

LEFT STAR AND RIGHT STAR PARTIAL ORDERING OF CONJUGATE UNITARY MATRICES

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ABSTRACT

The concept of Left star and Right star partial ordering of conjugate unitary matrices is introduced. Characterizations of Left star and Right star partial ordering of conjugate unitary matrices are obtained and derived some theorems.

Key Words: Unitary matrix, Conjugate normal matrix, conjugate unitary matrix and Left Star and Right Star partial ordering.

Mathematics Subject Classification: 15B10, 15B99.

INTRODUCTION

Two partial orderings in the set of complex matrices are introduced by combining each of the conditions $A^*A = A^*B$ and $AA^* = BA^*$, which we define the star partial ordering, with one of the conditions $\text{Range}(A) \subseteq \text{Range}(B)$ and $\text{Range}(A^*) \subseteq \text{Range}(B^*)$, defined by Baksalary J.K and Mitra S.K in [2].

Some characterizations of the star partial ordering and rank subtractivity for matrices was discussed by R.E.Hartwig and G.P.H.Styan in [6]. Also some characterizations of left star, right star and partial ordering between matrices of the same size are obtained Hongxing Wang and Jin Xu in [7].

Jorma.K.Merikoski and Xiaoji Liu found several characterizations of $A \leq^* B$, [8] in the case of normal matrices. A relationship between star and minus partial ordering was settled by J.K.Baksalary in [1]. In this paper we introduce the concept of left star and right star partial ordering of conjugate unitary matrices.

1.1 NOTATIONS

Let $C_{n \times n}$ be the space of $n \times n$ complex matrices of order n . For $A \in C_{n \times n}$. Let \bar{A} , A^* denote conjugate, conjugate transpose of a matrix A respectively. Also the permutation matrix V satisfies $V^T = \bar{V} = V^* = V$ and $V^2 = I$

A matrix $A \in C_{n \times n}$ is called unitary if $AA^* = A^*A = I$ [9]

A matrix $A \in C_{n \times n}$ is called normal if $AA^* = A^*A$ [5]

A matrix $A \in C_{n \times n}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [3]

A matrix $A \in C_{n \times n}$ is called is called conjugate unitary if $AA^* = \overline{A^*A} = I$ [4]

2. LEFT STAR AND RIGHT STAR PARTIAL ORDERING OF CONJUGATE UNITARY MATRICES

Definition 2.1: Star Partial Ordering:- For $A, B \in C_{n \times n}$, $A \leq_* B$ iff $A^*A = A^*B$ and $AA^* = BA^*$.

Definition 2.2: Left Star Partial Ordering: Let $A, B \in C_{n \times n}$, $A \leq^* B$ iff $A^*A = A^*B$ and $\text{Range}(A) \subseteq \text{Range}(B)$.

Definition 2.3: Right Star Partial Ordering:- Let $A, B \in C_{n \times n}$, $A \leq^* B$ if $AA^* = BA^*$ and $\text{Range}(A^*) \subseteq \text{Range}(B^*)$.

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Theorem 2.4: Let $AV * \leq VA$. If $A \in C_{n \times n}$ be conjugate unitary matrix and $VA^* = VA, AV = A^*V$ then \overline{VA} is involutory.

Proof:

Let $AV * \leq VA$.

$$\begin{aligned} AV * \leq VA \\ \Rightarrow (AV)^*(AV) &= (AV)^*(VA) \text{ and } \text{Range}(AV) \subseteq \text{Range}(VA) \\ \Rightarrow V^*A^*AV &= V^*A^*VA \\ \Rightarrow VA^*AV &= VA^*VA \quad (\because V^* = V) \\ \Rightarrow VA^*AV &= VA.VA \quad (\because VA^* = VA) \\ \Rightarrow VA^*AV &= (VA)^2 \end{aligned}$$

