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CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS

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ABSTRACT

Bose and Nair (1939) introduced a class of binary, equireplicate and proper designs, called 'Partially Balanced Incomplete Block Designs'. The Partially Balanced Incomplete Block Designs are available with smaller number of replications for many more numbers of treatments. This paper provides a new series of Partially Balanced Incomplete Block Designs.

Keywords: *Partially Balanced Incomplete Block Designs.*

1. INTRODUCTION

Balanced Incomplete Block Designs are not always suitable for varietal trials, since they require large number of replications and also that suitable designs are not available for all number of treatments. Bose and Nair (1939) introduced a class of binary, equireplicate and proper designs, called 'Partially Balanced Incomplete Block Designs'. The Partially Balanced Incomplete Block Designs are available with smaller number of replications for many more numbers of treatments. The numbers of replications of pair of treatments is not constant and are defined in general for m types of replications of different pairs of treatments where m is an integer.

The arrangement of 'v' treatments in 'b' blocks each of size 'k' (< v) and each treatment occurs in 'r' blocks such that n_1 pairs of treatments occurs in λ_1, n_2 pairs of treatments occurs in λ_2 times, … so on and n_m pairs of treatments occurs in λ_m times then the incomplete block design is said to be 'Partially Balanced Incomplete Block Design' (PBIBD) with m-associate classes The numbers v, b, r, k, $\lambda_1, \lambda_2, \dots, \lambda_m, n_1, n_2, \dots, n_m,$ P^i_{jk} (i, j, k = 1, 2, … m) are called the parameters of Partially Balanced Incomplete Block Design. Thus, there are 2m+4 parameters.

The parameters of m-associate class Partially Balanced Incomplete Block Design satisfy the following relations: vr = bk, $n_1 + n_2 + \dots + n_m = v-1$, $n_1\lambda_1 + n_2\lambda_2 + \dots + n_m\lambda_m = r(k-1)$, $\sum_{k=1}^{m} P^i_{jk} = n_j$ if $i \neq j$ and $\sum_{k=1}^{m} P^i_{jk} = n_j - 1$ if i = j; $P^i_{jk} = P^i_{kj}$, $n_i P^i_{jk} = n_j P^i_{jk} = n_k P^{\mathbf{k}}_{\mathbf{ij}}$. When number of associations is 2 then called 2- associate class Partially Balanced Incomplete Block Design.

2 . CONSTRUCTION OF NEW SERIES OF PBIBD'S

Two new series for the construction of two associate and m- associate class Partially Balanced Incomplete Block Designs.

Method 2.1: Let N be the incidence matrix of a Balanced Incomplete Block Design with parameters v, b, r, k, λ such that $v+\lambda \neq 2r$ and J be the matrix of unities. The combinatorial arrangement of the incidence matrix N is

$$N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix}$$

The resulting is a two associate class Partially Balanced Incomplete Block Design with incidence matrix as N′ with parameters $v' = 2v$, $b' = 2b$, $r' = b+r$, $k' = v+k$, $\lambda_1' = v+\lambda$, $\lambda_2' = 2r$.

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Theorem 2.1: A Partially Balanced Incomplete Block Design with parameters $v' = 2v$, $b'=2b$, $r' = b+r$, $k'=v+k$, $\lambda_1'= v+ \lambda$, $\lambda_2' = 2r$ can be constructed using the incidence matrix of a Partially Balanced Incomplete Block Design with parameters v, b, r, k, λ such that $v+\lambda \neq 2r$ with an arrangement of N and J as

$$N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix}$$

Proof: Let N_{vxb} be the incidence matrix of a Symmetric Balanced Incomplete Block Design with parameters v, b, r, k, λ such that $v+\lambda \neq 2r$. Let J be the matrix of unities of order vxb. It can be observed directly from the arrangement of N and J in N′ will provides an incidence matrix of a design with each treatment replicated b+r times and the block size is v+k. The 2v treatments can be partitioned into two groups each consisting of v treatments such that (i) any pair of treatments belonging to the same group occur together in $b+\lambda$ blocks, (ii) any pair of treatments belonging to different groups occur in 2r blocks. Therefore the resulting incidence matrix of a Partially Balanced Incomplete Block Design with parameters $v' = 2v$, $b'= 2b$, $r' = b+r$, $k'= v+k$, $\lambda_1'= v+ \lambda$, $\lambda_2' = 2r$.

The method is illustrated in the example 2.1

Example 2.1: Let N_{4x6} be the incidence matrix of Balanced Incomplete Block Design with parameters v = 4, b = 6, r = 3, k = 2, λ = 1.

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix};$$

The arrange the incidence matrices N and J in N′ as

$$N' = \begin{bmatrix} N & J \\ J & N \end{bmatrix} = \left[\begin{array}{cccccc|cccccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

The resulting design N′ is the incidence matrix of a SBIBD with parameters $v' = 8$, $b'= 12$, $r' = 9$, $k' = 6$, $\lambda_1' = 7$, $\lambda_2' = 6$.

