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ON THE BI-QUADRATIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS $2\left(x^4-y^4\right)=13\left(z^2-w^2\right)p^2$

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ABSTRACT

We obtain infinitely many non-zero integer quintuples (x, y, z, w, p) satisfying the homogenous bi-quadratic equation with five unknowns. Various interesting properties among the values of x, y, z, w and p are presented. Some relations between the solutions and special numbers are exhibited.

Key Words: *Integral solutions, Bi-quadratic equation with five unknowns, Special numbers.*

MSC Classification: *11D25.*

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- $\Pr_n$ - Pronic number of rank n.
- J_n - Jacobsthal number of rank n.
- G_n - Gnomonic number of rank n.
- S_r - Star number of rank n.

INTRODUCTION

There is a great interest for mathematicians since ambiguity in homogeneous and non-homogeneous Bi-Quadratic Diophantine Equations [1, 2, 4 and 16]. In this context, one may refer [3, 5-15] for varieties of problems on the bi-quadratic Diophantine equations with three, four and five variables. In this paper, bi-quadratic equation with five variables given by $2\left(x^4-y^4\right)=13\left(z^2-w^2\right)p^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special numbers are exhibited.

METHOD OF ANALYSIS

The diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$2\left(x^4-y^4\right)=13\left(z^2-w^2\right)p^2 \tag{1}$$

Introducing the transformations

$$x=u+v,\, y=u-v,\, z=4u+v,\, w=4u-v\,,\; u\neq v\neq 0 \tag{2}$$

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In (1), it leads to

$$u^2+v^2=13p^2 \tag{3}$$

(3) can be solved through different methods and we obtain different sets of integer solutions to (1).

Set-1:

Assume $p=a^2+b^2$ (4)

where a and b are non-zero distinct integers.

Write 13 as $13=(3+2i)(3-2i)$ (5)

Substituting (4) and (5) in (3) and applying the method of factorization, define

$$u+iv=(3+2i)(a+ib)^2$$

Equating the real and imaginary parts, we have

$$\begin{aligned} u&=3a^2-3b^2-4ab\\ v&=2a^2-2b^2+6ab \end{aligned} \tag{6}$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$x=5a^2-5b^2+2ab$$
$$y=a^2-b^2-10ab$$
$$z=14a^2-14b^2-10ab$$
$$w=10a^2-10b^2-22ab$$

Note-1:

Instead of (5), write 13 as

$$13=(2+3i)(2-3i)$$

Following the procedure as in set 1, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x=5a^2-5b^2-2ab$$
$$y=-a^2+b^2-10ab$$
$$z=11a^2-11b^2-20ab$$
$$w=5a^2-5b^2-28ab$$

Properties:

- $x(a,a+1)-5y(a,a+1)-52\,\mathrm{Pr}_a=0.$
- $11x(a,b)+w(a,b)=65*$difference of two squares.
- $w(a,a-1)-2x(a,a-1)+5S_a-4t_{4,a}\equiv 5 \pmod 4.$
- $x(1,2a-1)-y(1,2a-1)+z(1,2a-1)+12S_a-2G_a-4J_3=0.$
- $x(3a,a-1)-5y(3a,a-1)-26S_a+26=0.$

Set-2:

Rewrite 13 as $13=\dfrac{(18+i)(18-i)}{25}$ (7)

Substituting (4) and (7) in (3) and applying the method of factorization, define

$$u+iv=\frac{(18+i)}{5}(a+ib)^2$$

Equating the real and imaginary parts, we have

$$u=\frac{1}{5}\left\{18a^2-18b^2-2ab\right\}$$

$$v=\frac{1}{5}\left\{a^2-b^2+36ab\right\}$$

As our aim is to find integer solutions choosing $a=5A$, $b=5B$ in the above equations we get

$$\begin{aligned} u&=90A^2-90B^2-2AB \\ v&=5A^2-5B^2+180AB \end{aligned} \tag{8}$$

$$p=25\left(A^2+B^2\right) \tag{9}$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$x=95A^2-95B^2+170AB$$

$$y=85A^2-85B^2-190AB$$

$$z=365A^2-365B^2+140AB$$

$$w=355A^2-355B^2-220AB$$

$$p=25\left(A^2+B^2\right)$$

Note-2:
Equation (7) can also be written as

$$13=\frac{(1+18i)(1-18i)}{25}$$

Following the procedure as in set 2, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x=95A^2-95B^2-170AB$$

$$y=-85A^2+85B^2-190AB$$

$$z=110A^2-110B^2-710AB$$

$$w=-70A^2+70B^2-730AB$$

$$p=25\left(A^2+B^2\right)$$

Properties:

