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ON TRANS-SASAKIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

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ABSTRACT

The present paper deals with a study of trans-Sasakian manifolds satisfying certain curvature conditions on contact conformal curvature tensor. It is shown that the trans-Sasakian manifold satisfying the curvature conditions $C_0(\xi, X).R = 0,\ C_0(\xi, X).C_0 = 0$ *and* $\tilde{C}(\xi, X).C_0 = 0$ *is contact conformally semi-symmetric manifold.*

Keywords: *Trans-Sasakian manifold, contact conformal curvature tensor, semi-symmetric, concircular curvature tensor.*

MSC 2010: *53C15, 53C25, 53D15.*

1. INTRODUCTION

In [5], authors studied on the sixteen classes of almost Hermitian manifolds and their linear invariants. They considered unitary group $U(n)$ on a certain space W and studied that the representation of $U(n)$ on W has four irreducible components, $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$. Among these components $W_3 \oplus W_4$ corresponds to the class of Hermitian manifolds. A new class of almost contact metric structure, called trans-Sasakian structure was studied in [9] which is an analogue of a locally conformal Kaehler structure on an almost Hermitian manifold. An almost contact metric structure (φ, ξ, η, g) (where the symbols have their usual meanings) on M is trans-Sasakian [9] if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right) \tag{1.1}$$

for any vector field X on M, where G is the product metric on $M \times R.$ Trans-Sasakian manifold is the trans-Sasakian structure of type (α, β), where α and β are smooth functions on M. Trans-Sasakian manifold of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are cosympletic [1], α-Sasakian and β-Kenmotsu manifold [2,7] respectively. Trans-Sasakian manifolds have been studied in [3], [10] and by others.

In [6], contact conformal curvature tensor field was introduced and defined by Jeong *et al.* in a $(2n+1)$-dimensional Sasakian manifold.

2. PRELIMINARIES

Let M be a $(2n+1)$-dimensional differentiable manifold with an almost contact metric structure (φ, ξ, η, g), where φ is a (1, 1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that [1]

$$\varphi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$. The fundamental 2-form Φ of the almost contact metric structure (φ, ξ, η, g) is defined as

$$\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y), \tag{2.4}$$

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Since φ is a skew-symmetric with respect to g.

An almost contact metric manifold M is called trans-Sasakian manifold if [9]

$$(\nabla_X \varphi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X,Y)\xi - \eta(Y)\varphi X\}, \tag{2.5}$$

where ∇ is Levi-Civita connection of Riemannian metric g and α, β are smooth functions on M.

From (2.5) it follows that

$$\nabla_X \xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}, \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.7}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold M, the following relations hold [3]:

$$\begin{aligned} R(X,Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2(Y) \\ & + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2(X), \end{aligned} \tag{2.8}$$

$$\begin{aligned} R(\xi, X)Y = {} & (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X] + 2\alpha\beta[g(\varphi Y, X)\xi \\ & - \eta(Y)\varphi X] + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\alpha) \\ & + (Y\beta)[X - \eta(X)\xi] - g(\varphi X, \varphi Y)(grad\beta), \end{aligned} \tag{2.9}$$

$$2\alpha\beta + (\xi\alpha) = 0, \tag{2.10}$$

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\varphi X)\alpha) - (2n-1)(X\beta), \tag{2.11}$$

$$\eta(R(X,Y)Z) = -g(R(X,Y)\xi, Z), \tag{2.12}$$

$$\eta(R(X,Y)\xi) = \eta(R(\xi, X)\xi) = \eta(R(X,\xi)\xi = 0, \tag{2.13}$$

$$\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2 - (\xi\beta))g(\varphi X, \varphi Y). \tag{2.14}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold if we put $\varphi(grad\alpha) = (2n-1)grad\beta$, then some of the above relations are reduced to

$$(\xi\beta) = 0, \tag{2.15}$$

$$S(X,\xi) = 2n(\alpha^2 - \beta^2)\eta(X), \tag{2.16}$$

$$\eta(R(\xi, X)Y) = (\alpha^2 - \beta^2)g(\varphi X, \varphi Y), \tag{2.17}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)\{\eta(X)\xi - X\}, \tag{2.18}$$

$$R(X,\xi)\xi = -R(\xi, X)\xi. \tag{2.19}$$

We shall calculate all the results under the condition $\varphi(grad\ \alpha) = (2n-1)grad\beta$.