Note: If $b+\lambda=2r$ then N′ represents the incidence matrix of BIBD with parameters of BIBD with parameters
$v' = 2v = b'$, $r' = b+r = k'$ $\lambda' = v+ \lambda$.

Method 2.2: Let N be the incidence matrix of a Balanced Incomplete Block Design with parameters v, b, r, k, λ. Let $\overline{N}$ be the dual design of N. Arrange the incidence matrix and its dual in the form

$$N' = \begin{bmatrix} N & \overline{N} \\ \overline{N} & N \end{bmatrix}$$

The resulting design is the incidence matrix of three associate class Partially Balanced Incomplete Block Design with parameters $v' = 2v$, $b' = 2b$, $r' = b_1$, $k' = v$, $\lambda_1 = b-2r+2\lambda$, $\lambda_2 = 2r-2\lambda$, $\lambda_3 = 0$, $n_1 = v-1$, $n_2 = v-1$, $n_3 = 1$.

Theorem 2.2: A three associate class Partially Balanced Incomplete Block Design with parameters $v' = 2v$, $b' = 2b$, $r' = b_1$, $k' = v$, $\lambda_1 = b-2r+2\lambda$, $\lambda_2 = 2r-2\lambda$, $\lambda_3 = 0$, $n_1 = v-1$, $n_2 = v-1$, $n_3 = 1$ can be constructed using the combinatorial arrangement of N and its dual in N′, where N is the incidence matrix of Balanced Incomplete Block Design.

$$N' = \begin{bmatrix} N & \overline{N} \\ \overline{N} & N \end{bmatrix}.$$

Proof: let N_{vxb} be the incidence matrix of a Balanced Incomplete Block Design with parameters v, b, r, k, λ, .then the incidence matrix contains: the no of 1's in the each row r, the no of 1's in the each column k, the number of 0's in the each row b-k, the number of 1's in the each column -k. Let $\overline{N}_{vxb}$ be the complementary matrix of N contains, no of 1's in each row are b-r, no of 1's in each column are b-k, no of 0's in each row are r, no of 0's in each column are k.

In the incidence matrix N of Balanced Incomplete Block Design, the number of pairs (0, 0), (0, 1), (1, 0), (1, 1) are occurring in any two rows are b-2r+λ, r-λ, r-λ, λ respectively. Then $\overline{N}$ contains the no. of pairs are λ, r-λ, r-λ, b-2r+λ. The arrangement incidence matrix N and its complement matrix $\overline{N}$ in the form

$$N'_{2vx2b} = \begin{bmatrix} N & \overline{N} \\ \overline{N} & N \end{bmatrix}$$

As a results it will contains 2v treatments, 2b blocks, each block size is v and each treatment replicated b times and λ_1= b-2r+2λ, λ_2 = 2r-2λ, λ_3 = 0.

The method is illustrated in the example 2.2.

Example 2.2: Consider a Balanced Incomplete Block Design with parameters v = 5, b = 10, r = 6, k = 3, λ = 3 whose incidence matrix is N. where N and $\overline{N}$ are

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } \overline{N} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The arrangement of incidence matrices results to

$$N' = \left[\begin{array}{ccccccccc|ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

The resulting design is the incidence matrix of a Partially Balanced Incomplete Block Design with parameters v′ = 10, b′ = 20, r′ = 10, k′ = 5, λ_1 = 4, λ_2 = 6, λ_3 = 0, n_1 = 4, n_2 = 4, n_3 = 1.

4. EFFICIENCY AND OPTIMALITY CRIREIA OF PBIBD

4.1 Efficiency of PBIBD: The pattern of NN′ matrix is, all its diagonal elements equal to 'r' and off-diagonal elements are either 'λ_i's. Let $B_0 = I_v$. Let us define the association matrices $B_i = ((b_{jk}^{(i)}))$ i= 1, 2, … m, where $b_{jk}^{(i)} = 1$ if j^{th} and k^{th} treatments are i^{th} associates and = 0 otherwise. We know that $B_iJ_{v,1} = n_i J_{v,1}$ for i = 1, 2, … m and $\sum_{i=0}^{m} B_i = JJ'$ and $B_0, B_1, B_2, \dots B_m$ are linearly independent. Since B_jB_k can be interpreted as the number of treatments common to the j^{th} associates of α and k^{th} associate of β. Therefore NN′ can be expressed as

$$NN' = rB_0 + \lambda_1B_1 + \lambda_2B_2 + \dots + \lambda_mB_m.$$

Then the determinant of $|NN'| = rk\ (r-\theta_1)^{\alpha 1} \ldots (r-\theta_m)^{\alpha m},\ \sum_{t=1}^{m} \alpha_t = v-1.$

In particular when m=2,

$$NN' = rB_0 + \lambda_1 B_1 + \lambda_2 B_2.$$

$$\Rightarrow \quad |NN'| = rk\ (r - \theta_1)^{\alpha 1}.\ (r - \theta_2)^{\alpha 2}, \tag{4.3.1}$$

$\therefore$ The efficiency factors for PBIBD are

$E_1 = 1 - \dfrac{r-\lambda_1}{rk}$ if i^{th} and j^{th} are 1^{st} associates and

$E_2 = \dfrac{mn^2\lambda_2\{\lambda_1 + (m-1)\lambda_2\}}{rk\{\lambda_1 + (mn-1)\lambda_2\}}$ if i^{th} and j^{th} are 2^{nd} associates

If $\lambda_1 < \lambda_2$ then the pair of treatments (α_i, α_j) are second associates occur more number of times than the pair of treatments (α_i, α_m) which are first associates then $E_2 > E_1$ otherwise $E_2 < E_1$.