- $y(2,B)-x(2,B)-10\left\{t_{6,B}-t_{4,B}\right\}\equiv -40 \pmod{710}$.
- $x(A,A)+Z(A,A)-21t_{4,A}$ is a perfect square.
- $6\left\{y(A,A)+w(A,A)+446t_{4,A}\right\}$ is a nasty number.
- $z(2A-1,1)-w(2A-1,1)-10t_{4,2A-1}\equiv -370 \pmod{720}$.
- $z(A,A+1)-y(A,A+1)-330\,\mathrm{Pr}_A\equiv -280 \pmod{560}$.

Set-3:
One may write (3) as

$$u^2+v^2=13p^2*1 \tag{10}$$

Write 1 as

$$1=\frac{(4+3i)(4-3i)}{25} \tag{11}$$

Using (4) , (5) and (11) in (10) and applying the method of factorization, define

$$(u+iv)=(3+2i)\frac{(4+3i)}{5}(a+ib)^2 \tag{12}$$

Equating the real and imaginary parts of (12), we have

$$u=\frac{1}{5}\left\{6a^2-6b^2-34ab\right\}$$

$$v=\frac{1}{5}\left\{17a^2-17b^2+12ab\right\}$$

As our aim is to find integer solutions, choosing a=5A, b=5B in the above equations, we obtain

$$\left.\begin{aligned}u&=30A^2-30B^2-170AB\\ v&=85A^2-85B^2+60AB\end{aligned}\right\} \tag{13}$$

$$p=25\left(A^2+B^2\right) \tag{14}$$

In view of (2), the integer solutions of (1) are given by

$$x=115A^2-115B^2-110AB$$

$$y=-55A^2+55B^2-230AB$$

$$z=205A^2-205B^2-620AB$$

$$w=35A^2-35B^2-740AB$$

$$p=25\left(A^2+B^2\right)$$

Remark:
Instead of (2) , one may also introduce another set of transformations as

$$x=u+v,\ y=u-v,\ z=2u+2v,\ w=2u-2v\ ,\ (u\neq v\neq 0) \tag{15}$$

For this choice, the corresponding sets of distinct integer solutions to (1) are as represented below:

Set-4:
By substituting the equations (4) and (6) in (15) we obtain the integral solutions to (1) as given by

$$x=5a^2-5b^2+2ab$$

$$y=a^2-b^2-10ab$$

$$z=10a^2-10b^2+4ab$$

$$w=2a^2-2b^2-20ab$$

$$p=a^2+b^2$$

Set-5:
And also by substituting the equation (8) and (9) in (15), we get the integral solutions to (1) as given by

$$x=95A^2-95B^2+170AB$$

$$y=85A^2-85B^2-190AB$$

$$z=190A^2-190B^2+340AB$$

$$w=170A^2-170B^2-380AB$$

$$p=25\left(A^2+B^2\right)$$

Set-6:
By substituting the equation (13) and (14) in (15), we have the corresponding non-zero integral solutions to (1) as found to be

$$x=115A^2-115B^2-110AB$$

$$y=-55A^2+55B^2-230AB$$

$$z=230A^2-230B^2-220AB$$

$$w=-110A^2+110B^2-460AB$$
$$p=25\left(A^2+B^2\right)$$

Note-3:
It is worth to mention that, in (11), '1' maybe considered in general as

$$1=\frac{\left\{2mn+i\left(m^2-n^2\right)\right\}\left\{2mn-i\left(m^2-n^2\right)\right\}}{\left(m^2+n^2\right)^2}$$

(or)

$$1=\frac{\left\{\left(m^2-n^2\right)+i\,2mn\right\}\left\{\left(m^2-n^2\right)-i\,2mn\right\}}{\left(m^2+n^2\right)^2}$$

CONCLUSION

In this paper, we have made an attempt to find different patterns of non-zero distinct integer solutions to the bi-quadratic equation with five unknowns given by $2\left(x^4-y^4\right)=13\left(z^2-w^2\right)p^2$. As bi-quadratic equations are rich in variety, one may search for integer solutions to other choices of bi-quadratic equations with multivariates along with suitable properties.

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