Taking conjugate on both sides, we have,

$$\begin{aligned} \overline{VA^*AV} &= \overline{(VA)^2} \\ \overline{V^2} &= \overline{(VA)^2} \quad (\because \overline{A^*A} = I) \\ \overline{I} &= \overline{(VA)^2} \quad (\because V^2 = I) \\ I &= \overline{(VA)^2} \quad (\because \overline{I} = I) \end{aligned}$$

i. e., $(\overline{VA})^2 = I$

Again let $AV * \leq VA$

$$\begin{aligned} AV * \leq VA \Rightarrow (AV)^*(AV) &= (AV)^*(VA) \text{ and } \text{Range}(AV) \subseteq \text{Range}(VA) \\ \Rightarrow V^*A^*AV &= V^*A^*VA \\ \Rightarrow VA^*AV &= VA^*VA \quad (\because V^* = V) \\ \Rightarrow (VA^*)(AV) &= (VA^*)(VA) \\ \Rightarrow VAA^*V &= VA.VA \quad (\because VA^* = VA, AV = A^*V) \\ \Rightarrow V(AA^*)V &= (VA)^2 \\ \Rightarrow V^2 &= (VA)^2 \quad (\because AA^* = I) \\ \Rightarrow I &= (VA)^2 \quad (\because V^2 = I) \end{aligned}$$

Taking conjugate on both sides, we have,

$$\begin{aligned} \overline{I} &= \overline{(VA)^2} \\ \Rightarrow I &= \overline{(VA)^2} \quad (\because \overline{I} = I) \\ \text{i. e., } \overline{(VA)^2} &= I \\ \Rightarrow \overline{VA} &\text{ is involutory.} \end{aligned}$$

Theorem 2.5: Let $VA * \leq AV$. If $A \in C_{n \times n}$ be conjugate unitary matrix and $A^*V = AV, VA^* = VA$ then \overline{VA} is involutory.

Proof: Similar proof of Theorem 2.4.

Theorem 2.6: Let $AV \leq^* VA$. If $A \in C_{n \times n}$ be conjugate unitary matrix and $A^*V = AV, VA^* = VA$, then (VA) is involutory.

Proof:

Let $AV \leq^* VA$.

$$\begin{aligned} AV \leq^* VA \\ \Rightarrow (AV)(AV)^* &= (VA)(VA)^* \text{ and } \text{Range}(AV)^* \subseteq \text{Range}(VA)^* \\ \Rightarrow AVV^*A^* &= VAV^*A^* \\ \Rightarrow AVVA^* &= VAVA^* \quad (\because V^* = V) \\ \Rightarrow AV^2A^* &= VAVA \quad (\because VA^* = VA) \\ \Rightarrow AA^* &= (VA)^2 \quad (\because V^2 = I) \\ \Rightarrow I &= (VA)^2 \quad (\because AA^* = I) \end{aligned}$$

ie., $(VA)^2 = I$

Again let $AV \leq^* VA$.

$$\begin{aligned} AV \leq^* VA \Rightarrow (AV)(AV)^* &= (VA)(VA)^* \text{ and } \text{Range}(AV)^* \subseteq \text{Range}(VA)^* \\ \Rightarrow AVV^*A^* &= VAV^*A^* \\ \Rightarrow AVVA^* &= VAVA^* \quad (\because V^* = V) \\ (A^*V)(VA) &= (VA)(VA) \quad (\because A^*V = AV, VA^* = VA) \\ \Rightarrow A^*A &= (VA)^2 \quad (\because V^2 = I) \end{aligned}$$

Taking conjugate on both sides, we have,

$$\overline{A^*A} = \overline{(VA)^2} \\ \Rightarrow I = \overline{(VA)^2} \quad (\because \overline{A^*A} = I)$$

Again taking conjugate on both sides, we have,

$$\overline{I} = \overline{(VA)^2} \\ \Rightarrow I = \overline{(VA)^2} \quad (\because \overline{I} = I) \\ \text{ie., } (VA)^2 = I \\ \Rightarrow (VA) \text{ is involutory.}$$

Theorem 2.7: Let $(VA) \leq^* (AV)$. If $AV \leq VA$. If $A \in C_{n \times n}$ be conjugate unitary matrix and $A^*V = AV$, $VA^* = VA$ then (AV) is involutory.