In a $(2n+1)$-dimensional trans-Sasakian manifold the contact conformal curvature tensor field C_0 and the concircular curvature tensor $\tilde{C}$ of type (1, 3) are defined by

$$\begin{aligned} C_0(X,Y)Z = {} & R(X,Y)Z + \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ & - g(X,Z)QY + S(X,Z)\eta(Y)\xi - S(Y,Z)\eta(X)\xi \\ & + \eta(X)\eta(Z)QY - \eta(Y)\eta(Z)QX + S(\phi X, Z)\phi Y \\ & - S(\phi Y, Z)\phi X + g(X, \phi Z)Q(\phi Y) - g(Y, \phi Z)Q(\phi X) \\ & + 2g(X, \phi Y)Q(\phi Z) + 2S(\phi X, Y)\phi Z\} \\ & + \frac{1}{2n22(n+1)}\{2n^2 - n - 2 + \frac{(n+2)r}{2n}\}\{g(Y, \phi Z)\phi X \\ & - g(X, \phi Z)\phi Y - 2g(X, \phi Y)\phi Z\} + \frac{1}{2n(n+1)}\{n + 2 - \frac{(3n+2)r}{2n}\} \\ & \times\{g(Y,Z)X - g(X,Z)Y\} - \frac{1}{2n(n+1)}\{4n^2 + 5n + 2 - \frac{(3n+2)r}{2n}\} \\ & \times\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\} \end{aligned} \tag{2.20}$$

where R, S, Q and r denote the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively [6] and

$$\tilde{C}(U,V)Z = R(U,V)Z - \frac{r}{2n(2n+1)}\{g(V,Z)U - g(U,Z)V\} \tag{2.21}$$

for all $X,Y,Z,U,V \in \chi(M)$ respectively [4].

From (2.20), we have

$$C_0(X,Y)\xi = R(X,Y)\xi + (\alpha^2 - \beta^2 - 2)\{\eta(Y)X - \eta(X)Y\}, \tag{2.22}$$

$$C_0(\xi,X)Y = R(\xi,X)Y + (\alpha^2 - \beta^2 - 2)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.23}$$

$$\eta(C_0(X,Y)Z) = \eta(R(X,Y)Z) + (\alpha^2 - \beta^2 - 2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.24}$$

$$\eta(C_0(X,Y)\xi) = 0, \tag{2.25}$$

$$\eta(C_0(\xi,X)Y) = 2(\alpha^2 - \beta^2 - 1)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{2.26}$$

From (2.21), we also have

$$\tilde{C}(\xi,V)Z = R(\xi,V)Z - \frac{r}{2n(2n+1)}\{g(V,Z)\xi - \eta(Z)V\}. \tag{2.27}$$

Definition: A $(2n+1)$-dimensional trans-Sasakian manifold M is said to be an Einstein manifold if its Ricci tensor S of type (0, 2) is of the form

$$S(X,Y) = \lambda g(X,Y) \tag{2.28}$$

where λ is a scalar function on M.

