≈g_α\_CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we introduce a new class of sets called ≈g_α\_closed sets in topological spaces. We prove that this class lies between α\_closed sets and α\_g-closed sets and discuss some basic properties of ≈g_α\_closed sets.

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1. INTRODUCTION


2. PRELIMINARIES

Throughout this paper (X,τ) (or X) represents topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), cl(A), int(A) and A^c or X \ A denote the closure of A, the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition: 2.1 A subset A of a space (X,τ) is called

(i) Semi-open set [9] if A \subseteq cl(int(A));
(ii) preopen set [12] if A \subseteq int(cl(A));
(iii) α\_open set [13] if A \subseteq int(cl(int(A)));
(iv) semi-preopen set [1] if A \subseteq cl(int(cl(A))).

The complements of the above mentioned open sets are called their respective closed sets.

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The preclosure [14] (resp. semi-closure [6], $\alpha$-closure [13], semi-pre-closure [1]) of a subset $A$ of $X$, denoted by $pcl(A)$ (resp. $scl(A)$, $\alpha cl(A)$, $spcl(A)$), is defined to be the intersection of all preclosed (resp. semi-closed, $\alpha$-closed, semi-preclosed) sets of $(X, \tau)$ containing $A$. It is known that $pcl(A)$ (resp. $scl(A)$, $\alpha cl(A)$, $spcl(A)$) is a preclosed (resp. semi-closed, $\alpha$-closed, semi-preclosed) set.

**Definition: 2.2** A subset $A$ of a space $(X, \tau)$ is called

(i) a generalized closed (briefly g-closed) set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of g-closed set is called g-open set;

(ii) a semi-generalized closed (briefly sg-closed) set [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$. The complement of sg-closed set is called sg-open set;

(iii) a generalized semi-closed (briefly gs-closed) set [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of gs-closed set is called gs-open set;

(iv) an $\alpha$-generalized closed (briefly $\alpha$-g-closed) set [11] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of $\alpha$-g-closed set is called $\alpha$-g-open set;

(v) a generalized semi-preclosed (briefly gsp-closed) set [7] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of gsp-closed set is called gsp-open set;

(vi) a generalized preclosed (briefly gp-closed) set [14] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. The complement of gp-closed set is called gp-open set;

(vii) a $\tilde{g}$-closed set [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is sg-open in $(X, \tau)$. The complement of $\tilde{g}$-closed set is called $\tilde{g}$-open set;

(viii) a $A$-closed set [16] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tilde{g}$-open in $(X, \tau)$. The complement of $A$-closed set is called A-open set;

(ix) a $B$-closed set [16] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is A-open in $(X, \tau)$. The complement of $B$-closed set is called B-open set;

(x) a $\approx g$-closed set [16] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is B-open in $(X, \tau)$. The complement of $\approx g$-closed set is called $\approx g$-open set.

The collection of all $\approx g$-closed (resp. g-closed, gs-closed, gsp-closed, $\alpha$ g-closed, sg-closed, gp-closed, $\alpha$-closed, semi-closed) sets in $(X, \tau)$ is denoted by $\approx GC(X)$ (resp. $GC(X)$, $GS C(X)$, $GSP C(X)$, $\alpha GC(X)$, $SG C(X)$, $GP C(X)$, $\alpha C(X)$, $S C(X)$).

We denote the power set of $X$ by $P(X)$.

**Remark: 2.3** [16] For a topological space $(X, \tau)$, the following hold.

1. Every semi-open set is B-open but not conversely.
2. Every open set is B-open but not conversely.
3. Every open set is A-open but not conversely.
4. Every closed set is $\approx g$-closed but not conversely.

3. $\approx g$ - CLOSED SETS

We introduce the following definition.

**Definition: 3.1** A subset $A$ of a space $(X, \tau)$ is called an $\approx g$-closed set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is B-open in $(X, \tau)$.

The complement of $\approx g$-closed set is called $\approx g$-open set.

The collection of all $\approx g$-closed (resp. $\approx g$-open) sets in $(X, \tau)$ is denoted by $\approx G C(X)$ (resp. $G C(X)$, $G S C(X)$, $G P C(X)$, $\alpha G C(X)$, $S G C(X)$, $S P C(X)$).
Proposition: 3.2 Every $\alpha$-closed set is $\approx g_{\alpha}$-closed.

Proof: Let A be an $\alpha$-closed set and G be any B-open set containing A. Since A is $\alpha$-closed, we have $\alpha \text{cl}(A) = A \subseteq G$. Hence A is $\approx g_{\alpha}$-closed.

The converse of Proposition 3.2 need not be true as seen from the following example.

