International Journal of Mathematical Archive-2(9), 2011, Page: 1742-1746
JMA Available online through www.ijma.info ISSN 2229 – 5046

A Remark on Invariant Subspaces of Index One in p^t (μ)-Spaces

K. Hedayatian*

Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran

E-mail: hedayati@shirazu.ac.ir

(Received on: 26-06-11; Accepted on: 06-07-11)

ABSTRACT

In this paper, isometric equivalence of the multiplication operator M_z restricted to the invariant subspaces of index 1 in p^t (μ) are investigated. Also, the reducing subspaces of the operators M_z^i, $i \geq 1$ on p^2 (μ) are discussed.

[1]**2000 Mathematics Subject Classification:** *47B38, 47A15.*

Key words and Phrases: *Isometric equivalence of subspaces, $p^t(\mu)$, index 1, reducing subspaces. This research was in part supported by a grant from Shiraz University Research Council.*

1. INTRODUCTION:

Let μ be a finite, nonnegative measure on the closure of the open unit disc D. For $1 \leq t < \infty$, let $p^t(\mu)$ denote the closure of analytic polynomials in $L^t(\mu)$. A point $z \in \mathbb{C}$ is called a bounded point evaluation for $p^t(\mu)$ if there is a positive constant c such that $|p(z)| \leq c \| p \|_{L^t(\mu)}$ for all polynomials p; the collection of all such points is denoted bpe $(p^t(\mu))$. If $z \in \mathbb{C}$ and there are positive constants M and r such that $| p(w) | \leq M \| p \|_{L^t(\mu)}$, whenever p is a polynomial$_t$ and $|w - z| < r$, then z is called an analytic bounded point evaluation for $p(\mu)$. Denote the set of all such points by abpe $(p^t(\mu))$. Observe that abpe $(p^t(\mu))$ is an open subset of bpe $(p^t(\mu))$.

We will, assume that support $(\mu) = D$, $\mu(\{\lambda\}) = 0$ for every $\lambda \in D$, $abpe(p^t(\mu)) = D$, and $p^t(\mu)$ is irreducible, by which we mean that $p^t(\mu)$ contains no nontrivial characteristic functions. It is known that $p^t(\mu)$ has the division property at all $\lambda \in D$; that is, if $f \in p^t(\mu)$ and $\lambda \in D$ with $f(\lambda) = 0$, then, $f(z)/(z - \lambda) \in p^t(\mu)$. Multiplication by z defines a bounded linear operator on $p(\mu)$ which we will denote by S. An invariant subspace of $p^t(\mu)$ is a closed linear subspace $M \subseteq p^t(\mu)$ such that $SM \subseteq M$. It is also known that $S - \lambda I$ is bounded below for every $\lambda \in D$; so if M is an invariant subspace of $p^t(\mu)$ then $(S - \lambda I)M$ is closed. Furthermore, it follows from the Fredholm theory that dim $M/(S - \lambda I)M$ does not depend on $\lambda \in D$. The index of an invariant subspace of M is the dimension of M/SM. Recall that if $\mu = \frac{1}{\pi} A$ is the normalized Lebesgue measure on D then $p^t(\mu)$ is the Bergman space L_a^t.

***Corresponding author:** *K. Hedayatian**, ***E-mail:** *hedayati@shirazu.ac.ir*

2. ISOMETRICALLY EQUIVALENT INVARIANT SUBSPACES

Recall that an operator U is an isometry if $\|Uf\|=\|f\|$ for every f in $dom(U)$. Two invariant subspaces M and N of $p^t(\mu)$ are called isometrically equivalent if there is a linear isometry $U: M \to N$ such that $US|_M = S|_N U$. It follows from Beurling's theorem that all invariant subspaces of the multiplication by z, M_z, on the Hardy space H^2 are isometrically equivalent to one another. If the underlying space is a Hilbert space and U is a linear surjective isometry then M and N are called unitarily equivalent. Richter in [3] has shown that no two invariant subspaces of the Bergman space L_a^2 are unitarily equivalent to one another, unless they are equal. Also, he has proved that if M and N are invariant subspaces of M_z on the Dirichlet space D and $N \subseteq M$ then $M = N$.

Theorem 1: Suppose that M and N are two isometrically equivalent invariant subspaces of $p^t(\mu)$ both of indices 1. If $N \subseteq M$ then $M = N$.