3. TRANS-SASAKIAN MANIFOLD SATISFYING $C_0(\xi, X).S = 0$

Now, we prove the following result

Theorem 3.1: If a trans-Sasakian manifold M of dimension $(2n+1)$ satisfies the condition $C_0(\xi, X).S = 0,$ then the manifold is an Einstein manifold and its scalar curvature is $r = 2n(2n+1)(\alpha^2 - \beta^2).$

Proof: Let M be a $(2n+1)$-dimensional trans-Sasakian manifold which satisfies the condition

$$(C_0(\xi, X).S)(U,V) = 0. \tag{3.1}$$

(3.1) implies that

$$S(C_0(\xi, X)U,V) + S(U, C_0(\xi, X)V) = 0. \tag{3.2}$$

Putting $V = \xi$ in (3.2) and using (2.16) and (2.23), we obtain

$$S(X,U) = 2n(\alpha^2 - \beta^2)g(X,U). \tag{3.3}$$

Let $\{e_i\}, i = 1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$ in (3.3) and summing over $i, 1 \le i \le 2n+1,$ we get

$$r = 2n(2n+1)(\alpha^2 - \beta^2). \tag{3.4}$$

In view of (3.3) and (3.4) it follows that the manifold M is an Einstein manifold with scalar curvature $r = 2n(2n+1)(\alpha^2 - \beta^2).$ This completes the proof of the theorem.

4. TRANS-SASAKIAN MANIFOLD SATISFYING $\tilde{C}(\xi, X).C_0 = 0$

Theorem 4.1: Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. If M satisfies the condition $\tilde{C}(\xi, X).C_0 = 0,$ then the manifold is contact conformally semi-symmetric.

Proof: By definition we have

$$\tilde{C}(\xi, X)C_0(U,V)Z - C_0(\tilde{C}(\xi, X)U,V)Z - C_0(U,\tilde{C}(\xi, X)V)Z - C_0(U,V)\tilde{C}(\xi, X)Z = 0 \tag{4.1}$$

Using (2.21) in (4.1) we get

$$R(\xi, X)C_0(U,V)Z - \frac{r}{2n(2n+1)}\{g(X, C_0(U,V)Z)\xi$$

$$-\eta(C_0(U,V)Z)X\} - C_0\left(R(\xi,X)U - \frac{r}{2n(2n+1)}\{g(X,U)\xi - \eta(U)X\}, V\right)Z$$

$$-C_0\left(U, R(\xi,X)V - \frac{r}{2n(2n+1)}\{g(X,V)\xi - \eta(V)X\}\right)Z$$

$$-C_0(U,V)\left(R(\xi,X)Z - \frac{r}{2n(2n+1)}\{g(X,Z)\xi - \eta(Z)X\}\right) = 0$$

$$[R(\xi, X)C_0(U,V)Z - C_0(R(\xi, X)U,V)Z - C_0(U, R(\xi, X)V)Z$$

$$-C_0(U,V)R(\xi, X)Z] + \frac{r}{2n(2n+1)}[\eta(C_0(U,V)Z)X$$

Or,

$$-g(X, C_0(U,V)Z)\xi + g(X,U)C_0(\xi,V)Z - \eta(U)C_0(X,V)Z$$

$$+g(X,V)C_0(U,\xi)Z - \eta(V)C_0(U,X)Z + g(X,Z)C_0(U,V)\xi$$

$$-\eta(Z)C_0(U,V)X] = 0$$

Or,

$$\left(R(\xi,X).C_0\right)(U,V)Z + \frac{r}{2n(2n+1)}[\eta(C_0(U,V)Z)X$$

$$-g(X, C_0(U,V)Z)\xi + g(X,U)C_0(\xi,V)Z - \eta(U)C_0(X,V)Z$$

$$+g(X,V)C_0(U,\xi)Z - \eta(V)C_0(U,X)Z + g(X,Z)C_0(U,V)\xi \tag{4.2}$$

$$-\eta(Z)C_0(U,V)X] = 0.$$

Putting $Z = \xi$ in (4.2) and taking inner product on both sides by ξ we get

$$g(R(\xi, X).C_0(U,V)\xi, \xi) + \frac{r}{2n(2n+1)}[-g(X, C_0(U,V)\xi) - \eta(C_0(U,V)X] = 0. \tag{4.3}$$

Using (2.22) and (2.24) in (4.3), we get

$$g(R(\xi, X).C_0(U,V)\xi, \xi) + \frac{r}{2n(2n+1)}[-g(X, R(U,V)\xi) - \eta(R(U,V)X)]. \tag{4.4}$$

In view of (2.8), (2.12) and (4.4) we obtain

$$R(\xi, X).C_0(U,V)\xi = 0. \tag{4.5}$$

Equation (4.5) implies that the manifold is contact conformally semi-symmetric. This completes the proof of the theorem.