Proof: Similar proof of Theorem 2.6

Theorem 2.8: Let $A, B \in C_{n \times n}$ are conjugate unitary matrices and if $\text{Range}(A) = \text{Range}(A^*)$, $\text{Range}(B) = \text{Range}(B^*)$ then $A \leq B \Rightarrow A^{-1} \leq^* B^{-1}$

Proof: Let $A \leq B$ if $A^*A = A^*B$ and $\text{Range}(A) \subseteq \text{Range}(B)$
 \Rightarrow Pre and post multiply by V on both sides, we have,
 $VA^*AV = VA^*BV$

Taking conjugate on both sides, we have,

$$\overline{VA^*AV} = \overline{VA^*BV} \\ \Rightarrow \overline{VA^*A} \overline{V} = \overline{VA^*B} \overline{V} \\ \Rightarrow \overline{V} \overline{V} = \overline{VA^*B} \overline{V} \quad (\because \overline{A^*A} = I) \\ \Rightarrow \overline{V^2} = \overline{VA^*B} \overline{V} \\ \Rightarrow \overline{I} = \overline{VA^*B} \overline{V} \quad (\because V^2 = I)$$

Taking conjugate on both sides, we have,

$$I = VA^*BV \\ \Rightarrow \text{Pre and post multiplying by } V \text{ on both sides, we have,} \\ V^2 = A^*B \\ \Rightarrow I = A^*B \quad (\because V^2 = I) \\ \Rightarrow B^{-1} = A^* \\ \Rightarrow B^{-1} = A^{-1} \quad (\because AA^* = \overline{A^*A} = I \Rightarrow A^* = A^{-1})$$

Post multiplying by $(A^{-1})^*$ on both sides, we have,

$$(B^{-1})(A^{-1})^* = (A^{-1})(A^{-1})^* \\ \text{ie., } (A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^* \quad (1)$$

Given $\text{Range}(A) = \text{Range}(A^*)$, $\text{Range}(B) = \text{Range}(B^*)$

$$\Rightarrow \text{Range}(A^*) \subseteq \text{Range}(B^*) \quad (2)$$

Again let $A \leq B$ if $A^*A = A^*B$ and $\text{Range}(A) \subseteq \text{Range}(B)$

$$A^*A = A^*B$$

Pre multiplying by $(A^*)^{-1}$ on both sides, we have,

$$(A^*)^{-1}A^*A = (A^*)^{-1}A^*B \\ \Rightarrow A = B$$

Post multiplying by A^* on both sides, we have,

$$AA^* = BA^* \\ \Rightarrow I = BA^* \quad (\because AA^* = I) \\ \Rightarrow B^{-1} = A^* \\ \Rightarrow B^{-1} = A^{-1} \quad (\because AA^* = \overline{A^*A} = I \Rightarrow A^* = A^{-1})$$

Post multiplying $(A^{-1})^*$ by on both sides, we have,

$$(B^{-1})(A^{-1})^* = (A^{-1})(A^{-1})^* \\ \text{ie., } (A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^* \quad (3)$$

$$\begin{aligned} \text{Given } \text{Range}(A) &= \text{Range}(A^*), \text{Range}(B) = \text{Range}(B^*) \\ \Rightarrow \text{Range}(A^*) &\subseteq \text{Range}(B^*) \end{aligned} \quad (4)$$

Therefore from equations, (1), (2), (3) & (4), we have,
 $A \leq B \Rightarrow A^{-1} \leq B^{-1}$

Theorem 2.9: If $A, B \in C_{n \times n}$ are conjugate unitary matrices and $\text{Range}(A) = \text{Range}(A^*)$, $\text{Range}(B) = \text{Range}(B^*)$ then
 $A \leq B \Rightarrow A^{-1} \leq B^{-1}$

