)-subgroup of \( f(R) \) acted by \( G \).

Proof: It follows that \( f^{\mu} \) is a fuzzy left \( N \)-subset \( f(R) \) acted by \( G \)

Let \( A_1 = f^{\mu}(x * y_1) \); \( A_2 = f^{\mu}(x * y_2) \);

Let \( A_{12} = f^{\mu}(x * (n(y_1, y_2))) \) where \( y_1, y_1 \in f(R) \), \( n \) in \( N \), and \( x \) in \( G \).

Consider the set \( A_1 - A_2 = \{ s \in R: (x * s) = (x * a_1) - (x * a_2) \} \) for some \( (x * a_1) \in A_1 \), and \( (x * a_2) \in A_2 \).

If \( f((x * s) \in A_1 - A_2 \), then \( (x * s) = (x * a_1) - (x * a_2) \) for some \( (x * a_1) \in A_1 \), and \( (x * a_2) \in A_2 \).

It gets that \( f((x * s) = f((x * a_1) - f((x * a_2) = y_1 - y_2 \)

Thus \( (x * s) \in f^{\mu}(x * y_1) - (x * y_2)) = f^{\mu}(x * n(y_1 - y_2)) = A_{12}. \)
Implies that $A_1 - A_2 \subseteq A_2$.

It follows that
\[
\mu^f(x * n(y_1 - y_2)) = \sup \{\mu(x * s) : (x * s) \in \mathcal{F}^{-1}((x * n(y_1 - y_2))
= \sup \{\mu(x * s) : (x * s) \in A_{12}\}
\geq \sup \{\mu(x * s) / (x * s) \in A_1 - A_2\}
\geq \sup \{\mu(x * x_1) - (x * x_2) : x * x_1 \in A_1 and x * x_2 \in A_2\}.
\]

Since $T$ is continuous, and every $\epsilon > 0$, there exists $\delta > 0$ such that
\[
\sup \{\mu(x * x_1) \mid (x * x_1) \in A_1\} - (x * x_2) \in A_2 \leq \delta and
\sup \{\mu(x * x_2) : x * x_1 \in A_2\} - (x * x_2) \in A_2 \leq \delta, then we get
\]
\[
T\{(\sup \{\mu(x * x_1) : (x * x_1) \in A_1\}, (x * x_2) \in A_2) - T\{(x * x_1), (x * x_2)\}\} \leq \epsilon.
\]

Choose $(x * a_1) \in A_1$, and $(x * a_2) \in A_2$ with $\sup \{\mu(x * x_1) : (x * x_1) \in A_1\} - \mu(x * a_1) \leq \delta$ then it finds that
\[
T\{(\sup \{\mu(x * x_1) : (x * x_1) \in A_1\}, (x * x_2) \in A_2) - T\{(x * x_1), (x * x_2)\}\} \leq \epsilon.
\]

It becomes now that
\[
\mu^f(x * n(y_1 - y_2)) \geq \sup\{T(\mu(x * x_1) \in A_1 and (x * x_2) \in A_2) : (x * x_1) \in A_1; (x * x_2) \in A_2\}
\geq \sup \{\mu(x * x_1) : (x * x_1) \in A_1\}, \sup \{\mu(x * x_2) : (x * x_2) \in A_2\}\}
\geq T(\mu^f((x * y_1), \mu^f(x * y_2))
\]

Similarly, it can show $\mu^f(nx, q) \geq \mu^f(x, q)$. Hence $G$ acts fuzzy left $N$-subgroup $\mu^f$ of $f(R)$.

**Theorem 3.6:** Let $\mu$ is fuzzy left $N$-subgroup of a near-ring $R$ acted by a group $(G, \Delta)$. Then the fuzzy set $< \mu >$ is a fuzzy left $N$-subgroup of $R$ generated by $\mu$. Also $G$ acts on $< \mu >$ as the smallest fuzzy left $N$-subgroup containing it.

**Proof:** Let $u, v \in R; \mu(x * u) = t_1; \mu(x * v) = t_2; \mu(x * n(u-v)) = t$.

Let it possible $t = < \mu > (x * n(u-v))$
\[
\leq T \{< \mu > (x * (nu)), < \mu > ((x * (nv))
\leq T \{< \mu > (x * u), < \mu > ((x * v))
= T\{t_1, t_2\} = t_1 \text{ (say)}.
\]

Then $t_1 = < \mu > (x * u) = \sup \{k : (x * u) \in < \mu_k >\} \geq t_1$

Therefore there exists $k$ with $(x * u) \in < \mu_k >$.

