ANTI-ININVARIANT SUBMANIFOLDS OF (€) –SASAKIAN MANIFOLD

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(Received On: 08-01-18; Revised & Accepted On: 14-02-18)

ABSTRACT

The object of the present paper is to study anti-invariant submanifolds of (€) - Sasakian manifold M. It is shown that if M is totally umbilical then M is totally geodesic. Also results have been obtained connecting totally geodesicty and anti-invariance of M. Also we find the necessary and sufficient condition for anti-invariant submanifolds of (€)-Sasakian manifold to be T-invariant and anti-invariant and condition for integrability of the distribution D.

AMS Subject Classification: 53C15, 53C20, 53C50.

Key Words: Anti-invariant submanifold, (€)-Sasakian manifold, T-invariant, Totally geodesic, Totally umbilical, Integrable condition.

INTRODUCTION

The index of a metric is very important in differential geometry as it gives rise to vector fields such as space-like, time like, and light-like fields. K.L.Duggal and A.Bejancu [9], introduced and studied (€)-Sasakian manifolds with the help of these vector fields and further such manifolds were investigated by Xufeng and Xiaoli [2], and others ([1], [3], [4]) and study of light like submanifolds is carried out by ([6], [17], [25]) because Sasakian manifolds with indefinite metrics play crucial role in Physics. The research work on the geometry of invariant submanifolds of contact and complex manifolds is carried out by M.Kon [29], in 1973, C.S.Bagewadi [27], in 1982, K.Yano and M.Kon [28], in 1984 and other authors ([7],[8][10][20]). Also the study of geometry of anti-invariant submanifolds is carried out by ([11], [13], [14], [15], [16], [21], [22], [23], [24], [26]) invarious contact manifolds. Motivated by the studies of the above authors, we study antiinvariant submanifolds of (€)-Sasakian manifold. The paper is organised as follows: the section 1 consists of preliminaries of (€)-Sasakian manifold, and section 2 contains the results as stated in abstract.

1. PRELIMINARIES

A (2n+1) -dimensional differentiable manifold M endowed with an almost contact structure (φ, ξ, η), where φ is a tensor field of type (1, 1), η is a 1-form and ξ is a vector field on M Satisfying

\[ \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \]
\[ \eta(\phi X) = 0, \quad \eta(\xi) = 0 \]  

is called an almost contact manifold. If there exists a semi-Riemannian metric g satisfying,

\[ g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y) \]  

then (φ, ξ, η, g) is called an (ε)-almost contact metric structure and M is known as (ε)- almost contact manifold for all X,Y ∈ T(M) where ε= ±1, For an (ε)-almost contact manifold.

We also have

\[ \eta(X) = \epsilon g(\xi, X) \]

for all X∈TM, ε=g(ξ, ξ). Hence ξ is never a light like vector field on M and we have two classes of (ε)-Sasakian manifolds. when ε=-1 and the index of g is odd then M is time like Sasakian manifold and M is a space like Sasakian manifold when ε=-1 and the index of g is even. For ε= 1 and index of g is zero we obtain usual Sasakian manifold and for ε = 1 and index of g is one then M is a Lorentz - Sasakian manifold. If dη(X, Y) = g(φX, Y) then M is said to have (ε)-contact metric structure (φ, η, ξ, g). If moreover this structure is normal then the (ε)-contact metric structure is called (ε)-Sasakian manifold and the manifold endowed with this structure is called an (ε)- Sasakian manifold. Also the (ε)-contact metric structure is an (ε)-Sasakian structure if and only if

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Let $M$ be a submanifold of $\mathbb{M}$. Let $T_x(M)$ and $T^\perp_x(M)$ denote the tangent and normal space of $M$ at $x\in M$ respectively. The Gauss and Weingarten formulas are given by
\[
\nabla_X Y = \nabla_X Y + \sigma(Y, X)
\]
\[
\nabla_X N = - A_N X + \nabla^\perp X N
\]
for any vector fields $X, Y$ tangent to $M$ and any vector field $N$ normal to $M$, where $\nabla$ and $\nabla^\perp$ are the operators of covariant differentiation on $M$ and $\mathbb{M}$ respectively. The Gauss and Weingarten formulas are given by
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\[
\nabla_X Y = \nabla_X Y + \sigma(Y, X)
\]
\[
\nabla_X N = - A_N X + \nabla^\perp X N
\]
Remark 2.1: If M is totally geodesic then \( (\xi(\varphi X))^\perp = \sigma(X, \xi) = 0, \) i.e \( C(\varphi X)^\perp = 0 \)

i.e \( (\varphi X)^\perp = 0, \) \( \varphi X \) is tangent to M and hence M is invariant submanifold of \((\xi)\)-Sasakian manifold. Therefore M will also be \((\xi)\)-Sasakian manifold.

Theorem 2.2: Let M be a submanifold of a \((\xi)\)-Sasakian manifold \( \overline{M} \) tangent to the structure vector field \( \xi \) of \( \overline{M} \) then \( \xi \) is parallel with respect to the induced connection on M if and only if M is anti-invariant submanifold in \( \overline{M} \).

Proof: Suppose the structure vector field \( \xi \) is tangent to M.