Proof: Let $U: M \to N$ be a linear isometry such that $US|_M = S|_N U$ For any $\lambda \in D$, define K_λ^M to be the element in $\ker(S|_M - \lambda)^*$ such that $f(\lambda) = \langle f, K_\lambda^M \rangle$ for every $f \in M$. If $Z(M) = \{\lambda \in D : f(\lambda) = 0$, for each $f \in M\}$ then by Lemmas 2.1 and 3.1 of [2], for every $\lambda \in D \setminus Z(M)$, $\ker(S|_M - \lambda)^* = C.K_\lambda^M$. Similarly, $\ker(S|_N - \lambda)^* = C.K_\lambda^N$ for every $\lambda \in D \setminus Z(N) \subseteq D \setminus Z(M)$. Since U^* maps $\ker(S|_N - \lambda)^*$ into $\ker(S|_M - \lambda)^*$, we see that $U^* K_\lambda^N = \varphi(\lambda) K_\lambda^M$ for every $\lambda \in D \setminus Z(N)$. Thus,

$$(Uf)(\lambda) = \langle Uf, K_\lambda^N \rangle = \langle f, U^* K_\lambda^N \rangle = \langle f, \varphi(\lambda) K_\lambda^M \rangle$$
$$= \varphi(\lambda) \langle f, K_\lambda^M \rangle = \varphi(\lambda) f(\lambda)$$

for each $f \in M$ and $\lambda \in D \setminus Z(N)$.

Since $N \neq \{0\}$, the set $Z(N)$ is a countable subset of D with accumulation points only on the unit circle T. Thus, $D \setminus Z(N)$ is always a nonempty open set. Let f be a nonzero function in M. If f has a zero of order n at $z \in D \setminus (N)$ and φf has a zero of order m at z, then $(j-1)n \le jm$ for every $j \ge 1$, because $f^{j-1}(f \varphi^j) = (f \varphi)^j$ and $f \varphi^j$ is analytic on $D \setminus Z(N)$. Therefore,

$$\frac{j-1}{j} \le \frac{m}{n} \quad j = 1, 2, 3, \ldots,$$

Letting $j \to \infty$, we conclude that n ≤ m which implies that $\varphi = \dfrac{\varphi f}{f}$ is analytic on $D \setminus Z(N)$. Feom now on, suppose that f is a fixed nonzero element in M. Since $\|U\| = 1$, we have

$$|(\varphi^n(f)(\lambda)| = |\langle U^n f, K_\lambda^M \rangle| \le \|f\| \, \|K_\lambda^M\|$$

for each $n \ge 1$ and all $\lambda \in D \setminus Z(N)$; thus

$$|\varphi(\lambda)| |f(\lambda)|^{1/n} \le (\|f\|.\|K_\lambda^M\|)^{1/n}.$$

Letting $n \to \infty$, we see that $|\varphi(\lambda)| \le 1$ for all λ such that $f(\lambda) \neq 0$. Thus, $|\varphi(\lambda)| \le 1$ 1 on $D \setminus Z(N)$, and so can be extended to an analytic function on D. Hence, $(Ug)(\lambda) = \varphi(\lambda) g(\lambda)$ for every $g \in M$ and $\lambda \in D$. Put $\Omega = \{z : |\varphi(z)| < 1\}$. Then

$$\|\varphi^n f\|_p = \|f\|_p;$$

that is

$$\int_{\Omega} |\varphi|^{np} |f|^p \, d\mu = \int_{\Omega} |f|^p \, d\mu, \; n \geq 1.$$

Applying the dominated convergence theorem, we conclude that $\int_{\Omega} |f|^p \, d\mu = 0$. Consequently, if

$$\Omega_n = \{z \in \Omega : |f(z)| \geq \frac{1}{n}\}, \; n \geq 1$$

then

$$\frac{1}{n^p} \mu(\Omega_n) \leq \int_{\Omega_n} |f|^p \, d\mu \leq \int_{\Omega} |f|^p \, d\mu = 0.$$

Since f is a nonzero analytic function, $\Omega_0 = \{z \in \Omega : f(z) = 0\}$ is a countable set and so $\mu(\Omega_0) = 0$. Hence,

$$\mu(\Omega) \leq \mu(\bigcup_{n=0}^{\infty} \Omega_n) \leq \sum_{n=0}^{\infty} \mu(\Omega_n) = 0.$$

This shows that $|\varphi(z)| = 1$ a.e. $[\mu]$. The continuity of the function $\psi(z) = |\varphi(z)|$ on D implies that if $\psi(z_0) \neq 1$ for some z_0 in D then $\psi(z) \neq 1$ on a neighborhood of z_0, which contradicts the fact that $\sup p\mu = \overline{D}$. Therefore, $|\varphi(z)| = 1$ for all $z \in \mathbf{C}$ and the open mapping theorem implies that φ is a constant function. Hence, $M \subseteq N$.

