

FIXED POINT THEOREM FOR GENERALIZED R - WEAKLY COMMUTING MAPPING

PIYUSH KUMAR TRIPATHI*

Amity University Uttar Pradesh, (Amity School of Applied Sciences, Lucknow), India.

(Received On: 28-02-18; Revised & Accepted On: 02-04-18)

ABSTRACT

A number of authors have defined various contractive type self-mapping of metric spaces which are generalizations of well-known Banach contraction principle and have used the same technique. The contractive condition on maps produce suitable iterations, which give Cauchy sequence and a hypothesis of completeness in the range containing these sequences. The concept of generalized R- weakly commuting mappings has been used in probabilistic metric space. Using this we have obtained a number of fixed point theorems in the light of some new contractive conditions.

1. INTRODUCTION

Banach contraction principle had many applications but it is suffering from a drawback of assuming f to be continuous. In 1968, Kannan surpassed this hurdle. He took a number a with $0 < a < \frac{1}{2}$, such that for all $x, y \in X$, $d(fx, fy) \leq a[d(x, fx) + d(y, fy)]$. Then f has a unique fixed point. This result has been extended by Chatterjee [1] and others. In 1969, Meir and Keeler obtained a remarkable generalization of the Banach contraction principle. They proved that if for every $\varepsilon > 0$ there exist $\delta > 0$ such that, $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon$. Then f has a fixed point. Many authors generalized the above result and gave very useful results in process of the study. Bonsall initiated the study of fixed points of sequence of mappings. Let F be a family of self-mappings on a set X . An element $x \in X$ is called a fixed point of F if $fx = x$, for every $f \in F$. It is obvious that if a map $f \in F$ commutes with any $s \in F$ and has unique fixed point, then so is true for F . After studying the idea of Bonsall, Singh [7], Stojakovic [10] has also obtained some results on sequence of contraction mappings. Most of the papers dealing with fixed points for sequence of maps fall into one of three categories. The first category assume that each pair f_i, f_j satisfies the same contractive condition and concludes that $\{f_n\}$ has a common fixed point. The second category assume that each f_n satisfies the same contractive condition and $\{f_n\}$ converges point wise to a limit function f . The conclusion is that f has a fixed point p , which is the limit of each of the fixed points p_n of f_n . The third type assume that each f_n has a fixed point p_n , and that $\{f_n\}$ converges uniformly to a function which satisfies a particular contractive condition, with p is a fixed point of f . The conclusion is that $f_n \rightarrow p$. Rhoades extended a fourth class of theorems for a sequence of maps. In this the function f_i, f_j satisfy pair wise contraction principle, but with different contractive constants. The conclusion is that the sequence has a common unique fixed point. The sequences produce a limit point, which becomes a fixed point of the mapping. The contractive condition on mapping has two roles; first they assure that certain iterations are Cauchy, and second, they assure the uniqueness of fixed point.

Some common fixed point theorems using sequence which are not necessarily obtained as a sequence of iterates of certain mappings are motivated by a result of Jungck [3]. He proved that a continuous self-mapping f of a complete metric space (X, d) has a fixed point provided there exists $q \in (0, 1)$ and a mapping $g : X \rightarrow X$ which commute with f and satisfies

- (a) $g(X) \subseteq f(X)$
- (b) $d(gx, gy) \leq qd(fx, fy)$, for all $x, y \in X$. Then g and f have unique common fixed point.

Corresponding Author: Piyush Kumar Tripathi*
Amity University Uttar Pradesh, (Amity School of Applied Sciences, Lucknow), India.

The above theorem promoted the commutative maps as a tool for generalizing some of the results. Subsequently, using commuting- map concept, a variety of variations and generalizations of the above theorem were obtained by Yeh [12], Park [5], Singh and Pant [8]. In 1982, Sessa [6] introduce a generalization of the commuting map-concept by saying that maps $f, g : (X, d) \rightarrow (X, d)$ are weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for $x \in X$.

This property is strictly weaker than the commutativity. In 1986, Jungck [4] introduced the concept of compatibility, which is weaker than weakly commutativity.