5. TRANS-SASAKIAN MANIFOLD SATISFYING $C_0(\xi, X).C_0 = 0$

Theorem 5.1: Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. The manifold M satisfying the condition $C_0(\xi, X).C_0 = 0$ is contact conformally semi-symmetric if

$$(\alpha^2 - \beta^2)\{2g(V,Z)X - g(X,V)Z - g(X,Z)V\} - g(R(\xi,V)Z, X)\xi - R(X,V)Z = 0.$$

Proof: Let M be a trans-Sasakian manifold of dimension $(2n+1)$ which satisfies the condition $C_0(\xi, X).C_0(U,V)Z = 0$, then by definition we have

$$C_0(\xi, X)C_0(U,V)Z - C_0(C_0(\xi, X)U,V)Z - C_0(U, C_0(\xi, X)V)Z - C_0(U,V)C_0(\xi, X)Z = 0 \tag{5.1}$$

Using (2.23) in (5.1) we get

$$\begin{aligned} &R(\xi, X).C_0(U,V)Z + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U,V)Z)\xi \\ &- \eta(C_0(U,V)Z)X - g(X,U)C_0(\xi,V)Z + \eta(U)C_0(X,V)Z \\ &- g(X,V)C_0(U,\xi)Z + \eta(V)C_0(U,X)Z - g(X,Z)C_0(U,V)\xi \\ &+ \eta(Z)C_0(U,V)X] = 0. \end{aligned} \tag{5.2}$$

Taking inner product on both sides of (5.2) by ξ and using (2.25) we obtain

$$\begin{aligned} &g(R(\xi, X).C_0(U,V)Z,\xi) + (\alpha^2 - \beta^2 - 2)[g(X, C_0(U,V)Z) \\ &- \eta(X)\eta(C_0(U,V)Z) - g(X,U)\eta(C_0(\xi,V)Z) \\ &+ \eta(U)\eta(C_0(X,V)Z) - g(X,V)\eta(C_0(U,\xi)Z) \\ &+ \eta(V)\eta(C_0(U,X)Z) - \eta(Z)\eta(C_0(U,V)X)] = 0. \end{aligned} \tag{5.3}$$

Putting $U = \xi$ in (5.3) and using (2.23), (2.24) and (2.26), we get

$$\begin{aligned} &g(R(\xi, X).C_0(\xi,V)Z,\xi) + (\alpha^2 - \beta^2 - 2)[g(R(\xi,V)Z, X) \\ &+ \eta(R(X,V)Z) + (\alpha^2 - \beta^2)\{\eta(Z)g(X,V) - 2\eta(X)g(V,Z) \\ &+ \eta(V)g(X,Z)\}] = 0. \end{aligned} \tag{5.4}$$

Equation (5.4) implies that

$$\begin{aligned} R(\xi, X).C_0(\xi,V)Z = (\alpha^2 - \beta^2 - 2)[&(\alpha^2 - \beta^2)\{2g(V,Z)X \\ &- g(X,V)Z - g(X,Z)V\} - g(R(\xi,V)Z, X)\xi \\ &- R(X,V)Z]. \end{aligned} \tag{5.5}$$

From (5.5) it follows that the manifold M is contact conformally semi-symmetric if the right hand side vanishes i.e., if

$$(\alpha^2 - \beta^2)\{2g(V,Z)X - g(X,V)Z - g(X,Z)V\} - g(R(\xi,V)Z, X)\xi - R(X,V)Z = 0 \tag{5.6}$$

This completes the proof of the theorem.

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