Also $t_2 = < \mu > (x * v) = \sup \{k : (x * v) \in < \mu_k >\} \geq t_2$

Therefore there exists $m > t$ with $(x * v) \in < \mu_m >$.

Without loss of generality, assume that $k, m$ with $< \mu_k > \subseteq < \mu_m >$.
hen $u, v \in < \mu_k >$, which is a contradiction since $m > t$. Therefore $t \geq t_1$.

Consequently,
\[
\mu^f(x * n(u - v)) \geq T \{< \mu > (x * u), \mu > (x * v)\}
\]

(1)

Now let, if possible $t_3 = \{< \mu > (x * (nu)) \leq < \mu > (x * u) = t_1$.
Then $t_1 = < \mu > (x * u) = \sup \{k : (x * u) \in < \mu_k >\} > t_3$

So there exists $k$ with $x * u \in < \mu_k >$, and $t_1 > k > t_3$ so that $n(x * u) \in < \mu_k > \subseteq < \mu_\epsilon >$, which is a contradiction. Hence $t_3 = < \mu > (x * (nu)) \geq < \mu > (x * u) = t_2$

(2)

The equations (1) and (2) yield that $G$ acts on fuzzy left $N$-subgroup $< \mu >$ of $R$.

Finally to show that $< \mu >$ is the smallest fuzzy left $N$-subgroup containing $\mu$ acted by $G$.

For this, assume that $G$ acts on fuzzy left $N$-subgroup $Q$ of $R$ such that $\mu \subseteq Q$, and show that $< \mu > \subseteq Q$. Let it possible, $t = < \mu > (x * u) \geq Q(x * u)$ for some $x$ in $G$, and $u$ in $R$.

Let $\epsilon > 0$ be given. Then $t = \mu = \sup \{k : (x * u) \in < \mu_k > and t - \epsilon \leq k < t\}$ implies that $(x * u) \in < \mu > \subseteq < \mu_{k - \epsilon} > for all \epsilon > 0.
Now \( a = a_1(x \ast x_1) + a_2(x \ast x_2) + a_3(x \ast x_3) + \ldots + a_n(x \ast x_n) \) where \( a_i \in \mathbb{N} \), and \( (x \ast x_i) \) belongs to \((t-\varepsilon)\). \( (x \ast x_i) \in \mu_{t-\varepsilon} \) implies that \( \mu(x \ast x_i) \geq t - \varepsilon \). Thus \( Q(x \ast u) \geq t - \varepsilon \) for all \( \varepsilon > 0 \).

So that \( Q(x \ast u) \geq T \{ Q(x \ast u_1), Q(x \ast u_2), \ldots, Q(x \ast u_n) \} \geq t - \varepsilon \) for all \( \varepsilon > 0 \).

Hence \( Q(x \ast u) = t \) which is a contradiction to our supposition.

**Theorem 3.7:** Let a group \((G, \Delta)\) acts a fuzzy left \( N \)-subgroup \( \mu \) of a near ring \( R \) and \( \mu^+ \) be a fuzzy set in \( R \) defined by \( \mu^+(x \ast u) = \mu(x \ast u) + 1 - \mu(x \ast 0) \) for all \( u \) in \( R \), and \( x \) in \( G \). Then \( G \) acts on a normal fuzzy \( N \)-subgroup \( \mu^+ \) of \( R \) containing \( \mu^+ \).

**Proof:** Let \( u, v \in R \), and \( x \in G \). We have

(i). \( \mu^+(x \ast (u-v)) = \mu(x \ast (u-v)) + 1 - \mu(x \ast 0) \)
\[ \geq \{ \mu(x \ast u) + 1 - \mu(x \ast 0) \} \]
\[ \geq T \{ \mu(x \ast u) + 1 - \mu(x \ast 0), \mu(x \ast v) + 1 - \mu(x \ast 0) \} \]

(ii). \( \mu^+(x \ast nu) = \mu(x \ast nu) + 1 - \mu(x \ast 0) \)
\[ \geq \mu(x \ast u) + 1 - \mu(x \ast 0) \]
\[ = \mu^+(x \ast u) \]

**REFERENCES**

1. Abou.Zoid, S., On Fuzzy sub near rings and ideals, Fuzzy sets and systems, 44,(1991),139-146.