The above concept of S. Sessa [6] was generalized by Singh and Pant [8] by introducing the definition of weakly commuting mappings in probabilistic metric space. In the sequel Dimri and Gairola [2] defined R-weakly commuting mappings in probabilistic metric space and proved some common fixed point theorems.

The above developments motivated Piyush Kumar Tripathi [11] to introduce the definition of generalized R- weakly commuting mappings in probabilistic metric space, generalizing the definition of R- weakly commuting mappings defined by Dimmri and Gairola [2] and Singh and Pant [8]. As a consequence of this definition, Piyush Kumar Tripathi [11] proved some common fixed point theorems using a lemma of Singh and Pant [8].

Definition [8]: Two self mappings f and g on a probabilistic metric space X will be called weakly commuting if

$$F_{fgp, gfp}(x) \geq F_{fp, gp}(x) \quad \forall p \in X \text{ and } x > 0.$$

In 1998 R.C. Dimiri and U.C. Gairola [2] revised the above definition of S.L. Singh and B.D. Pant and called R-weakly commuting mappings.

Definition [2]: Two self mappings f and g on a probabilistic metric space X is said to R- weakly commuting if there exist a real number $R > 0$ such that $F_{fgp, gfp}(Rx) \geq F_{fp, gp}(x) \quad \forall p \in X \text{ and } x > 0$.

Piyush Kumar Tripathi [11] defined generalized R-weakly commuting mappings as,

Definition[11]: Two self mappings f and g on a probabilistic metric space X be called generalized R- weakly commuting if there exist a real number $R > 0$ such that

$$F_{fgp, gfp}(Rx) \geq F_{fp, gp}(x) \quad \forall p, q \in X \text{ and } x > 0.$$

From the definition of generalized R- weakly commuting mappings we notice that generalized R- weakly commuting mappings implies R- weakly commuting mappings without holding converse.

Following is the useful lemma proved by S.L. Singh and B.D. Pant [8].

1.1 Lemma: Suppose $\{p_n\}$ is a sequence in Menger probabilistic metric space (X, F, t) , where t is continuous and $t(x, x) \geq x \quad \forall x \in [0, 1]$. If $\exists k \in (0, 1)$ such that $\forall x > 0$ and positive integer n , $F_{p_n, p_{n+1}}(kx) \geq F_{p_{n-1}, p_n}(x)$. Then $\{p_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

2.1 Theorem: Suppose (X, F, t) be a complete Menger probabilistic metric space, where $F_{p, q}$ is strictly increasing distribution function in $(0, 1)$, $\forall p \neq q$. Let f and g are two pair of self mappings on X , satisfying.

(a) if $\exists k \in (0, 1)$ s.t $F_{fp, fq}(kx) \geq F_{gp, gq}(x) \quad \forall p, q \in X \text{ and } x > 0$

(b) $f(X) \subset g(X)$ and f is continuous.

(c) f and g are generalized R - weakly commuting mappings.

Then f and g have unique common fixed point.

Proof: Let $p_0 \in X$. Choose $p_1 \in X$ such that $f(p_0) = g(p_1)$, this can be done because $f(X) \subset g(X)$, so we can construct a sequence $\{p_n\}$ such that $f(p_n) = g(p_{n+1})$. Then

$$F_{fp_n, fp_{n+1}}(kx) \geq F_{gp_n, gp_{n+1}}(x) = F_{fp_{n-1}, fp_n}(x)$$

i.e. $F_{fp_n, fp_{n+1}}(kx) \geq F_{fp_{n-1}, fp_n}$, so by lemma 2.1.1 [S 12] $\{fp_n\}$ is a Cauchy sequence. Since (X, F, t) is complete so $fp_n \rightarrow z \in X$, $g_n \rightarrow gz$. Since f is continuous so $ffp_n \rightarrow fz \in X$ and $fgp_n \rightarrow fz$.