Example: 3.3 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a, b\}, X\}$. Then $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{a, b, c\}\}$, $\alpha C(X) = \{\emptyset, \{a, c\}\}$, and $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}\}$. We have $A = \{a, c\}$ is $\approx g_{\alpha}$-closed set but not $\alpha$-closed.

Proposition: 3.4 Every $\approx g_{\alpha}$-closed set is $\approx g_{\alpha}$-closed.

Proof: Let A be a $\approx g_{\alpha}$-closed set and G be any B-open set containing A. Since A is $\approx g$-closed, we have $G \supseteq \text{cl}(A) \supseteq \alpha \text{cl}(A)$. Hence A is $\approx g_{\alpha}$-closed.

The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{c\}, X\}$. Then $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We have $A = \{a\}$ is $\approx g_{\alpha}$-closed set but not $\approx g_{\alpha}$-closed.

Proposition: 3.6 Every $\approx g_{\alpha}$-closed set is $\approx g_{\alpha}$-closed.

Proof: Let A be an $\approx g_{\alpha}$-closed set and G be any open set containing A. Since any open set is B-open, we have $\approx \text{cl}(A) \subseteq \alpha \text{cl}(A) \subseteq G$. Hence A is $\approx g_{\alpha}$-closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{c\}, X\}$. Then $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We have $A = \{a\}$ is both $g_{\alpha}$-closed and $\approx g_{\alpha}$-closed set but not $\approx g_{\alpha}$-closed.

Proposition: 3.8 Every $\approx g_{\alpha}$-closed set is $\approx g_{\alpha}$-closed.

Proof: Let A be an $\approx g_{\alpha}$-closed set and G be any open set containing A. Since any open set is B-open, we have $\approx \text{cl}(A) \subseteq \alpha \text{cl}(A) \subseteq G$. Hence A is $\approx g_{\alpha}$-closed.

The converse of Proposition 3.8 need not be true as seen from the following example.

Example: 3.9 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{c\}, X\}$. Then $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\approx G_{\alpha} C(X) = \{\emptyset, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We have $A = \{a\}$ is both $g_{\alpha}$-closed and $\approx g_{\alpha}$-closed set but not $\approx g_{\alpha}$-closed.

Proposition: 3.10 Every $\approx g_{\alpha}$-closed set is $\approx g_{\alpha}$-closed.

Proof: Let A be an $\approx g_{\alpha}$-closed set and G be any open set containing A. Since any open set is B-open, we have $\approx \text{cl}(A) \subseteq \alpha \text{cl}(A) \subseteq G$. Hence A is $\approx g_{\alpha}$-closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11 Let X and $\tau$ be as in Example 3.5. Then $G_{\alpha} C(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We have $A = \{a\}$ is $g_{\alpha}$-closed set but not $\approx g_{\alpha}$-closed.
Proof: Let A be an $\approx g_\alpha$-closed set and G be any open set containing A. Since any open set is B-open, we have $pcl\ (A) \subseteq \alpha{cl(A)} \subseteq G$. Hence A is gp-closed.

The converse of Proposition 3.12 need not be true as seen from the following example.

Example: 3.13 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $G C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ and $GP C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. We have $A = \{a, b\}$ is gp-closed set but not $\approx g_\alpha$-closed.

Proposition: 3.14 Every $\approx g$-closed set is g-closed.

Proof: Let A be a $\approx g$-closed set and G be any open set containing A. Since any open set is B-open, we have $G \supseteq cl(A)$. Hence A is g-closed.

The converse of Proposition 3.14 need not be true as seen from the following example.

Example: 3.15 Let X and $\tau$ be as in Example 3.13. Then $G C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $GP C(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. We have $A = \{a, b\}$ is g-closed set but not $\approx g$-closed.

Remark: 3.16 The following examples show that $\approx g_\alpha$-closedness is independent of semi-closedness and g-closedness.

Example: 3.17 Let X and $\tau$ be as in Example 3.9. Then $SC(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. We have $A = \{b\}$ is semi-closed set but not $\approx g_\alpha$-closed.

Example: 3.18 Let X and $\tau$ be as in Example 3.3. Then $SC(X) = \{\phi, \{c\}, X\}$. We have $A = \{b, c\}$ is $\approx g_\alpha$-closed set but not semi-closed.

Example: 3.19 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $G C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ and $GP C(X) = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. We have (i) $A = \{b\}$ is $\approx g_\alpha$-closed set but not g-closed.
(ii) $B = \{a, c\}$ is g-closed set but not $\approx g_\alpha$-closed.

Remark: 3.20 From the above discussions and the known results in [7, 14, 16, 18], we obtain the following diagram, where $A \rightarrow B$ represents A implies B but not conversely.