Proof: Similar proof of Theorem 2.8

Theorem 2.10: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, then $A \leq B$, $B \leq C$
 $\Rightarrow A \leq C$

Proof:

$$\begin{aligned} A \leq B &\Rightarrow A^*A = A^*B \text{ and } \text{Range}(A) \subseteq \text{Range}(B) \\ B \leq C &\Rightarrow B^*B = B^*C \text{ and } \text{Range}(B) \subseteq \text{Range}(C) \\ B^* \leq C &\Rightarrow B^*B = B^*C \\ &\Rightarrow B^*AA^*B = B^*C \quad (\because AA^* = I) \\ &\Rightarrow (B^*A)(A^*A) = (B^*C) \quad (\because A^*B = A^*A) \\ &\Rightarrow B^*AA^*A = B^*C \end{aligned}$$

$$\begin{aligned} \text{Pre multiplying by } (B^*)^{-1} \text{ on both sides, we have,} \\ (B^*)^{-1}B^*AA^*A &= (B^*)^{-1}B^*C \\ \Rightarrow AA^*A &= C \end{aligned}$$

$$\begin{aligned} \text{Taking conjugate on both side, we have,} \\ \overline{A} \overline{A^*A} &= \overline{C} \\ \Rightarrow \overline{A} &= \overline{C} \quad (\because \overline{A^*A} = I) \end{aligned}$$

$$\begin{aligned} \text{Again taking conjugate, on both side, we have,} \\ A &= C \end{aligned}$$

$$\begin{aligned} \text{Pre multiplying by } A^* \text{ on both sides we have,} \\ A^*A &= A^*C \end{aligned} \quad (1)$$

$$\begin{aligned} \text{Given } \text{Range}(A) &\subseteq \text{Range}(B) \text{ and } \text{Range}(B) \subseteq \text{Range}(C) \\ \Rightarrow \text{Range}(A) &\subseteq \text{Range}(C) \end{aligned} \quad (2)$$

$$\begin{aligned} \text{Again let } B^* \leq C &\Rightarrow B^*B = B^*C \text{ and } \text{Range}(B) \subseteq \text{Range}(C) \\ B^* \leq C &\Rightarrow B^*B = B^*C \\ &\Rightarrow B^*AA^*B = B^*C \quad (\because AA^* = I) \\ &\Rightarrow (B^*A)(A^*A) = (B^*C) \quad (\because A^*B = A^*A) \\ &\Rightarrow B^*AA^*A = B^*C \\ &\Rightarrow B^*A = B^*C \quad (\because AA^* = I) \end{aligned}$$

$$\begin{aligned} \text{Pre multiplying by } (B^*)^{-1} \text{ on both sides, we have,} \\ A &= C \end{aligned}$$

$$\begin{aligned} \text{Pre multiplying by } A^* \text{ on both sides we have,} \\ A^*A &= A^*C \end{aligned} \quad (3)$$

$$\begin{aligned} \text{Given } \text{Range}(A) &\subseteq \text{Range}(B) \text{ and } \text{Range}(B) \subseteq \text{Range}(C) \\ \Rightarrow \text{Range}(A) &\subseteq \text{Range}(C) \end{aligned} \quad (4)$$

$$\begin{aligned} \text{Therefore from eqns. (1), (2), (3) \& (4), we have,} \\ A \leq C \end{aligned}$$

$$\text{That is } A \leq B, B \leq C \Rightarrow A \leq C$$

Theorem 2.11: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, then $A \leq B$, $B \leq C$
 $\Rightarrow A \leq C$

Proof: Similar proof of Theorem 2.10

Theorem 2.12: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, then $A \leq^* B \Rightarrow A^* \leq B^*$