Again since f and g are generalized R- weakly commuting mappings so $F_{fgp_n, gfp_n}(Rx) \geq F_{fp_n, gp_n}(x)$. Taking $n \rightarrow \infty$, we get $F_{fz, gfp_n}(Rx) \geq F_{z, z}(x) = 1$,

Therefore $F_{fz, gfp_n}(Rx) = 1$, so $gfp_n \rightarrow fz$.

Claim: z is a common fixed point of f and g . First we prove that $z = f(z)$ otherwise $F_{fp_n, ffp_n}(kx) \geq F_{gp_n, gfp_n}$, taking $n \rightarrow \infty$ we get $F_{z, fz}(kx) \geq F_{z, fz}(x)$ which is not possible because $F_{p, q}$ is strictly increasing distribution function in $(0, 1)$, $\forall p \neq q$, but in our case $kx < x$.

Therefore $z = f(z)$. Since $f(X) \subset g(X)$ so $\exists z_1 \in X$ such that $z = fz = gz_1$.

Now,

$F_{ffp_n, fz_1}(kx) \geq F_{gfp_n, gz_1}(x)$. Taking $n \rightarrow \infty$, we get $F_{fz, fz_1}(kx) \geq F_{fz, fz_1}(x) = 1$, that is

$F_{fz, fz_1}(kx) = 1$, i.e. we get $z = fz = fz_1 = gz_1$.

Again,

$F_{fz, gz}(Rx) = F_{fgz_1, ggz_1}(Rx) \geq F_{fz_1, gz_1}(x) = 1$, so $F_{fz, gz}(Rx) = 1 \Rightarrow fz = g = gz$.

Therefore z is common fixed point of f and g . For uniqueness suppose z and z' are two common fixed point of f and g .

Then $F_{z, z'}(kx) = F_{fz, fz'}(kx) \geq F_{gz, gz'}(x) = F_{z, z'}(x) \Rightarrow F_{z, z'}(kx) \geq F_{z, z'}(x)$.

Which is not possible because $F_{p, q}$ is strictly increasing distribution function in $(0, 1)$ $\forall p \neq q$, but in our case $kx < x$.

Therefore z is unique common fixed point of f and g .

REFERENCES

1. Chatterjee A. K. and Singh M. R. *Common fixed points for four maps on a metric space*, Bull. Cal. Math. Soc. 81 (1989), 466A-466B.
2. Dimri R. C. and Gairola U. C. *A fixed point theorem for generalized nonlinear contraction*, Gurukula Kangri Vijnan Patrika Aryabhata 1 (2) (1998), 155-160.
3. Jungck G. *Commuting mappings and fixed points*, Amer. Math. Monthly 83 (1976), 261-263. MR 53#4031
4. Jungck G. *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9 (1986), 771-779. MR 87m:54122
5. Park S. *A coincidence theorem*, Bull. Acad. Polon. Sci. Math. 29 (1981), 487-489.
6. Sessa S. *On a weak commutativity condition of mappings in fixed point contractions*, Publ. Inst. Math. (N. S.) (Beograd) 32 (1982), 149-153. MR 85f:54107
7. Singh S. L. *Application of a common fixed point theorem*, Math. Sem. Notes. Kobe Univ. 6 (1978), 37-40. MR0606178 (82e:54058).
8. Singh S. L. and Pant B. D. *Fixed point theorems for commuting mappings in probabilistic metric spaces*, Honam Math. J. 5 (1983), 139-150.
9. Singh S. L. and Pant B. D. *Common fixed points of weakly commuting mappings on non-Archimedean Menger spaces*, The Vikram Math. J. 6 (1985), 27-31.
10. Stojaković M. *A common fixed point theorem for the commuting mappings*, Indian J. Pure Appl. Math. 17 (4) (1986), 466-475.
11. Tripathi Piyush and Pathak Ajita. *Generalized R – Weakly Commuting Mappings in Non Archimedean Menger Space*, International Journal of Mathematical Archive 2(1) (2011), 165-168.
12. Yeh C.C. *Common fixed point of continuous mappings in metric spaces*, Publ. De Inst. Math. 27 (41) (1981), 21-25.

Source of support: Nil, Conflict of interest: None Declared.

[Copy right © 2018. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]