None of the above implications is reversible as shown in the above examples and in the related papers [7, 14, 16, 18].

4. PROPERTIES OF $\approx g_\alpha$-CLOSED SETS

In this section, we discuss some basic properties of $\approx g_\alpha$-closed sets.

Definition: 4.1 [16] The intersection of all B-open subsets in $(X, \tau)$ containing A is called the B-kernel of A and denoted by $B-k(A)$.

Lemma: 4.2 A subset A of $(X, \tau)$ is $\approx g_\alpha$-closed if and only if $\alpha{cl(A)} \subseteq B-k(A)$. 
Proof: Suppose that \( A \) is \( g_\alpha \)-closed. Then \( \alpha cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( B \)-open. Let \( x \in \alpha cl(A) \). If \( x \notin B-ker(A) \), then there is an \( B \)-open set \( U \) containing \( A \) such that \( x \notin U \). Since \( U \) is an \( B \)-open set containing \( A \), we have \( x \notin \alpha cl(A) \) and this is a contradiction.

Conversely, let \( \alpha cl(A) \subseteq B-ker(A) \). If \( U \) is any \( B \)-open set containing \( A \), then \( \alpha cl(A) \subseteq B-ker(A) \subseteq U \). Therefore, \( A \) is \( g_\alpha \)-closed.

Proposition: 4.3 If \( A \) and \( B \) are \( g_\alpha \)-closed sets in \((X, \tau)\), then \( A \cup B \) is \( g_\alpha \)-closed in \((X, \tau)\).

Proof: If \( A \cup B \subseteq G \) and \( G \) is \( B \)-open, then \( A \subseteq G \) and \( B \subseteq G \). Since \( A \) and \( B \) are \( g_\alpha \)-closed, \( G \supseteq \alpha cl(A) \) and \( G \supseteq \alpha cl(B) \) and hence \( G \supseteq \alpha cl(A) \cup \alpha cl(B) = \alpha cl(A \cup B) \). Thus \( A \cup B \) is \( g_\alpha \)-closed in \((X, \tau)\).

Proposition: 4.4 If a set \( A \) is \( g_\alpha \)-closed in \((X, \tau)\) and \( A \subseteq B \subseteq \alpha cl(A) \), then \( B \) is \( g_\alpha \)-closed in \((X, \tau)\).

Proof: Let \( G \) be an \( B \)-open set in \((X, \tau)\) such that \( B \subseteq G \). Then \( A \subseteq G \). Since \( A \) is an \( g_\alpha \)-closed set, \( \alpha cl(A) \subseteq G \). Also \( \alpha cl(B) = \alpha cl(A) \subseteq G \). Hence \( B \) is also an \( g_\alpha \)-closed in \((X, \tau)\).

Proposition: 4.5 If \( A \) is \( B \)-open and \( \alpha cl(A) \subseteq \alpha cl(A) \), then \( A \) is \( \alpha \)-closed in \((X, \tau)\).

Proof: Since \( A \) is \( B \)-open and \( \alpha cl(A) \subseteq A \) and hence \( A \) is \( \alpha \)-closed in \((X, \tau)\).

Definition: 4.7 A subset \( A \) of a space \((X, \tau)\) is called \( \Lambda_B \)-set if \( A = B-ker(A) \).

Definition: 4.8 A subset \( A \) of a space \((X, \tau)\) is called \( \lambda_B \)-closed if \( A = L \cap F \) where \( L \) is a \( \Lambda_B \)-set and \( F \) is \( \alpha \)-closed.

The complement of \( \lambda_B \)-closed set is called \( \lambda_B \)-open set.

The collection of all \( \lambda_B \)-closed (resp. \( \lambda_B \)-open) sets in \((X, \tau)\) is denoted by \( \lambda_B C(X) \) (resp. \( \lambda_B O(X) \)).

Lemma: 4.9 For a subset \( A \) of a topological space \((X, \tau)\), the following conditions are equivalent.

(i) \( A \) is \( \lambda_B \)-closed.
(ii) \( A = L \cap cl(A) \) where \( L \) is a \( \lambda_B \)-set.
(iii) \( A = B-ker(A) \cap cl(A) \).

Lemma: 4.10

(i) Every \( \alpha \)-closed set is \( \lambda_B \)-closed but not conversely.
(ii) Every \( \lambda_B \)-set is \( \lambda_B \)-closed but not conversely.

The separate converses of Lemma 4.10 need not be true as seen from the following examples.

Example: 4.11 Let \( X \) and \( \tau \) be as in Example 3.13. Then \( \alpha C(X) = \{ \phi, \{ b \}, \{ c \}, \{ b, c \}, X \} \) and \( \lambda_B C(X) = P(X) \). We have \( A = \{ a \} \) is \( \lambda_B \)-closed set but not \( \alpha \)-closed.