Proof:

$$A \leq^* B \Rightarrow AA^* = BA^* \text{ and } \text{Range}(A^*) \subseteq \text{Range}(B^*)$$

$$AA^* = BA^*$$

Post multiplying by A on both sides, we have,

$$AA^*A = BA^*A$$

$$\Rightarrow A = BA^*A \quad (\because AA^* = I)$$

Taking conjugate on both sides, we have,

$$\Rightarrow \bar{A} = \bar{B} \overline{A^*A}$$

$$\Rightarrow \bar{A} = \bar{B} \quad (\because \overline{A^*A} = I)$$

Again taking conjugate on both sides, we have,

$$\Rightarrow A = B$$

Pre multiplying by B^* on both sides, we have,

$$\Rightarrow B^*A = B^*B$$

Taking conjugate on both sides, we have,

$$\Rightarrow \overline{B^*A} = I \quad (\because \overline{B^*B} = I)$$

Again taking conjugate on both sides, we have,

$$\Rightarrow B^*A = \bar{I}$$

$$\Rightarrow B^*A = I \quad (\because \bar{I} = I)$$

Post multiplying by A^* on both sides, we have,

$$\Rightarrow B^*AA^* = A^*$$

$$\Rightarrow B^* = A^* \quad (\because AA^* = I)$$

Pre multiplying by B on both sides, we have,

$$\Rightarrow BB^* = BA^*$$

$$\Rightarrow I = BA^* \quad (\because BB^* = I)$$

$$\Rightarrow AA^* = BA^* \quad (\because AA^* = I)$$

Taking conjugate transpose on both sides, we have,

$$\Rightarrow (AA^*)^* = (BA^*)^* \tag{1}$$

$$\Rightarrow (A^*)^*A^* = (A^*)^*B^* \tag{2}$$

$$\Rightarrow \text{Range}(A^*) \subseteq \text{Range}(B^*)$$

Therefore from eqns. (1) & (2), we have,

$$A \leq^* B \Rightarrow A^* \leq B^*$$

Theorem 2.13: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, then $A \leq B \Rightarrow A^* \leq^* B^*$

Proof: Similar proof of Theorem 2.12

Theorem 2.14: If $A, B \in C_{n \times n}$ are conjugate unitary matrices and $\text{Range}(A) \subseteq \text{Range}(A^*A)$, $\text{Range}(B) \subseteq \text{Range}(B^*B)$ then $A \leq B \Rightarrow A^*A \leq^* B^*B$

Proof: Let $A \leq B \Rightarrow A^*A = A^*B$ and $\text{Range}(A) \subseteq \text{Range}(B)$

$$A^*A = A^*B$$

Taking conjugate transpose on both sides, we have,

$$(A^*A)^* = (A^*B)^*$$

$$\Rightarrow A^*A = B^*A$$

$$\Rightarrow A^*AA^*A = B^*BB^*A \quad (\because AA^* = I, BB^* = I)$$

$$\Rightarrow (A^*A)(A^*A) = (B^*B)(B^*A)$$

Taking conjugate transpose on both sides, we have,

$$\begin{aligned}
 &\Rightarrow ((A^*A)(A^*A))^* = ((B^*B)(B^*A))^* \\
 &\Rightarrow (A^*A)^*(A^*A)^* = (B^*A)^*(B^*B)^* \\
 &\Rightarrow (A^*A)^*(A^*A)^* = (A^*B)(B^*B) \\
 &\Rightarrow (A^*A)^*(A^*A) = (A^*A)(B^*B) \quad (\because (A^*A)^* = (A^*A)) \\
 &\Rightarrow (A^*A)^*(A^*A) = (A^*A)^*(B^*B) \quad (\because (A^*A)^* = (A^*A))
 \end{aligned} \tag{1}$$

Again taking $A^*A = A^*B$

Taking conjugate on both sides, we have,

$$\begin{aligned}
 &\overline{A^*A} = \overline{A^*B} \\
 &\overline{I A^*A} = \overline{I A^*B} \\
 &\Rightarrow \overline{A^*A A^*A} = \overline{B^*B A^*B} \quad (\because \overline{A^*A} = I, \overline{B^*B} = I)
 \end{aligned}$$