By Gauss formula

\[
-C(\varphi X) = \nabla_\xi X = \nabla X \xi + \sigma(X, \xi)
\]

Next suppose \( \xi \) is parallel w.r.t induced connection on M, then we have \( \nabla X \xi = 0 \) from equation (2.1) we have

\[
-C(\varphi X) = \sigma(X, \xi) \quad \text{i.e} \quad \varphi X = -C \sigma(X, \xi)
\]

Hence \( C(\varphi X) \) is normal to M, \( \varphi X \in T^\perp x(M) \) Thus M is anti-invariant.

Conversely: suppose M is anti-invariant, then by definition of anti-invariant if \( X \in T_x(M) \) Then \( \varphi X \in T^\perp x(M) \) so \( \varphi X = \sigma(X, \xi) \) for convenience we choose \( -C(\varphi X) = \sigma(X, \xi) \)

Hence from (2.1), We have \( \nabla X \xi = 0 \)

This shows that \( \xi \) is parallel w.r.to the induced connection on M.

Hence the theorem.

Theorem 2.3: Let M be a sub manifold of \((\xi)\)-Sasakian manifold \( \overline{M} \) If \( \xi \) is normal to M then M is totally geodesic if and only if M is anti-invariant submanifold.

Proof: Suppose \( \xi \) is normal to M then Weingarten formula implies

\[
[\overline{\nabla}_X \xi] = -A(\xi, X) + \overline{\nabla}^\perp X \xi
\]

Using (1.4) and (2.2) we have

\[
g(-C(\varphi X), Y) = g(\overline{\nabla}^\perp X \xi, Y) = g(-A(\xi, X) + \overline{\nabla}^\perp X \xi, Y) = -g(A(\xi, X), Y)
\]

for any X and Y tangent on M, that is,

\[
g(\varphi X, Y) = g(A(\xi, X), Y)
\]

(2.3)

Interchange X and Y in the above and adding and by virtue of (1.2) we have

\[
g(A(\xi, X) + A(\xi, Y), Y) = 0
\]

and

\[
g(\sigma(X, Y), \xi) = g(A(\xi, X), Y)
\]

and \( A(\xi) \) is symmetric we must have

\[
g(A(\xi, X), Y) = 0
\]

If \( M \) is totally geodesic, then \( \sigma(X, Y) = 0, \) i.e \( A(\xi)X = 0, \) then by (1.4) - \( C(\varphi X) = \overline{\nabla}^\perp X \xi \)

Hence M is anti-invariant

Conversely: suppose M is anti-invariant then \( C(\varphi X) \in T^\perp x(M) \) then from (2.2) we get

\[
-g(A(\xi, X), Y) = 0 \quad \text{i.e} \quad g(\sigma(X, Y), \xi) = 0
\]

i.e \( \sigma(X, Y) = 0 \)

Hence M is totally geodesic.

We have the following known result;

Proposition 2.1: [11] Let M be a submanifold tangent to the structure vector field \( \xi \) of a normal almost para contact metric manifold with constant \( c(c \neq 3) \) Then M is \( T \)-invariant if and only if M is invariant or anti-invariant.

On the basis of the above we can prove the following Theorem.

Theorem 2.4: Let M be a submanifold tangent to \( \xi \) the structure vector field of \((\xi)\)-Sasakian manifold \( \overline{M} \) with constant \( k (k \neq 3) \) then M is \( T \)-invariant if and only if M is invariant or anti-invariant.

Proof: Easily follows from the Proposition 2.1
Theorem 2.5: Let M be an anti-invariant submanifold tangent to ξ the structure vector field of \((\cdot)-\)Sasakian manifold \(\bar{M}\) with constant k. If \(A_NX = 0\) for any \(N \in T^\perp_x(M)\), then \(\varphi(T_x(M))\) is parallel w. r. t the normal connection.

Proof: To show that \(\varphi(T_x(M))\) is parallel w. r. t to the normal connection, \(V^\perp\) we have to show that for every local section \(\varphi Y \in \varphi(T_x(M))\) is also a local section in \(\varphi(T_x(M))\).

Using Gauss and Weingarten formula
\[
\begin{align*}
V^\perp_x \varphi Y &= \nabla_X \varphi Y + \sigma(X)\varphi Y + \xi g(\varphi X, Y)
\end{align*}
\]

By virtue of (1.5).

Since, \(A_NX = 0\) for any \(N \in T^\perp_x(M)\) we have
\[
\begin{align*}
g(\nabla_X \varphi Y, N) &= g(\varphi X, Y)g(\xi, N) - \xi g(\varphi X, Y) + g(\sigma(X)\varphi Y, N) + g(A_{\varphi Y} X, N)
\end{align*}
\]

Hence \(g(\nabla_X \varphi Y, N) = 0\)

Hence the result.

If \(D\) denotes the orthogonal subspace of \(T\-bar{M}\) to \(\xi\) then we can write \(T\-\bar{M} = D \oplus \{\xi\}\). We prove the following Theorem.

Theorem 2.6: Let \(M\) be a submanifold of an \((\cdot)-\)Sasakian manifold \(\bar{M}\) then \(M\) is anti-invariant if and only if \(D\) is integrable.

Proof: Let \(X,Y \in D\) then \(X,Y \in T\-\bar{M}\)
\[
g([X,Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)
\]

Using (1.4) we have
\[
\begin{align*}
g([X,Y], \xi) &= -g(\nabla_X Y, \xi) + g(\nabla_Y X, \xi) - g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)
\end{align*}
\]

Hence the theorem.

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