Example: 4.12 Let \( X \) and \( \tau \) be as in Example 3.13. Then \( \Lambda_B \)-sets are \( \phi, \{ a \}, \{ a, b \}, \{ a, c \} \) and \( X \); and \( \lambda_B C(X) = P(X) \). We have \( A = \{ b \} \) is \( \lambda_B \)-closed set but not \( \lambda_B \)-set.

Theorem: 4.13 For a subset \( A \) of a topological space \((X, \tau)\), the following conditions are equivalent.
(i) A is $\alpha$-closed.
(ii) A is $\approx g_\alpha$-closed and $\lambda_B$-closed.

**Proof:** (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). Since A is $\approx g_\alpha$-closed, so by Lemma 4.2, $\alpha cl(A) \subseteq B ker(A)$. Since A is $\lambda_B$-closed, so by Lemma 4.9, $A = B ker(A) \cap \alpha cl(A) = \alpha cl(A)$. Hence A is $\alpha$-closed.

**Remark:** 4.14 The following examples show that the concepts of $\approx g_\alpha$-closed sets and $\lambda_B$-closed sets are independent of each other.

**Example:** 4.15 Let X and $\tau$ be as in Example 3.13. Then $\approx G_\alpha C(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $\lambda_B C(X) = P(X)$. We have $A = \{a\}$ is $\lambda_B$-closed set but not $\approx g_\alpha$-closed.

**Example:** 4.16 Let X and $\tau$ be as in Example 3.3. Then $\approx G_\alpha C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\lambda_B C(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, X\}$. We have $A = \{a\}$ is $\lambda_B$-closed set but not $\approx g_\alpha$-closed.

5. PROPERTIES OF $\approx g_\alpha lc^*$-SETS

**Definition:** 5.1 Let $(X, \tau)$ be a topological space. A subset A of X is called $\approx g_\alpha lc^*$-set if A = $M \cap N$ where M is B-open and N is $\lambda_B$-closed in $(X, \tau)$.

**Example:** 5.2 Let X and $\tau$ be as in Example 3.3. Then $\approx g_\alpha lc^*$-set.

**Proposition:** 5.3 Every B-open set is $\approx g_\alpha lc^*$-set but not conversely.

**Example:** 5.4 Let X and $\tau$ be as in Example 3.3. Then $\approx g_\alpha lc^*$-sets are $\emptyset, \{a\}, \{b\}, \{a, b\}$ and X; and B-open sets are $\emptyset, \{a\}, \{b\}, \{a, b\}$ and X. We have $A = \{c\}$ is $\approx g_\alpha lc^*$-set but not B-open.

**Proposition:** 5.5 Every $\alpha$-closed set is $\approx g_\alpha lc^*$-set but not conversely.

**Example:** 5.6 Let X and $\tau$ be as in Example 3.3. Then $\approx g_\alpha lc^*$-sets are $\emptyset, \{a\}, \{b\}, \{a, b\}$ and X; and $\alpha C(X) = \{\emptyset, \{c\}, X\}$. We have $A = \{a\}$ is $\approx g_\alpha lc^*$-set but not $\alpha$-closed.

**Theorem:** 5.7 Let $(X, \tau)$ be a topological space and A a subset of X. Then, A is $\alpha$-closed if and only if it is $\approx g_\alpha$-closed and $\approx g_\alpha lc^*$-set.

**Proof:** Let A be an $\alpha$-closed. By Propositions 3.2 and 5.5, A is $\approx g_\alpha$-closed and $\approx g_\alpha lc^*$-set.

Conversely, let A = M $\cap$ N. Then M is B-open and N is $\alpha$-closed. Since A is $\approx g_\alpha$-closed, $\alpha cl(A) \subseteq M$. Also $\alpha cl(A) \subseteq \alpha cl(N) = N$. We have $\alpha cl(A) \subseteq M \cap N = A$. Hence A is $\alpha$-closed.

**Remark:** 5.8 The following example shows that the concepts of $\approx g_\alpha$-closed sets and $\approx g_\alpha lc^*$-sets are independent of each other.

**Example:** 5.9 Let X and $\tau$ be as in Example 3.3. Then $\approx g_\alpha lc^*$-sets are $\emptyset, \{a\}, \{b\}, \{a, b\}$ and X; and $\approx G_\alpha C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. We have

(i) $A = \{a\}$ is $\approx g_\alpha lc^*$-set but not $\approx g_\alpha$-closed.

(ii) $A = \{a, c\}$ is $\approx g_\alpha$-closed set but not $\approx g_\alpha lc^*$-set.
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