Again taking conjugate on both sides, we have,

$$\begin{aligned}
 &\Rightarrow (A^*A)(A^*A) = (B^*B)(A^*B) \\
 &\Rightarrow (A^*A)(A^*A) = (B^*B)(A^*A) \quad (\because A^*B = A^*A)
 \end{aligned}$$

Taking conjugate transpose on both sides, we have,

$$\begin{aligned}
 &\Rightarrow ((A^*A)(A^*A))^* = ((B^*B)(A^*A))^* \\
 &\Rightarrow (A^*A)^*(A^*A)^* = (A^*A)^*(B^*B)^* \\
 &\Rightarrow (A^*A)^*(A^*A) = (A^*A)^*(B^*B) \quad (\because (A^*A)^* = (A^*A), (B^*B)^* = (B^*B))
 \end{aligned} \tag{2}$$

Given $\text{Range}(A) \subseteq \text{Range}(A^*A)$, $\text{Range}(B) \subseteq \text{Range}(B^*B)$

$$\Rightarrow \text{Range}(A^*A) \subseteq \text{Range}(B^*B) \tag{3}$$

Therefore, from equations (1), (2) and (3), we have,

$$A^* \leq B \Rightarrow A^*A^* \leq B^*B$$

Theorem 2.15: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, and $\text{Range}(A) \subseteq \text{Range}(A^2)$, $\text{Range}(B) \subseteq \text{Range}(B^2)$ then $A^* \leq B \Rightarrow A^2 \leq B^2$

Proof: Let $A^* \leq B$

$$\begin{aligned}
 &A^* \leq B \Rightarrow A^*A = A^*B \text{ and } \text{Range}(A) \subseteq \text{Range}(B) \\
 &A^*A = A^*B
 \end{aligned}$$

Pre multiplying by A on both sides, we have,

$$\begin{aligned}
 &AA^*A = AA^*B \\
 &A = B \quad (\because AA^* = I) \\
 &A.I = B.I \\
 &AAA^* = BBB^* \quad (\because AA^* = I, BB^* = I) \\
 &A^2A^* = B^2B^*
 \end{aligned}$$

Post multiplying by B on both sides, we have,

$$\begin{aligned}
 &A^2A^*B = B^2B^*B \\
 &A^2A^*A = B^2B^*B \quad (\because A^*B = A^*A)
 \end{aligned}$$

Taking conjugate on both sides, we have,

$$\begin{aligned}
 &\overline{A^2 A^*A} = \overline{B^2 B^*B} \\
 &\overline{A^2} = \overline{B^2} \quad (\because \overline{A^*A} = I, \overline{B^*B} = I)
 \end{aligned}$$

Again taking conjugate on both sides, we have,

$$A^2 = B^2$$

Pre multiplying by $(A^2)^*$ on both sides we have,

$$(A^2)^*A^2 = (A^2)^*B^2 \tag{1}$$

Given $\text{Range}(A) \subseteq \text{Range}(A^2)$ and $\text{Range}(B) \subseteq \text{Range}(B^2)$

$$\Rightarrow \text{Range}(A^2) \subseteq \text{Range}(B^2) \tag{2}$$

Therefore, from equations (1) and (2) we have

$$A^2 * \leq B^2$$

That is $A * \leq B \Rightarrow A^2 * \leq B^2$

Theorem 2.16: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, and $\text{Range}(A^*) \subseteq \text{Range}(A^*)^2$, $\text{Range}(B^*) \subseteq \text{Range}(B^*)^2$ then $A \leq^* B \Rightarrow A^2 \leq^* B^2$

Proof: Similar proof of Theorem 2.16.

CONCLUSION

Left star and Right star partial ordering of conjugate unitary matrix was defined and theorems related to left star and right star partial ordering of conjugate unitary matrices are derived.

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