We now give some direct consequences of the above theorem.

Corollary 1: Suppose that $\mu(\partial D) > 0$, and M and N are two nonzero isometrically equivalent invariant subspaces of $p^t(\mu)$ such that $N \subseteq M$. Then $M = N$.

Proof: By Theorem A of [1], M and N have index 1, and so the result follows using Theorem 1.

We remark that there are various subclasses of invariant subspaces of $p^t(\mu)$ having index 1 (see, for example, [2]). In particular, it is shown that any invariant subspace of the Bergman space $L_a^p, p \geq 1$, generated by functions in L_a^{2p}, must have index 1. Recall that for a family $\{M_\gamma\}_{\gamma \in \Gamma}$ of invariant subspaces of $p^t(\mu)$, $\bigvee_{\gamma \in \Gamma} \mu_\gamma$ is the smallest closed subspace that contains each M_γ.

Corollary 2: Suppose that M and N are two distinct invariant subspaces of $p^t(\mu)$ of index 1 such that $M \cap N \neq \{0\}$. Then M and N are not isometrically equivalent to $M \vee N$.

Proof: By Corollary 3.12 of [2], $M \vee N$ has index 1. So in light of Theorem 1, we get the result.

Corollary 3: Let $\{M_\gamma\}_{\gamma \in \Gamma}$ be a family of invariant subspaces of $p^t(\mu)$ of index 1, and M_{γ_0} be a nonzero element of this family such that $M_{\gamma_0} \vee M_\gamma$ has index 1 for all $\gamma \in \Gamma$. If there is $\gamma_1 \in \Gamma$ so that M_{γ_1} and $\bigvee_{\gamma \in \Gamma} M_\gamma$ are isometrically equivalent then $M_{\gamma_1} = \bigvee_{\gamma \in \Gamma} M_\gamma$.

Proof: By Theorem 3.13 (b) of [2], $\vee_{\gamma \in \Gamma} M_\gamma$ has index 1. So using Theorem 1, the result follows.

Corollary 4: Suppose that $\{M_\gamma\}_{\gamma\in\Gamma}$ is a family of invariant subspaces of $p^t(\mu)$ of index 1. If M_{γ_0} and $\bigcap_{\gamma\in\Gamma} M_\gamma$ are isometrically equivalent then $\bigcap_{\gamma\in\Gamma} M_\gamma = M_{\gamma_0}$.

Proof: By Theorem 3.16 of [2], $\bigcap_{\gamma\in\Gamma} M_\gamma$ is of index 1. Now, apply Theorem 1.

For a subspace N of $p^t(\mu)$ recall that $N^\perp$ is the annihilator of N in $(p^t(\mu))^*$; i.e., $N^\perp = \{x \in (p^t(\mu))^* : x(g) = 0, \forall g \in N\}$. Keeping this in mind, we bring another consequence of the preceding theorem.

Corollary 5: Suppose that the spectrum of the operator S is $\sigma(S) = D$. Let M and N be two isometrically equivalent, invariant subspaces of $p^t(\mu)$ such that $N \subseteq M$ and N has index 1. Moreover, suppose that there is a unimodular complex number λ which is not in $\sigma(S^*|_{N^\perp})$. Then $M = N$.

Proof: It follows from Corollary 4.8 of [2] that M has index 1, so again the result follows using Theorem 1.

3. REDUCING SUBSPACES

For $i \geq 1$, we recall that an invariant subspace M of S^i in $p^2(\mu)$ is called a reducing subspace of S^i if S^i $M^\perp \subseteq M^\perp$. Also, the operator S is irreducible if the only reducing subspaces of S are the trivial subspaces.