The efficiency of a particular group of Group Divisible Partially Balanced Incomplete Block Designs efficiencies are evaluated and presented in Table 3.1.

Table-3.1: The Efficiencies of PBIBDs

	v	**r**	**k**	**b**	**m**	**n**	λ_1	λ_2	E_1	E_2	**E**
1	6	2	4	3	3	2	2	1	1.00	0.86	0.88
2	6	4	4	6	3	2	4	2	1.00	0.86	0.88
3	6	6	4	9	3	2	6	3	1.00	0.86	0.88
4	6	8	4	12	3	2	8	4	1.00	0.86	0.88
5	6	10	4	15	3	2	10	5	1.00	0.86	0.88
6	8	3	4	6	4	2	3	1	1.00	0.80	0.82
7	8	6	4	12	4	2	6	2	1.00	0.80	0.82
8	8	9	4	18	4	2	9	3	1.00	0.77	0.82
9	8	3	6	4	4	2	3	2	1.00	0.94	0.95
10	8	6	6	8	4	2	6	4	1.00	0.94	0.95
11	8	9	6	12	4	2	9	6	1.00	0.94	0.95
12	9	2	6	3	3	3	2	1	1.00	0.90	0.92
13	9	4	6	6	3	3	4	2	1.00	0.90	0.92
14	9	6	6	9	3	3	6	3	1.00	0.90	0.92
15	9	8	6	12	3	3	8	4	1.00	0.90	0.92
16	9	10	6	15	3	3	10	5	1.00	0.90	0.92
17	10	4	4	10	5	2	4	1	1.00	0.77	0.79
18	10	4	8	5	5	2	4	3	1.00	0.97	0.97
19	10	6	6	10	5	2	6	3	1.00	0.91	0.92
20	10	8	4	20	5	2	8	2	1.00	0.75	0.79
21	10	8	8	10	5	2	8	6	1.00	0.97	0.97
22	12	2	8	3	3	4	2	1	1.00	0.92	0.94
23	12	5	4	15	6	2	5	1	1.00	0.75	0.77
24	12	10	4	30	6	2	10	2	1.00	0.74	0.77
25	12	3	6	6	4	3	3	1	1.00	0.86	0.88
21	12	4	8	6	3	4	4	2	1.00	0.92	0.94
22	12	5	6	10	6	2	5	2	1.00	0.89	0.90
23	12	6	6	12	4	3	6	2	1.00	0.86	0.88
24	12	9	6	18	4	3	9	3	1.00	0.86	0.89
25	12	10	6	20	6	2	10	4	1.00	0.89	0.90
26	12	10	8	15	3	4	10	5	1.00	0.92	0.94
27	12	10	8	15	6	2	10	6	1.00	0.95	0.95

28	14	3	6	7	7	2	3	1	1.00	0.88	0.87
29	14	6	4	21	7	2	6	1	1.00	0.73	0.75
30	14	6	6	14	7	2	6	2	1.00	0.88	0.87
31	14	9	6	21	7	2	9	3	1.00	0.88	0.87
32	15	4	6	10	5	3	4	1	1.00	0.83	0.85
33	15	8	6	20	5	3	8	2	1.00	0.83	0.85
34	16	3	8	6	4	4	3	1	1.00	0.89	0.91
35	16	6	8	12	4	4	6	2	1.00	0.89	0.91
36	16	7	4	28	8	2	7	1	1.00	0.72	0.74
37	18	7	8	14	8	2	7	3	1.00	0.92	0.93
38	18	4	6	12	9	2	4	1	1.00	0.86	0.86
39	18	5	6	15	6	3	5	1	1.00	0.82	0.84
40	18	8	6	24	9	2	8	2	1.00	0.86	0.86
41	18	8	4	36	9	2	8	1	1.00	0.71	0.73
42	20	10	6	30	6	3	10	2	1.00	0.82	0.84
43	20	8	8	18	8	2	8	3	1.00	0.82	0.92
44	20	9	6	30	10	2	9	2	1.00	0.85	0.86
45	21	9	4	45	10	2	9	1	1.00	0.71	0.73
46	22	6	6	21	7	3	6	1	1.00	0.81	0.82
47	24	10	4	55	11	2	10	1	1.00	0.71	0.72
48	26	7	6	28	8	3	7	1	1.00	0.80	0.81
49	27	6	6	26	13	2	6	1	1.00	0.84	0.84

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