From now on, we assume that support $(\mu) \subseteq \overline{D}$ and $\langle z^n, z^m\rangle = 0$ for $n \neq m$, where $\langle .,.\rangle$ denotes the inner product in $p^2(\mu)$. For example, if v is a probability measure on [0, 1] and μ is the measure defined on $\overline{D}$ by

$$d\mu(re^{i\theta}) = \frac{1}{2\pi} d\theta dv(r),$$

then a routine calculation reveals that $\langle z^n, z^m\rangle = 0, n \neq m$. Moreover, $\|1\|_{L^2(\mu)} = 1$.The next theorem is on the reducing subspaces of $S^i, i \geq 1$.

Theorem 2: Let $w_n = \|z^n\|^{1/2}_{L^2(\mu)}, n \geq 1$ and

$$\underline{\lim} w_n = \sup_n \frac{w_{n+1}}{w_n} = 1.$$

Then S, the operator of multiplication by z on $p^2(\mu)$, is irreducible. Moreover, for $i > 1$ one of the following results holds:

(i) If for every pair of distinct nonnegative integers m, n with $0 \leq n \leq i-1$ and $0 \leq m \leq i-1$ there exists some integer $k > 0$ such that

$$\frac{w_{n+ki}}{w_n} \neq \frac{w_{m+ki}}{w_m}$$

then S^i has exactly 2^i distinct reducing subspaces.

(ii) If there exist two distinct integers n, m such that $0 \leq n \leq i-1$ and $0 \leq m \leq i-1$ and

$$\frac{w_{n+ki}}{w_n}=\frac{w_{m+ki}}{w_m}$$

for all integers k > 0, then S^i has infinitely many distinct reducing subspaces.

Proof: The elements $e_n=\frac{z^n}{w_n^2}, n\geq 0$ form an orthonormal basis for $p^2(\mu)$. So $Se_n=(w_{n+1}/w_n)^2 e_{n+1}$; i.e., S is a weighted shift with weight sequence $V_n=(W_{n+1}/W_n)^2$. Thus, Corollary 2 of [4] implies that S is irreducible. Without loss of generality, assume that $w_0=1$. By Proposition7 of [4] the operator S is unitarily equivalent to M_z on the space of $H_w^2, (w=\{w_0,w_1,w_2,\ldots\})$ which is the space of all formal power series $\sum_{n=0}^{\infty}\widehat{f}(n)z^n$ such that $\sum_{n=0}^{\infty}|\widehat{f}(n)|^2 w_n<\infty$. The mentioned proposition also implies that

$$\|S\|=\sup_{n\geq 0} v_n=1,$$

and so $w_n^2\leq w_0^2=1$, for all $n\geq 0$. Therefore

$$1=\underline{\lim}\sqrt[n]{w_n^2}\leq\overline{\lim}\sqrt[n]{w_n^2}\leq 1$$

which implies that $\sqrt[n]{w_n}$ converges to 1 as $n\to+\infty$. Now, if $f(z)=\sum_{n=0}^{\infty}\widehat{f}(n)z^n\in H_w^2$, then

$$\overline{\lim}\sqrt[n]{|\widehat{f}(n)|^2}=\overline{\lim}\sqrt[n]{|\widehat{f}(n)|^2 w_n}\leq 1.$$

Consequently, $\overline{\lim}\sqrt[n]{|\widehat{f}(n)|}\leq 1$. Hence, $f(z)$ is analytic on D. Now, the result follows from Theorems B and C of [5].

Remark: It is easily seen that if $d\mu(re^{i\theta})=\frac{1}{2\pi}d\theta dm(r)$, where m is the Lebesgue measure on [0, 1], then condition (i) of Theorem 2 holds. Hence, for every $i>1$, the operators S^i has exactly 2^i distinct reducing subspaces.

REFERENCES

[1] Aleman A., Richter S., Sundberg C., Nontangential limits in $p^t(\mu)$- spaces and the index of invariant subspaces, Annals of Mathematics, 169 (2009), 449-490.

[2] Richter S., Invariant subspaces in Banach spaces of analytic functions, Trans, Amer. Math. Soc. 304 (1987), 585-615.

[3] Richter S., Unitary equivalence of invariant subspaces of Bergman and Dirichlet spaces, Pacific J. Math, 133 (1988), 151-156.

[4] Shields A. L., Weighted shift operators and analytic function theory, Math. Surveys, Vol. 13, Amer. Math. Soc., Providence, 1974.

[5] Stessin M., Zhu K., Reducing subspaces of weighted shift operators, Proc. Amer. Math. Soc. 130 (2002), 2631-2639.

****

